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# Resolving actions of compact Lie groups

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A general process for the desingularization of smooth actions of compact Lie groups is described. If $G$ is a compact Lie group, it is shown that there is naturally associated to any compact $G$ manifold $M$ a compact $G \times (Z/2)^p$ manifold on which $G$ acts principally. Here $Z/2$ denotes the cyclic group of order two and $p + 1$ is the number of orbit types of the $G$ action on $M$.

## 1. Introduction

Let $G$ be a compact Lie group. In this note we show that there is naturally associated to any compact $G$ manifold $M$ a compact $G \times (Z/2)^p$ manifold $\hat{M}$ on which $G$ acts principally. Here $Z/2$ denotes the cyclic group of order two and $p + 1$ is the number of orbit types of the action of $G$ on $M$ (see §2). We call $\hat{M}$ a resolution of (the $G$ action on) $M$. Our method of construction of $\hat{M}$ is a modification of the familiar "blowing up" transformation of algebraic geometry and is closely related to the polar coordinate transformation as used, for example, by Ruelle and Takens in [4].

Our process of resolution is basic to a study of ours on the linearizations, modulo $G$, of equivariant diffeomorphisms close to the identity map and the construction of a *maximal family of slices for a* $G$ action. However, we feel that our results on resolutions may be of wider interest, with possible applications to the classification theory of smooth actions, and so we are presenting them separately.

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## 2. Blowing up $G$ manifolds

We start by recalling some facts about smooth (that is $C^\infty$ ) actions of a compact Lie group on a compact differential manifold. We refer to Bredon's text [2], especially Chapter 6, for full details and proofs.

Given a compact connected differential manifold $M$ and the action of a compact Lie group $G$ on $M$, we let $G(x)$ denote the $G$ orbit through $x$ and $G_x$ the isotropy subgroup of $G$ at $x$, $x \in M$. $G(x)$ is equivariantly diffeomorphic to the homogeneous space $G/G_x$. The isotropy subgroup at $gx$, $g \in G$, is conjugate to that at $x$ and indeed is obviously equal to $gG_xg^{-1}$. We say that $x, y \in M$ are of the same orbit type if $G_x$ and $G_y$ are conjugate subgroups of $G$ or, equivalently, if $G(x)$ and $G(y)$ are equivariantly diffeomorphic. The equality of orbit type partitions $M$ into points of the same orbit type. If $M$ is compact, this partition is finite. We write

$$M = \bigcup_{i \in I} M_i ,$$

where $M_i$ are the equivalence classes of points of the same orbit type. We define $\mathrm{orb} : M \to I$ by $\mathrm{orb}(x) = i$, $x \in M_i$.

There is defined a natural partial order on $I$ by $i < j$ if there exist $x \in M_i$, $y \in M_j$, such that $G_x \supset G_y$ (strict inclusion). We say that $x$ is of minimal orbit type if there does not exist $y \in M$ such that $\mathrm{orb}(y) < \mathrm{orb}(x)$. The finiteness of $I$ implies there exists at least one minimal orbit type. We may similarly define a maximal orbit type. In this case it may be shown that there exists precisely one maximal orbit type, say $N$, and that $M_N$ is an open dense subset of $M$ (connected if $G$ is connected).

For convenience we shall label orbit types by integers and write

$$M = \bigcup_{1 \le i \le N} M_i ,$$

where $\mathrm{orb}(x) < \mathrm{orb}(y)$ implies that if $x \in M_i$ and $y \in M_j$ then $i < j$ (the converse need not be true: if $i < j$ (as integers) then $G_x$ and $G_y$

need not be related. For example, $M_1$ and $M_2$ may both be minimal orbit types). We shall say that the action is principal if there exists only one orbit type and that it is free if $G_x = \{e\}$ for all $x \in M$.

Let $\xi$ be a riemannian metric on $M$. Averaging over $G$ using Haar measure we may assume that $\xi$ is $G$ invariant. We call $M$, together with a $G$ action and equivariant riemannian metric, a riemannian $G$ manifold. In the sequel, we assume $M$ is a riemannian $G$ manifold. Apart from the notation introduced above, we let $\mathrm{diff}_G^k(M)$ denote the space of $C^k$ equivariant diffeomorphisms of $M$, $1 \leq k \leq \infty$. In case $k < \infty$, we give $\mathrm{diff}_G^k(M)$ the $C^k$ topology ([1], [3]).

DEFINITION. A resolution of $M$ consists of a $G \times (Z/2)^{N-1}$ manifold $\hat{M}$ and a $C^\infty$ map $\pi : \hat{M} \to M$ and homomorphism $\phi : \mathrm{diff}_G^\infty(M) \to \mathrm{diff}^\infty(\hat{M})$ such that:

(1) if we give $M$ the trivial $(Z/2)^{N-1}$ action, $\pi$ is $G \times (Z/2)^{N-1}$ equivariant;

(2) the generators $f_1, \ldots, f_{N-1}$ of the $(Z/2)^{N-1}$ action on $\hat{M}$ may be indexed so that

$$\pi^{-1}(M_j) = \mathrm{fix}(f_j) \setminus \bigcup_{i<j} \mathrm{fix}(f_i), \quad 1 \leq j \leq N-1,$$

$$= \text{free part of the } (Z/2)^{N-1} \text{ action}, \quad j = N;$$

(3) $G$ acts principally on $\hat{M}$;

(4) for all $f \in \mathrm{diff}_G^\infty(M)$, $\phi(f)$ is a $C^\infty$ $G \times (Z/2)^{N-1}$ invariant map covering $f$.

REMARK. It follows from (2) of the definition that $\pi|\pi^{-1}(M_N)$ is a $2^{N-1}$ fold covering map of $M_N$.

THEOREM A. *Every compact $G$ manifold has a resolution. Moreover, for the resolution we construct we may require that the extension map*

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$\phi : \text{diff}_G^\infty(M) \to \text{diff}^\infty(\hat{M})$ *extends to a continuous map*

$$\phi : \text{diff}_G^{k+N-1}(M) \to \text{diff}^k(\hat{M}) , \quad r \geq 0 .$$

Before starting the proof of Theorem A, we prove a simple and presumably well known result which is special to actions by finite groups of odd order.

**THEOREM B.** *Let $G$ be a finite group of odd order acting on $M$. Then there exists a principal $G$ manifold $\tilde{M}$ and a $C^\infty$ equivariant map $\pi : \tilde{M} \to M$ such that $\pi^{-1}(M_N)$ is open and dense in $\tilde{M}$ and $\pi$ maps $\pi^{-1}(M_N)$ diffeomorphically onto $M_N$.*

Proof. We shall successively blow up the submanifolds $M_1, \ldots, M_{N-1}$. The techniques we use are well known and standard in equivariant differential topology and so we only outline the main details. Let $E_1 \to M_1$ denote the normal bundle of $M_1$ and choose $r > 0$ so that the disc bundle $E_1(r) = \{v \in E_1 : \|v\| < r\}$ is embedded as a tubular neighbourhood $Q(r)$ of $M_1$ by the exponential map. Choosing $r$ smaller if necessary, we may also require that $\partial Q(r)$ is a codimension one sub-manifold of $M$ equivariantly diffeomorphic to the unit sphere bundle $S(E_1)$ of $E_1$. Define $\gamma : S(E_1) \times R \to R$ by $\gamma(\theta, t) = \exp(t\theta)$. $Z/2$ acts freely on $S(E_1) \times R$ as multiplication by $-1$ (on both factors) and this action commutes with $\gamma$. If $X$ is any $Z/2$ invariant subset of $R$, we let $P(E_1, X)$ denote the orbit space of the induced $Z/2$ action on $S(E_1) \times X$. $\gamma$ restricts to a $C^\infty$ diffeomorphism of $P(E_1, \{-r, +r\})$ with $\partial Q(r)$ and, in the usual way, we may form the $G$ manifold

$$\tilde{M}_1 = (M \backslash Q(r)) \underset{\gamma}{\cup} P(E_1, [-r, +r]) .$$

$\pi_1 : \tilde{M}_1 \to M$ is defined to be the identity on $M \backslash Q(r)$ and $\gamma$ on $P(E_1, [-r, +r])$. Clearly $\pi_1$ is a diffeomorphism of $\pi_1^{-1}(M_1)$. Since $G$ is of odd order, it does not contain any $Z/2$ subgroups. Hence the orbit types of $G$ on $P(E_1, [-r, +r])$ are the same as those of $G$ on

$S(E_1) \times [-r, +r]$, which in turn are a subset of the orbit types $2, \ldots, N$. Hence no orbit of type $1$ appears in $\tilde{M}_1$. Iterating this process we may remove orbits of types $2$ up to $N-1$. //

**Proof of Theorem A.** As in the proof of Theorem B, we form the unit sphere bundle $S(E_1)$ of the normal bundle $E_1$ of $M_1$ and choose $r > 0$ so that exp embeds the disc bundle of radius $r^2$ as a tubular neighbourhood $Q(r)$ of $M_1$ with smooth boundary $\partial Q(r)$. Let $\gamma : S(E_1) \times R \to M$ be the map $(\theta, t) \longmapsto \exp(t^2\theta)$. $\gamma$ is $G \times (Z/2)$ invariant if we take the $Z/2$ action on $S(E_1) \times R$ defined by $(\theta, t) \to (\theta, -t)$ and the trivial $Z/2$ action on $M$. We define $\hat{M}_1$ to be

$$(M \backslash Q(r)) \underset{\gamma_-}{\cup} (S(E_1) \times [-r, +r]) \underset{\gamma_+}{\cup} (M \backslash Q(r)) ,$$

where $\gamma_{\pm} = \gamma | S(E_1) \times \{\pm r\}$ identifies $\partial Q(r) = \partial(M \backslash Q(r))$ to $S(E_1) \times \{\pm r\}$. The $Z/2$ action on $S(E_1) \times [-r, +r]$ extends in the obvious way to $\hat{M}_1$ and, since $\gamma$ is $G \times (Z/2)$ invariant, we see that $\hat{M}_1$ is a $G \times (Z/2)$ manifold. $\pi_1 : \hat{M}_1 \to M$ is defined to be the identity on either of the components $M \backslash Q(r)$ and $\gamma$ on $S(E_1) \times [-r, +r]$. Set

$$M_1^{\pm} = S(E_1) \times (0, \pm r] \underset{\gamma_{\pm}}{\cup} (M \backslash Q(r)) .$$

$\pi_1$ restricts to an equivariant diffeomorphism $\pi_1^{\pm}$ of $M_1^{\pm}$ onto $M \backslash M_1$. The only orbit types that can occur for the $G$ action on $\hat{M}_1$ are $2, \ldots, N$. If we let $\alpha_1^1$ be the involution generating the $Z/2$ action on $\hat{M}_1$, $\alpha_1$ is $C^\infty$, equivariant and has fixed point set $\pi_1^{-1}(M_1)$.

Let $f \in \mathrm{diff}_G^k(M)$. Then $f(M_j) = M_j$, $j = 1, \ldots, N$. We define $\hat{f}_1 : \hat{M}_1^{\pm} \to M_1^{\pm}$ by

$$\hat{f}_1 = (\pi_1^{\pm})^{-1} f \pi_1^{\pm} .$$

Clearly $f_1$ is $C^k$ on $M_1^{\pm}$. Choose $s > 0$ so that $f(Q(s)) \subset Q(r)$.

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Suppose $s < r$. Then for $0 < |t| < s$, we see that, relative to the coordinates on $\hat{M}_1$ given by $S(E_1) \times [-r, +r]$,

$$\hat{f}_1(t, \theta) = \left(\frac{\exp^{-1}(f\exp(t^2\theta))}{\|\exp^{-1}(f\exp(t^2\theta))\|}, \operatorname{sign}(t)\|\exp^{-1}(f \exp(t^2\theta))\|^{\frac{1}{2}}\right) .$$

$\hat{f}_1$ is certainly $G \times (Z/2)$ invariant. We claim that $\hat{f}_1$ extends as a $C^{k-1}$ $G \times (Z/2)$ invariant map across $\pi_1^{-1}(M_1)$. For this it is clearly enough to show that there exists a $C^{k-1}$ map $g : S(E_1) \times (-s, +s) \to E_1$ such that $g \neq 0$ and

$$\exp^{-1}(f \exp(t^2\theta)) = t^2 g(\theta, t) , \quad t \neq 0 , \quad \theta \in S(E_1) .$$

Fix a $C^\infty$ embedding of $M$ into some $R^n$. Such an embedding induces an embedding of $TM$ in $R^{2n}$ and, by restriction, of $E_1$ into $R^{2n}$. For $\theta \in S(E_1)$, consider the map $\rho_\theta : (-s, +s) \to R^{2n}$ defined by

$$\rho_\theta(t) = \exp^{-1}(f \exp(t\theta)) .$$

Then

$$\rho_\theta(t^2) = \int_0^1 \frac{\partial}{\partial u}\left(\rho_\theta(ut^2)\right)du$$
$$= t^2 \int_0^1 \rho_\theta(ut^2)du .$$

Therefore, if we define $g(\theta, t) = \int_0^1 \rho_\theta(ut^2)du$, $g$ will satisfy our requirements. Moreover, it is easily verified that if $(f_n)$ is a convergent sequence in the $C^k$ topology, the corresponding sequence $(g_n)$ will be convergent in the $C^{k-1}$ topology. In other words the map $\operatorname{diff}_G^k(M) \to \operatorname{diff}^{k-1}(\hat{M}_1)$ ; $f \to \hat{f}_1$ is continuous. The map is obviously a homomorphism.

Suppose inductively that we have performed $j$ successive polar blow ups to obtain a $G \times (Z/2)^j$ manifold $\hat{M}_j$ with $G$ orbit types $j+1, \dots, N$, $j+1 < N$. Denote the generators of the $(Z/2)^j$ action by $\alpha_j^1, \dots, \alpha_j^j$. . As above we choose an equivalent riemannian metric on $\hat{M}_j$ and polar blow up the set of points of orbit type $j + 1$ to obtain a new $G$ manifold $\hat{M}_{j+1}$. We set $\alpha_{j+1}^i = \hat{\alpha}_j^i$, $1 \le i \le j$, and let $\alpha_{j+1}^{j+1}$ denote the generator of the $Z/2$ action originating from the polar blow up of $\hat{M}_j$. Now since $\alpha_j^i$ is $G$ invariant $\alpha_{j+1}^i$ commutes with $\alpha_{j+1}^{j+1}$, $1 \le i \le j$. Also $\alpha_{j+1}^i$ and $\alpha_{j+1}^k$ commute for $1 \le i, k \le j$ since the lifting map is a homomorphism. Hence we have a $G \times (Z/2)^{j+1}$ action of $\hat{M}_{j+1}$ with $\alpha_{j+1}^1, \dots, \alpha_{j+1}^{j+1}$ generators of the $(Z/2)^{j+1}$ action. Similarly, if we have shown inductively that a $C^k$ $G$ diffeomorphism $f$ of $M$ lifts to a $C^{k-j}$ $G \times (Z/2)^j$ diffeomorphism $\hat{f}_j$ of $\hat{M}_j$, then $\hat{f}_j$ lifts to a $C^{k-j-1}$ diffeomorphism $\hat{f}_{j+1}$ of $\hat{M}_{j+1}$. Hence the inductive step is completed and we may take $\hat{M} = \hat{M}_{N-1}$, $\phi(f) = \hat{f}_{N-1}$. The generators of the $(Z/2)^{N-1}$ action on $\hat{M}$ are $\alpha_{N-1}^j$, $1 \le j \le N-1$. //

REMARKS. In the sequel we shall refer to the resolution of $M$ constructed in Theorem A as the polar resolution of $M$. We call the manifold $\hat{M}_j$ obtained in the proof of Theorem A after $j$ successive polar blow ups the $j$-fold polar blow up of $M$. We denote the $i$th orbit type of $\hat{M}_j$ by $(\hat{M}_j)_i$. Thus we will have $(\hat{M}_j)_i = \emptyset$, $1 \le i \le j$, and $\hat{M}_{N-1} = (\hat{M}_{N-1})_{N-1} = \hat{M}$.

EXAMPLE. Take the circle action on $S^3$ induced from scalar multiplication on $C^2$ by $(e^{pi\phi}, e^{qi\phi})$, $p, q \in Z^+$. If $p = q$, the action on $S^3$ is principal. If $p|q$ or $q|p$, $p \ne q$, then there are two orbit types with corresponding isotropy subgroups $Z/p$, $Z/q$. Suppose that $q > p$. Then the minimal orbit type consists of a single

$S^1$ orbit with isotropy subgroup $Z/q$ . Resolving, we find

$$\hat{S}^3 = D^2 \times S^1 \underset{\text{id}}{\cup} D^2 \times S^1 = S^2 \times S^1 .$$

The circle action on $D^2 \times S^1$ is $(z, y) \mapsto \left(ze^{iq\phi}, ye^{ip\phi}\right)$ , $z \in D^2$ , $y \in S^1 \subset C$ , and the involution is reflection in $\partial D^2 \subset S^2$ with fixed set $\partial D^2 \subset S^1 = T^2$ .

Finally suppose $p \nmid q$ , $q \nmid p$ . There are now three orbit types with isotropy subgroups $Z/p$, $Z/q$, $Z/m$ , where $m$ is the highest common factor of $p$ and $q$ . In this case $\hat{S}^3 = T^3$ and the circle action on $T^3 = S^1 \times S^1 \times S^1 \subset C^3$ is given by $(\alpha, \beta, \gamma) \mapsto \left(e^{pi\phi}\alpha, e^{qi\phi}\beta, \gamma\right)$ , and the involutions are

$$f_1(\alpha, \beta, \gamma) = (\alpha, \beta, \overline{\gamma}) ,$$

$$f_2(\alpha, \beta, \gamma) = (\alpha, \beta, -\overline{\gamma}) .$$

LEMMA C. *Up to* $G \times (Z/2)^j$ *diffeomorphism, the manifolds* $\hat{M}_j$ *constructed in the proof of Theorem* A *are independent of choices of riemannian metrics on* $M, \ldots, \hat{M}_{j-1}$ .

Proof. It is enough to prove this for $j = 1$ , since the general case follows by iteration. Let $M$ have equivariant riemannian metrics $\xi$ and $\xi'$ . Following the notation and assumptions of the proof of Theorem A, we define

$$\gamma_1 : S(E_1) \times [-r, +r] \to S(E_1)' \times R$$

by

$$\gamma_1(\theta, t) = \left(H(t^2\theta)/\|H(t^2\theta)\|', \text{sign}(t)\left(\|H(t^2\theta)\|'\right)^{\frac{1}{2}}\right),$$

where $H(t^2\theta) = (\exp')^{-1}(\exp(t^2\theta))$ and $\exp'$, $\|\ \|'$ , and $S(E_1)'$ are the exponential norm and unit sphere bundle corresponding to $\xi'$ . As in Theorem A, $\gamma_1$ is $C^\infty$ . Taking $r$ smaller if necessary, we may assume that the image of $\gamma_1$ lies in $S(E_1)' \times [-r', +r']$ . We extend $\gamma_1$ to a

$G \times (Z/2)$ diffeomorphism of $\hat{M}_1$ by setting $\gamma_1 = (\pi_1)^{\pm}(\pi_1)^{\pm}$ outside $S(E_1) \times [-r, +r]$ . //

**COROLLARY.** *The polar resolution of* $M$ *is independent of the choice of metrics up to* $G \times (Z/2)^{N-1}$ *diffeomorphism.*

## 3. Blowing down $G$ manifolds

It is not hard to prove a converse to Theorem A and in this final section we shall indicate how this may be done.

**DEFINITION.** A $G$ sphere bundle is a quadruple $(X, E, \Sigma, \rho)$ consisting of a riemannian $G$ vector bundle $\pi : E \to \Sigma$ , where $\Sigma$ is a principal $G$ manifold, and an equivariant diffeomorphism $\rho : S(E) \to X$ , where $S(E)$ is the unit sphere bundle of $E$ . We usually refer to the "$G$ sphere bundle $X$ ".

Let $\pi : E \to \Sigma$ be a riemannian $G$ vector bundle and $N(S(E))$ denote the normal bundle of $S(E)$ in $E$ . Clearly $N(S(E))$ is a trivial line bundle over $S(E)$ . $N(S(E))$ has a natural $Z/2$ action induced by scalar multiplication by $-1$ in the fibres and the action has fixed set $S(E)$ - the zero section of $N(S(E))$ . Since the $Z/2$ action commutes with the $G$ action on $N(S(E))$ , we see that $N(S(E))$ has the structure of a $G \times (Z/2)$ bundle over $S(E)$ . If we take the product of the standard $Z/2$ action on $R$ with the $G$ action on $S(E)$ , then $S(E) \times R$ and $N(S(E))$ are isomorphic as $G \times (Z/2)$ bundles.

**PROPOSITION.** *Let* $N$ *be a compact connected* $G \times (Z/2)$ *manifold and* $f$ *be the generator of the* $Z/2$ *action on* $N$ . *Suppose that*

(1) $\mathrm{fix}(f)$ *is a* $G$ *sphere bundle; that is* $\mathrm{fix}(f)$ *is associated to a quadruple* $(\mathrm{fix}(f), E, \Sigma, \rho)$ *;*

(2) $\rho^*(N(\mathrm{fix}(f)))$ *and* $N(S(E))$ *are isomorphic as* $G \times (Z/2)$ *bundles,*

(3) $N\backslash\mathrm{fix}(f)$ *has two connected components* $N_1, N_2$ *, and* $f(N_1) = N_2$ .

*Then there exists a unique, up to* $G$ *diffeomorphism,* $G$ *manifold* $M$ *such that* $N$ *is* $G \times (Z/2)$ *diffeomorphic to the polar blow up of* $M$

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*along a minimal orbit type.*

Proof. Essentially a reversal of the argument of Theorem A. Fix $a > 0$ and give $N$ an equivariant riemannian metric. Since $S(E) \times R$ and $\rho^* N(\mathrm{fix}(f))$ are isomorphic as $G \times (Z/2)$ bundles, there exists a $G \times (Z/2)$ diffeomorphism $\gamma$ of $S(E) \times (-a, +a)$ onto a tubular neighbourhood $Q$ of $\mathrm{fix}(f)$. Here we suppose that $Q$ has smooth boundary which is the image of a sphere bundle of $N(\mathrm{fix}(f))$ by the exponential map. Regarding $\Sigma$ as the zero section of $E$, the normal bundle of $\Sigma$ is isomorphic to $E$ as a $G$ bundle and consequently. polar blown up along $\Sigma$ is $G \times (Z/2)$ diffeomorphic to $Q$. We now construct the required manifold $M$ by identifying the boundaries of $N_1 \backslash Q$ and the disc bundle of $E$ of radius $a$ using the map $\gamma$. //

Suppose $(X, E, \Sigma, \rho)$ is a $G$ sphere bundle. We may resolve the $G$ space $X$ to $\hat{X}$ as in Theorem A. If $X$ has $r$ orbit types, this will require $r - 1$ steps and $\hat{X}$ will be a $G \times (Z/2)^{r-1}$ manifold on which $G$ acts principally. We call $\hat{X}$ the "resolved $G$ sphere bundle $(X, E, \Sigma, \rho)$ ". We let $\widehat{N(S(E))}$ and $\widehat{S(E)}$ denote the polar resolutions of $N(S(E))$ and $S(E)$ respectively. Since $N(S(E)) \cong S(E) \times R$, $\widehat{N(S(E))} \cong \widehat{S(E)} \times R$ and $\widehat{S(E)}$ is of codimension one in $\widehat{N(S(E))}$. In case we have a $(Z/2)^q$ action on $(X, E, \Sigma, \rho)$ which commutes with $G$, we shall refer to $X$ as a resolved $G \times (Z/2)^q$ sphere bundle, it being understood that we do not resolve the $(Z/2)^q$ action.

Given a $G \times (Z/2)^p$ action on $N$, suppose that $\{f_1, \ldots, f_p\}$ is the set of generators for the $(Z/2)^p$ action. Observe that $\mathrm{fix}(f_j)$ is left invariant by $G \times (Z/2)^p$. It follows that $N(\mathrm{fix}(f_j))$ has the structure of a $G \times (Z/2)^p$ bundle over $\mathrm{fix}(f_j)$, $1 \le j \le p$.

THEOREM D. *Let $N$ be a compact connected $G \times (Z/2)^p$ manifold on which $G$ acts principally. Suppose that we can find an ordering $\{f_1, \ldots, f_p\}$ of the set of generators of the $(Z/2)^p$ action such that*

(1) *each submanifold $\mathrm{fix}(f_j)$ is a resolved $G \times (Z/2)^j$*

*sphere bundle* $(X_j, E_j, \Sigma_j, \rho_j)$ *, and the generators of the* $(Z/2)^{j-1}$ *action are* $(f_1, \ldots, f_{j-1})$ *,*

(2) $N(S(E_j))$ *and* $N(\mathrm{fix}(f_j))$ *are isomorphic as* $G \times (Z/2)^p$ *bundles,*

(3) $N \setminus \bigcup_{i=j}^{p} \mathrm{fix}(f_i)$ *has* $2^{p-j+1}$ *connected components* $N_u^j$ *,* $1 \le u \le 2^{p-j+1}$*, and* $\{f_j, \ldots, f_p\}$ *acts transitively on the set of components and, given* $s$ *,* $1 \le s \le p$ *, each* $N_u^j$ *,* $j < s$ *, is contained wholly within some* $N_k^s$ *.*

*Then there exists a unique, up to* $G$ *diffeomorphism,* $G$ *manifold* $M$ *such that* $N$ *is* $G \times (Z/2)^p$ *diffeomorphic to the polar resolution of* $M$ *.*

Proof. The proof follows straightforwardly by repeated application of the proposition and we omit details.

REMARKS. 1. If $(X_j, E_j, \Sigma_j, \rho_j)$ has less than $p - j + 1$ orbit types we nevertheless resolve $p - j$ times, doubling up when the orbit type is empty.

2. Since $S(E)$ is of codimension one in $N(S(E))$ , condition (2) of Theorem D implies that $\mathrm{fix}(f_j)$ is of codimension one, $1 \le j \le p$ .

3. We require $N$ to be connected to avoid exceptional cases where $M$ is a $G$ manifold with a minimal orbit type of codimension one and trivial normal bundle. In such cases the polar resolution of $M$ ceases to be connected. We leave the formulation of the appropriate version of Theorem D to the reader.

4. Theorem D implies that if $M$ and $M'$ have $G \times (Z/2)^p$ diffeomorphic polar resolutions then $M$ is $G$ diffeomorphic to $M'$ .

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