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GENERALIZED SIEGEL MODULAR FORMS AND COHOMOLOGY OF LOCALLY SYMMETRIC VARIETIES

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ABSTRACT. We generalize Siegel modular forms and construct an exact sequence for the cohomology of locally symmetric varieties which plays the role of the Eichler-Shimura isomorphism for such generalized Siegel modular forms.

1. **Introduction.** Let G be a semisimple Lie group over $\mathbb{R}$, and let K be a maximal compact subgroup of G. We assume that the associated symmetric space has a G-invariant complex structure. Let $\rho: G \longrightarrow \mathrm{Sp}(m,\mathbb{R})$ be a homomorphism, and let $\tau: D \longrightarrow \mathcal{H}_m$ be a holomorphic embedding of D into the Siegel upper half space of degree m satisfying $\tau(gz) = \rho(g)\tau(z)$ for all $g \in G$ and $z \in D$. If Γ is a torsion-free cocompact arithmetic subgroup of G, such a pair (ρ,τ) determines a Kuga fiber variety over the locally symmetric variety $X = \Gamma \setminus D$ whose fibers are polarized abelian varieties (see *e.g.* [4], [6], [9], [12]).

For each $\gamma \in \Gamma$, let the map $z \longmapsto J(\gamma, z)$ be the Jacobian determinant of the holomorphic map of D into itself. We denote by $j: \mathrm{Sp}(m,\mathbb{R}) \times \mathcal{H}_m \longrightarrow \mathbb{C}$ the automorphy factor given by

$$j(g,Z) = CZ + D$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(m,\mathbb{R})$ and $Z \in \mathcal{H}_m$. Then the space $S_k(\Gamma,\tau,\rho)$ of generalized Siegel modular forms consists of holomorphic functions $f: D \longrightarrow \mathbb{C}$ satisfying

$$f(\gamma z) = j\big(\rho(\gamma),\tau(z)\big)^k J(\gamma,z)^{-1} f(z)$$

for all $z \in D$ and $\gamma \in \Gamma$.

In this paper we construct a certain line bundle $\mathcal{L}$ and a vector bundle $\mathcal{V}_k$ over X for a nonnegative integer k such that there is an embedding $\widetilde{\mathcal{V}}_k \hookrightarrow \underline{\mathcal{L}}^{-k}$ of the sheaf $\widetilde{\mathcal{V}}_k$ of locally constant sections of $\mathcal{V}_k$ into the sheaf $\underline{\mathcal{L}}^{-k}$ of locally constant sections of the k-th tensor power of the dual bundle $\mathcal{L}^{-1}$ of $\mathcal{L}$. We then show that there exists a natural exact sequence of the form

$$0 \longrightarrow H^{n-1}(X, \underline{\mathcal{L}}^{-k}/\widetilde{\mathcal{V}}_k) \longrightarrow H^n(X,\widetilde{\mathcal{V}}_k) \longrightarrow S_k(\Gamma,\tau,\rho) \longrightarrow 0,$$

where n is the complex dimension of $X = \Gamma \setminus D$. Such an exact sequence was constructed by Nenashev [11] for $G = \mathrm{Sp}(m,\mathbb{R})$ and a congruence subgroup $\Gamma \subset \mathrm{Sp}(m,\mathbb{Z})$ as a

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generalization of the Eichler-Shimura isomorphism for elliptic modular forms (see *e.g.* [1], [8]).

2. **Vector bundles on locally symmetric varieties.** In this section we construct a vector bundle $\mathcal{V}_k$ and a line bundle $\mathcal{L}$ over a locally symmetric variety by extending the construction of Nenashev [11] of such bundles over Siegel modular varieties. We fix a positive integer m and set

$$\Psi_m = \left\{ \begin{pmatrix} U \\ V \end{pmatrix} \;\middle|\; U, V \in M_m(\mathbb{C}),\ {}^tUV = {}^tVU,\ \text{rank} \begin{pmatrix} U \\ V \end{pmatrix} = m \right\},$$

where $M_m(\mathbb{C})$ denotes the set of $m \times m$ matrices with entries in $\mathbb{C}$. Given an element $\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \in \Psi_m$ and a nonnegative integer k we define $\eta_k \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$ to be the map

$$\eta_k \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} : \Psi_m \longrightarrow \mathbb{C}$$

on Ψ_m given by

$$\eta_k \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \det{}^k \begin{pmatrix} U_0 & U \\ V_0 & V \end{pmatrix} \text{ for all } \begin{pmatrix} U \\ V \end{pmatrix} \in \Psi_m.$$

Let W_k be the vector space over $\mathbb{C}$ generated by the functions $\phi: \Psi_m \longrightarrow \mathbb{C}$ of the form $\eta_k \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$ for $\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \in \Psi_m$. Then the real symplectic group $\mathrm{Sp}(m, \mathbb{R})$ acts on W_k by

$$\sigma \cdot \eta_k \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \eta_k \left(\sigma \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \right)$$

for $\sigma \in \mathrm{Sp}(m, \mathbb{R})$. The group $\mathrm{Sp}(m, \mathbb{R})$ also acts on the Siegel upper half space

$$\mathcal{H}_m = \{ Z \in M_m(\mathbb{C}) \mid {}^tZ = Z,\ \Im Z \gg 0 \}$$

of degree m by

$$\sigma \cdot Z = (AZ + B)(CZ + D)^{-1} \text{ for } \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(m, \mathbb{R})$$

and $Z \in \mathcal{H}_m$. We set

$$j(\sigma, Z) = \det(CZ + D).$$

Then $j: \mathrm{Sp}(m, \mathbb{R}) \times \mathcal{H}_m \longrightarrow \mathbb{C}$ is an automorphy factor, *i.e.*, it satisfies

$$j(\sigma\mu, Z) = j(\sigma, \mu Z) j(\mu, Z)$$

for $\sigma, \mu \in \mathrm{Sp}(m, \mathbb{R})$ and $Z \in \mathcal{H}_m$. If $Z \in \mathcal{H}_m$ and if I_m denotes the $m \times m$ identity matrix, then the matrix $\begin{pmatrix} Z \\ I_m \end{pmatrix}$ is an element of Ψ_m. Now we define the map $\eta_k: \mathcal{H}_m \longrightarrow W_k$ by

$$\eta_k(Z) = \eta_k \begin{pmatrix} Z \\ I_m \end{pmatrix} \in W_k$$

for all $Z \in \mathcal{H}_m$. From the action of $\mathrm{Sp}(m, \mathbb{R})$ on W_k described above we have

$$\sigma \eta_k(Z) = \eta_k \left(\sigma \begin{pmatrix} Z \\ I_m \end{pmatrix} \right)$$

for $\sigma \in \mathrm{Sp}(m, \mathbb{R})$ and $Z \in \mathcal{H}_m$.

LEMMA 2.1. *The map η_k satisfies*

$$\eta_k(\sigma Z) = j(\sigma, Z)^{-k}\sigma\eta_k(Z)$$

for all $\sigma \in \mathrm{Sp}(m, \mathbb{R})$ and $Z \in \mathcal{H}_m$.

PROOF. See [11, (2.1.4)]. ■

Let G be a semisimple Lie group over $\mathbb{R}$, K a maximal compact subgroup of G, and $D = G/K$ the associated Riemannian symmetric space. We assume that D has a G-invariant complex structure. Let $\rho: G \longrightarrow \mathrm{Sp}(m, \mathbb{R})$ be a homomorphism and $\tau: D \longrightarrow \mathcal{H}_m$ a holomorphic embedding such that

$$\tau(gz) = \rho(g)\tau(z)$$

for all $g \in G$ and $z \in D$ (see [12] for detailed descriptions and applications of maps ρ and τ of this type). Then G acts on W_k by

$$g \cdot \eta_k \begin{pmatrix} U \\ V \end{pmatrix} = \eta_k\left(\rho(g)\begin{pmatrix} U \\ V \end{pmatrix}\right).$$

We define the map $\xi_k: D \longrightarrow W_k$ by $\xi_k(z) = \eta_k\big(\tau(z)\big)$ for all $z \in D$. Then for each $g \in G$ and $z \in D$ we obtain

$$\begin{aligned}\xi_k(gz) &= \eta_k\big(\tau(gz)\big) = \eta_k\big(\rho(g)\tau(z)\big) \\ &= j\big(\rho(g), \tau(z)\big)^{-k}\rho(g)\eta_k\big(\tau(z)\big) \\ &= j\big(\rho(g), \tau(z)\big)^{-k}\rho(g)\xi_k(z)\end{aligned}$$

by using Lemma 2.1.

LEMMA 2.2. *Let Δ be a subset of D whose image $\tau(\Delta)$ under τ contains a nonempty subset of $\mathcal{H}_m$. Then the set*

$$\{\xi_k(z) \mid z \in \Delta\}$$

is a linear span of the complex vector space W_k.

PROOF. Let Δ' be a nonempty open subset of $\mathcal{H}_m$ that is contained in $\tau(\Delta)$. Then it follows from [11, Lemma 2.1.1] that the set

$$\{\eta_k(z') \mid z' \in \Delta'\}$$

is a linear span of W_k. However, since $\xi_k(z) = \eta_k\big(\tau(z)\big)$ for $z \in D$, we have

$$\{\xi_k(z) \mid z \in \Delta\} \supset \{\eta_k(z') \mid z' \in \Delta'\};$$

hence the lemma follows. ■

Let Γ be a torsion-free cocompact arithmetic subgroup of G. Then Γ acts on the space $D \times W_k$ by

$$\gamma \cdot (z, x) = \big(\gamma z, \rho(\gamma)x\big)$$

for all $\gamma \in \Gamma$ and $(z, x) \in D \times W_k$. The group Γ also acts on $D \times \mathbb{C}$ by

$$\gamma \cdot (z, \lambda) = \Big(\gamma z, j\big(\rho(\gamma), \tau(z)\big)\lambda\Big)$$

for $\gamma \in \Gamma$ and $(z, \lambda) \in D \times \mathbb{C}$. We set

$$\mathcal{V}_k = \Gamma \setminus D \times W_k, \quad \mathcal{L} = \Gamma \setminus D \times \mathbb{C},$$

where the quotients are taken with respect to the actions of Γ described above. Then the natural projection map $D \longrightarrow \Gamma \setminus D$ induces the structures of a vector bundle on $\mathcal{V}_k$ and a line bundle on $\mathcal{L}$ over the locally symmetric space $X = \Gamma \setminus D$. Since the map $\xi_k : D \longrightarrow W_k$ satisfies

$$\xi_k(gz) = j\big(\rho(g), \tau(z)\big)^{-k} \rho(g)\xi_k(z)$$

for all $g \in \Gamma \subset G$, the map ξ_k can be regarded as a holomorphic section of the vector bundle

$$\mathcal{V}_k \otimes \mathcal{L}^{-k}$$

over $X = \Gamma \setminus D$, where $\mathcal{L}^{-k}$ denotes the k-fold tensor power $(\mathcal{L}^{-1})^{\otimes k}$ of the dual bundle $\mathcal{L}^{-1}$ of $\mathcal{L}$.

Now we define a bilinear pairing $\langle\, , \rangle : W_k \times W_k \longrightarrow \mathbb{C}$ on W_k obtained by extending linearly to the whole vector space W_k the map

$$\left\langle \eta_k \begin{pmatrix} U \\ V \end{pmatrix}, \eta_k \begin{pmatrix} U' \\ V' \end{pmatrix} \right\rangle = \det{}^k \begin{pmatrix} U & U' \\ V & V' \end{pmatrix}$$

for the generators $\eta_k \begin{pmatrix} U \\ V \end{pmatrix}$ and $\eta_k \begin{pmatrix} U' \\ V' \end{pmatrix}$ of W_k with $\begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} U' \\ V' \end{pmatrix} \in \Psi_m$.

LEMMA 2.3. *The bilinear pairing $\langle\, , \rangle$ on W_k is nondegenerate and is invariant under the action of G on W_k.*

PROOF. The nondegeneracy follows from [11, Lemma 2.3.1]. For the G-invariance, we have

$$\begin{aligned} \left\langle g \cdot \eta_k \begin{pmatrix} U \\ V \end{pmatrix}, g \cdot \eta_k \begin{pmatrix} U' \\ V' \end{pmatrix} \right\rangle &= \left\langle \eta_k \left(\rho(g) \begin{pmatrix} U \\ V \end{pmatrix} \right), \eta_k \left(\rho(g) \begin{pmatrix} U' \\ V' \end{pmatrix} \right) \right\rangle \\ &= \det{}^k \left(\rho(g) \begin{pmatrix} U & U' \\ V & V' \end{pmatrix} \right) = \det{}^k \begin{pmatrix} U & U' \\ V & V' \end{pmatrix} \\ &= \left\langle \eta_k \begin{pmatrix} U \\ V \end{pmatrix}, \eta_k \begin{pmatrix} U' \\ V' \end{pmatrix} \right\rangle \end{aligned}$$

for each $g \in G$, since $\det\big(\rho(g)\big) = 1$ due to the fact that $\rho(g) \in \mathrm{Sp}(m, \mathbb{R})$. Therefore $\langle\, , \rangle$ is G-invariant. ∎

Since the pairing $\langle\, , \rangle$ is G-invariant, it induces a fiber-wise pairing

$$\langle\, , \rangle : \mathcal{V}_k \times \mathcal{V}_k \longrightarrow X \times \mathbb{C}$$

on the vector bundle V_k over X. We shall now introduce a fiber-wise conjugation operation on V_k which will be used in the next section. If $x \in W_k$, we define the conjugate $\bar{x}$ by

$$\bar{x}\begin{pmatrix} U \\ V \end{pmatrix} = \overline{x\begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix}}, \quad \begin{pmatrix} U \\ V \end{pmatrix} \in \Psi_m.$$

Thus we have a conjugation on W_k, which induces a fiber-wise conjugation operation on the vector bundle V_k over X.

3. **The cohomology.** Let $J\colon \Gamma \times D \longrightarrow \mathbb{C}$ be the Jacobian determinant on the hermitian symmetric space D, *i.e.*, for each $\gamma \in \Gamma$ the function $z \longmapsto J(\gamma, z)$ is the determinant of the Jacobian matrix of the holomorphic map $z \longmapsto \gamma z$ of the complex manifold D. We set

$$\mathcal{J} = \Gamma \setminus D \times \mathbb{C},$$

where the quotient is taken with respect to the action of Γ on $D \times \mathbb{C}$ given by

$$\gamma \cdot (z, \lambda) = \big(\gamma z, J(\gamma, z)\lambda\big)$$

for $\gamma \in \Gamma$ and $(z, \lambda) \in D \times \mathbb{C}$. If $\mathcal{K}_X = \wedge^n T^*(X)$ is the canonical bundle on X (see *e.g.* [13, p. 218]), then $\mathcal{K}_X$ can be identified with the dual bundle $\mathcal{J}^{-1}$ of $\mathcal{J}$. If $\mathcal{B}$ is a vector bundle over X, then we shall denote by $\underline{\mathcal{B}}$ the sheaf of locally constant sections of $\mathcal{B}$.

DEFINITION 3.1. Let $\tau\colon D \longrightarrow \mathcal{H}_m$, $\rho\colon G \longrightarrow \mathrm{Sp}(m, \mathbb{R})$, $\Gamma \subset G$ and $j\colon \mathrm{Sp}(m, \mathbb{R}) \times \mathcal{H}_m \longrightarrow \mathbb{C}$ be as in Section 2, and let $J\colon \Gamma \times D \longrightarrow \mathbb{C}$ be the Jacobian determinant described above. Given a nonnegative integer k, a *generalized Siegel modular form on D of type* (k, τ, ρ) *for* Γ is a holomorphic function $f\colon D \longrightarrow \mathbb{C}$ such that

$$f(\gamma z) = j\big(\rho(\gamma), \tau(z)\big)^k J(\gamma, z)^{-1} f(z)$$

for all $\gamma \in \Gamma$ and $z \in D$. We shall denote by $S_k(\Gamma, \tau, \rho)$ the space of all generalized Siegel modular forms on D of type (k, τ, ρ) for Γ.

REMARK 3.2. Generalized Siegel modular forms in Definition 3.1 generalize mixed automorphic forms of type $(2, k)$ described in [5], [7] and certain types of mixed Siegel modular forms studied in [9] in the case of cocompact Γ. Certain aspects of more general automorphic forms were studied in [10].

LEMMA 3.3. *The space of generalized Siegel modular forms $S_k(\Gamma, \tau, \rho)$ is canonically isomorphic to the space $H^0(X, \underline{\mathcal{L}}^k \otimes \mathcal{J}^{-1})$ of sections of the sheaf $\underline{\mathcal{L}}^k \otimes \mathcal{J}^{-1}$.*

PROOF. The lemma follows easily from the construction of the bundles $\mathcal{L}^k$ and $\mathcal{J}$ and the fact that the section of the bundle $\mathcal{L}^k$ can be identified with a function $f\colon D \longrightarrow \mathbb{C}$ satisfying $f(gz) = j\big(\rho(\gamma), \tau(z)\big)^k$ for all $\gamma \in \Gamma$ and $z \in D$. ∎

Let $n = \dim_{\mathbb{C}} X$ be the complex dimension of the locally symmetric space X, then we obtain the Serre duality

$$H^n(X, \underline{\mathcal{L}}^{-k}) \times H^0(X, \underline{\mathcal{L}}^k \otimes \mathcal{J}^{-1}) \longrightarrow \mathbb{C}$$

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that is given by

$$([\omega], \varphi) \longmapsto \int_X \varphi\omega \wedge dz,$$

where $[\omega] \in H^n(X, \mathcal{L}^{-k})$ is the cohomology class represented by a differential form ω of type $(0, n)$ with coefficients in $\mathcal{L}^{-k}$ and φ is a section of the sheaf $\mathcal{L}^k \otimes \mathcal{J}^{-1}$ over X. On the other hand, if ξ_k is the section of $\widetilde{V}_k \otimes \mathcal{L}^{-k}$ constructed in Section 2, we can define an inner product $\langle\!\langle\, , \rangle\!\rangle$ on the space $H^0(X, \mathcal{L}^k \otimes \mathcal{J}^{-1})$ of sections of $\mathcal{L}^k \otimes \mathcal{J}^{-1}$ by

$$\langle\!\langle f, g\rangle\!\rangle = \int_X f\bar{g}\langle\bar{\xi}_k, \xi_k\rangle\, d\bar{z} \wedge dz$$

for all $f, g \in H^0(X, \mathcal{L}^k \otimes \mathcal{J}^{-1})$, where $\bar{g}$, $\bar{\xi}_k$ are the conjugates of g, ξ_k, respectively, and $\langle\, , \rangle: \mathcal{V}_k \times \mathcal{V}_k \longrightarrow X \times \mathbb{C}$ is the fiber-wise pairing in described in Section 2. Then to each cohomology class $[\omega] \in H^n(X, \mathcal{L}^{-k})$ represented by differential n-form ω we can associate a unique section ψ_ω of $\mathcal{L}^k \otimes \mathcal{J}^{-1}$ satisfying

$$\langle\!\langle \varphi, \bar{\psi}_\omega\rangle\!\rangle = \int_X \varphi\omega \wedge dz$$

for all $\varphi \in H^0(X, \mathcal{L}^k \otimes \mathcal{J}^{-1})$. Thus we obtain an antilinear isomorphism

$$H^n(X, \mathcal{L}^{-k}) \cong H^0(X, \mathcal{L}^k \otimes \mathcal{J}^{-1})$$

given by $[\omega] \longmapsto \psi_\omega$.

Since the fiber-wise pairing $\langle\, , \rangle: \mathcal{V}_k \times \mathcal{V}_k \longrightarrow X \times \mathbb{C}$ induces a map

$$\langle\, , \rangle: \widetilde{V}_k \times (\widetilde{V}_k \otimes \mathcal{L}^{-k}) \longrightarrow \mathcal{L}^{-k},$$

we obtain a map $\nu: \widetilde{V}_k \longrightarrow \mathcal{L}^{-k}$ given by $\nu(s) = \langle\bar{s}, \xi_k\rangle$.

LEMMA 3.4. *The map $\nu: \widetilde{V}_k \longrightarrow \mathcal{L}^{-k}$ is injective.*

PROOF. Suppose $\langle\bar{s}, \xi_k\rangle = 0$ with $s \in \Gamma(U, \widetilde{V}_k)$ for an open set $U \subset X$. Recall that the bundle $\mathcal{V}_k$ can be considered as the quotient of the trivial vector bundle $D \times W_k \longrightarrow D$ by Γ with respect to the action

$$\gamma \cdot (z, x) = \big(\gamma z, \rho(\gamma)x\big)$$

for $\gamma \in \Gamma$ and $(z, x) \in D \times W_k$. Let $s' \in \Gamma\big(\pi^{-1}(U), D \times W_k\big)$ be a locally constant section of the bundle $D \times W_k$ on $\pi^{-1}(U)$, where $\pi: D \longrightarrow X$ is a natural projection. Given a point $v \in \pi^{-1}(U)$ there is a neighborhood $U' \subset D$ of v such that $s' = \mu$ on U' for some $\mu \in W_k$. Then we have $\langle\bar{\mu}, \xi_k(z)\rangle = 0$ for all $z \in U'$. Since τ is an embedding, $\tau(U')$ is an open set in $\mathcal{H}_m$; hence by Lemma 2.2 the set $\{\xi_k(z) \mid z \in U'\}$ generates W_k. Thus it follows that $\bar{\mu} = 0$ and $\mu = 0$. Therefore ν is injective. ∎

The embedding $\nu: \widetilde{V}_k \longrightarrow \mathcal{L}^{-k}$ of $\widetilde{V}_k$ into $\mathcal{L}^{-k}$ thus induces a map $\nu_*: H^n(X, \widetilde{V}_k) \longrightarrow H^n(X, \mathcal{L}^{-k})$ and therefore a map

$$\phi_*: H^*(X, \widetilde{V}_k) \longrightarrow S_k(\Gamma, \tau, \rho),$$

if we use the canonical isomorphisms

$$H^0(X, \mathcal{L}^k \otimes J^{-1}) \cong S_k(\Gamma, \tau, \rho)$$

in Lemma 3.3 and the antilinear isomorphism

$$H^n(X, \mathcal{L}^{-k}) \cong H^0(X, \mathcal{L}^k \otimes J^{-1})$$

described above.

Now we shall construct a mapping $\phi^*: S_k(\Gamma, \tau, \rho) \longrightarrow H^n(X, \widetilde{\mathcal{V}}_k)$. Since $S_k(\Gamma, \tau, \rho)$ is canonically identified with $H^0(X, \mathcal{L}^k \otimes J^{-1})$ and ξ_k is a section of the vector bundle $\mathcal{L}^{-k} \otimes \mathcal{V}_k$ over X, for each $f \in S_k(\Gamma, \tau, \rho)$ the differential form $f\xi_k\, dz$ is a differential n-form on X with values in the vector bundle $\mathcal{V}_k$. Thus by the de Rham theory $f\xi_k\, dz$ determines a cocycle in $H^n(X, \widetilde{\mathcal{V}}_k)$. We set

$$\phi^*(f) = [f\xi_k\, dz] \in H^n(X, \widetilde{\mathcal{V}}_k).$$

PROPOSITION 3.5. *The composite $\phi_* \circ \phi^*$ is the identity map on the space $H^n(X, \widetilde{\mathcal{V}}_k)$.*

PROOF. First, we extend the morphism $\nu: \widetilde{\mathcal{V}}_k \longrightarrow \mathcal{L}^{-k}$ to a map $\nu: \widetilde{\mathcal{V}}_k \otimes \mathcal{A}^p \longrightarrow \mathcal{L}^{-k} \otimes \mathcal{A}^p$ for each p by

$$\nu(\omega) = \langle \overline{\omega_{(p,0)}}, \xi_k \rangle \in \Gamma(U, \mathcal{L}^{-k} \otimes \mathcal{A}^{0,p})$$

for each $\omega \in \Gamma(U, \widetilde{\mathcal{V}}_k \otimes \mathcal{A}^p)$, where $\mathcal{A}^p$ is the sheaf of differential p-forms on X, U is an open subset of X, and $\omega_{(p,0)}$ is the $(p,0)$-component of ω. Let f be an element of $S_k(\Gamma, \tau, \rho)$ regarded as a section of the sheaf $\mathcal{L}^k \otimes J^{-1}$. Then the differential form $f\xi_k\, dz$ becomes a section of $\widetilde{\mathcal{V}}_k \otimes \mathcal{A}^n$ and we have $\phi^* f = [f\xi_k\, dz]$. Since $f\xi_k\, dz$ is holomorphic, we obtain $(f\xi_k\, dz)_{(n,0)} = f\xi_k\, dz$ and

$$\nu_*\phi^* f = [\langle \overline{f\xi_k\, dz}, \xi_k \rangle] \in H^n(X, \mathcal{L}^{-k}).$$

From the antilinear isomorphism $H^n(X, \mathcal{L}^{-k}) \xrightarrow{\approx} S_k(\Gamma, \tau, \rho)$ it follows that there is an element $f_1 \in S_k(\Gamma, \tau, \rho)$ such that for each $g \in S_k(\Gamma, \tau, \rho)$ we have

$$\begin{aligned} \int_X g\bar{f_1} \langle \bar{\xi_k}, \xi_k \rangle\, d\bar{z} \wedge dz &= \int_X g \langle \overline{f\xi_k\, dz}, \xi_k \rangle \wedge dz \\ &= \int_X g\bar{f} \langle \bar{\xi_k}, \xi_k \rangle\, d\bar{z} \wedge dz, \end{aligned}$$

i.e., $\langle\langle g, f_1 \rangle\rangle = \langle\langle g, f \rangle\rangle$. Thus we obtain $\phi_*(\phi^* f) = f_1 = f$, and the proposition follows. ∎

From the embedding $\nu: \widetilde{\mathcal{V}}_k \longrightarrow \mathcal{L}^{-k}$ in Lemma 3.4 we obtain the short exact sequence

$$0 \longrightarrow \widetilde{\mathcal{V}}_k \longrightarrow \mathcal{L}^{-k} \longrightarrow \mathcal{L}^{-k} / \widetilde{\mathcal{V}}_k \longrightarrow 0,$$

which induces the long exact sequence

$$\begin{aligned} \cdots &\longrightarrow H^{n-1}(X, \widetilde{\mathcal{V}}_k) \longrightarrow H^{n-1}(X, \mathcal{L}^{-k}) \longrightarrow H^{n-1}(X, \mathcal{L}^{-k} / \widetilde{\mathcal{V}}_k) \\ &\longrightarrow H^n(X, \widetilde{\mathcal{V}}_k) \longrightarrow H^n(X, \mathcal{L}^{-k}) \longrightarrow H^n(X, \mathcal{L}^{-k} / \widetilde{\mathcal{V}}_k) \longrightarrow \cdots \end{aligned}$$

on the cohomology of $X = \Gamma \backslash D$.

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THEOREM 3.6. *Let* $\rho: G \longrightarrow \mathrm{Sp}(m,\mathbb{R})$, $\tau: D \longrightarrow \mathcal{H}_m$ *and* $X = \Gamma \backslash \mathcal{H}$ *be as in Section 2, and assume that* $\rho(\Gamma)$ *is contained in an arithmetic subgroup of* $\mathrm{Sp}(m,\mathbb{Q})$*. If* $n = \dim_{\mathbb{C}} X$*, then we have* $H^{n-1}(X, \mathcal{L}^{-k}) = 0$*.*

PROOF. Let Γ' be an arithmetic subgroup of $\mathrm{Sp}(m,\mathbb{Q})$ that contains $\rho(\Gamma)$, and let Y be the corresponding Siegel modular variety $\Gamma' \backslash \mathcal{H}_m$. Then we can consider the Baily-Borel compactification Y^* of Y (*cf.* [2]). The holomorphic embedding $\tau: D \longrightarrow \mathcal{H}_m$ induces an embedding $\tau_X: X \longrightarrow Y \subset Y^*$. Let $(\tau_X)_*\mathcal{L}$ be the direct image sheaf on Y obtained from the invertible sheaf $\mathcal{L}$ on X via the map τ_X. By [2] there is a positive integer N such that $\big((\tau_X)_*\mathcal{L}\big)^N$ defines a map of Y^* into a complex projective space whose restriction to Y is an embedding. Thus it follows that $\mathcal{L}^N$ defines an embedding of X into the same projective space, and consequently $\mathcal{L}$ is an ample invertible sheaf on X. Then $\mathcal{L}^{-k}$ is also an ample invertible sheaf, and therefore by Kodaira's vanishing theorem (see *e.g.* [3, Remark III.7.15]) we have

$$H^j(X, \mathcal{L}^{-k}) = 0 \text{ for } j < n.$$

Hence the proposition follows. ■

THEOREM 3.7. *If* $n = \dim_{\mathbb{C}} X$*, then there is an exact sequence*

$$0 \longrightarrow H^{n-1}(X, \mathcal{L}^{-k}/\widetilde{\mathcal{V}}_k) \longrightarrow H^n(X, \widetilde{\mathcal{V}}_k) \longrightarrow S_k(\Gamma, \tau, \rho) \longrightarrow 0.$$

PROOF. By Proposition 3.5 the map ϕ_* is surjective; hence we have

$$H^n(X, \mathcal{L}^{-k}/\widetilde{\mathcal{V}}_k) = 0$$

in the long exact sequence described above. Therefore the theorem follows from Theorem 3.6 and the antilinear isomorphism

$$H^n(X, \mathcal{L}^{-k}) \cong S_k(\Gamma, \tau, \rho).$$

■

REMARK 3.8. Nenashev [11] obtained the exact sequence in Theorem 3.7 when X is a Siegel modular variety and showed that his exact sequence is a generalization of the Eichler-Shimura isomorphism (see *e.g.* [1]) for elliptic modular forms. The exact sequence in Theorem 3.7 also generalizes a result of [8] in the case of cocompact Γ.

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