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Bogomolov conjecture over function fields for stable curves with only irreducible fibers

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Abstract. Let K be a function field and C a non-isotrivial curve of genus $g \geqslant 2$ over K. In this paper, we will show that if C has a global stable model with only geometrically irreducible fibers, then Bogomolov conjecture over function fields holds.

Key words: Stable curves, Arakelov Geometry.

1. Introduction

Let k be a field, X a smooth projective surface over k, Y a smooth projective curve over k, and $f : X \to Y$ a generically smooth semistable curve of genus $g \geqslant 2$ over Y. Let K be the function field of Y, $\overline{K}$ the algebraic closure of K, and C the generic fiber of f. For $D \in \operatorname{Pic}^1(C)(\overline{K})$, let $j_D : C_{\overline{K}} \to \operatorname{Pic}^0(C)_{\overline{K}}$ be an embedding defined by $j_D(x) = x - D$. Then, we have the following conjecture due to Bogomolov.

CONJECTURE 1.1 (Bogomolov conjecture over function fields). If f is non-isotrivial, then, for any embedding j_D, the image $j_D(C(\overline{K}))$ is discrete in terms of the semi-norm $\| \ \|_{NT}$ given by the Neron–Tate height pairing on $\operatorname{Pic}^0(C)(\overline{K})$, i.e., for any point $P \in \operatorname{Pic}^0(C)(\overline{K})$, there is a positive number ε such that the set

$$\{x \in C(\overline{K}) \mid \|j_D(x) - P\|_{NT} \leqslant \varepsilon\}$$

is finite.

In this paper, we will prove the above conjecture under the assumption that the stable model of $f : X \to Y$ has only geometrically irreducible fibers.

THEOREM 1.2 *If the stable model of $f : X \to Y$ has only geometrically irreducible fibers, then Conjecture 1.1 holds. More strongly, there is a positive number A with the following properties.*

(1) $A \geqslant \sqrt{\frac{g-1}{12g(2g+1)}\delta}$, *where δ is the number of singularities in singular fibers of $f_{\bar{k}} : X_{\bar{k}} \to Y_{\bar{k}}$.*

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(2) *For any small positive number* ε, *the set*

$$\left\{x \in C(\overline{K}) \mid \|j_D(x) - P\|_{NT} \leqslant (1-\varepsilon)A\right\}$$

is finite for any embedding j_D *and any point* $P \in \mathrm{Pic}^0(C)(\overline{K})$.

Our proof of Theorem 1.2 is based on the admissible pairing on semistable curves due to S. Zhang (cf. Section 2), Cornalba–Harris–Xiao's inequality over an arbitrary field (cf. Theorem 4.1) and an exact calculation of a Green function on a certain metrized graph (cf. Lemma 3.2). The estimation of a Green function also gives the following result, which strengthen S. Zhang's theorem [9].

THEOREM 1.3 (cf. Corollary 6.3) *Let* K *be a number field,* O_K *the ring of integers,* $f\colon X \to \mathrm{Spec}(O_K)$ *a regular semistable arithmetic surface of genus* $g \geqslant 2$ *over* O_K. *If* f *is not smooth, then*

$$(\omega^{Ar}_{X/O_K} \cdot \omega^{Ar}_{X/O_K}) \geqslant \frac{\log 2}{6(g-1)}.$$

2. Metrized graph, green function and admissible pairing

In this section, we recall several facts of metrized graphs, Green functions and the admissible pairing on semistable curves. Details can be found in Zhang's paper [9].

Let G be a locally metrized and compact topological space. We say G is a metrized graph if, for any $x \in G$, there is a positive number ε, a positive integer $d = v(x)$ (which is called the valence at x), and an open neighborhood U of x such that U is isometric to

$$\{t\, \mathrm{e}^{(2\pi\sqrt{-1}k)/d} \in \mathbb{C} \mid 0 \leqslant t < \varepsilon, k \in \mathbb{Z}\}.$$

Let $\mathrm{Div}(G)$ be a free abelian group generated by points of G. An element of $\mathrm{Div}(G)$ is called a *divisor* on G. Let $F(G)$ be the set of all piecewise smooth real valued functions on G. For $f \in F(G)$, we can define the functional $\delta(f)$ on $F(G)$ associated with f as follows. If $x \in G$ and $v(x) = n$, then $\delta(f)(x)$ is given by

$$(\delta(f)(x), g) = g(x)\sum_{i=1}^{n}\lim_{x_i \to 0} f'(x_i),$$

where $g \in F(G)$ and x_i is the arc-length parameter of one branch starting from x. $\delta(f)$ is called the Dirac function associated with f. Moreover, for $f \in F(G)$, the functional f'' on $F(G)$ is defined by

$$(f'', g) = \int_G f'' g \,\mathrm{d}\mu,$$

where $\mathrm{d}\mu$ is the Lebesgue measure arising from an arc-length parameter of G. The Laplacian Δ for $f \in F(G)$ is defined by

$$\Delta(f) = -f'' - \delta(f).$$

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Let $Q(G)$ be a subset of $F(G)$ consisting of piecewise quadratic polynomial functions. Let V be a set of vertices of G such that $G \setminus V$ is a disjoint union of open segments. Let E be the collection of segments in $G \setminus V$. We denote by $Q(G, V)$ the subspace of $Q(G)$ consisting of functions whose restriction to each edge in E are quadratic polynomial functions, and by $M(G, V)$ the vector space of measures on G generated by Dirac functions δ_v at $v \in V$ and by Lebesgue measures on edges $e \in E$ arising from the arc-length parameter. The fundamental theorem is the following existence of the admissible metric and the Green function.

THEOREM 2.1 ([9, Theorem 3.2]). *Let $D = \sum_{x \in G} d_x x$ be a divisor on G such that the support of D is in V. If G is connected and* $\deg(D) \neq -2$, *then there are a unique measure $\mu \in M(G, V)$ and a unique function g_μ on $G \times G$ with the following properties.*

(1) $\int_G \mu = 1$.
(2) $g_\mu(x, y)$ *is symmetric and continuous on $G \times G$.*
(3) *For a fixed $x \in G$, $g_\mu(x, y) \in Q(G)$. Moreover, if $x \in V$, then $g_\mu(x, y) \in Q(G, V)$.*
(4) *For a fixed $x \in G$, $\Delta_y(g_\mu(x, y)) = \delta_x - \mu$.*
(5) *For a fixed $x \in G$, $\int_G g_\mu(x, y)\mu(y) = 0$.*
(6) $g_\mu(D, y) + g_\mu(y, y)$ *is a constant for all $y \in G$, where $g_\mu(D, y) = \sum_{x \in G} d_x g_\mu(x, y)$.*

Furthermore, if $d_x \geqslant v(x) - 2$ for all $x \in G$, then μ is positive.

The measure μ in Theorem 2.1 is called the *admissible metric* with respect to D and g_μ is called the *Green function* with respect to μ. The constant $g_\mu(D, y) + g_\mu(y, y)$ is denoted by $c(G, D)$.

Let k be an algebraically closed field, X a smooth projective surface over k, Y a smooth projective curve over k, and $f : X \to Y$ a generically smooth semistable curve of genus $g \geqslant 1$ over Y. Let $\mathrm{CV}(f)$ $(\subset Y)$ be the set of all critical values of f, i.e., $y \in \mathrm{CV}(f)$ if and only if $f^{-1}(y)$ is singular. For $y \in \mathrm{CV}(f)$, let G_y be the metrized graph of $f^{-1}(y)$ defined as follows. The set of vertices V_y of G_y is indexed by irreducible components of the fiber $f^{-1}(y)$ and singularities of $f^{-1}(y)$ correspond to edges of length 1. We denote by C_v the irreducible curve corresponding to a vertex v in V_y. Let K_y be the divisor on G_y given by

$$K_y = \sum_{v \in V_y} (\omega_{X/Y} \cdot C_v) v.$$

Let μ_y be the admissible metric with respect to K_y and g_{μ_y} the Green function of μ_y. The admissible dualizing sheaf $\omega^a_{X/Y}$ is defined by

$$\omega^a_{X/Y} = \omega_{X/Y} - \sum_{y \in \mathrm{CV}(f)} c(G_y, K_y) f^{-1}(y).$$

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Here we define a new pairing $(D \cdot E)_a$ for $D, E \in \mathrm{Div}(X) \otimes \mathbb{R}$ by

$$(D \cdot E)_a = (D \cdot E) + \sum_{y \in \mathrm{CV}(f)} \left\{ \sum_{v, v' \in V_y} (D \cdot C_v) g_{\mu_y}(v, v') (E \cdot C_{v'}) \right\}.$$

This pairing is called the *admissible pairing*. It has lots of properties. For our purpose, the following are important. (Details can be found in [9].)

(1) (Adjunction formula) If B is a section of f, then $(\omega^a_{X/Y} + B \cdot B)_a = 0$.
(2) (Intersection with a fiber) If D is an $\mathbb{R}$-divisor with degree 0 along general fibers, then $(D \cdot Z)_a = 0$ for all vertical curves Z. (cf. Proposition A.3)
(3) (Compatibility with base changes) The admissible pairing is compatible with base changes. Namely, let $\pi : Y' \to Y$ be a finite morphism of smooth projective curves, and X' the minimal resolution of the fiber product of $X \times_Y Y'$. The induced morphisms are denoted as follows:

$$\begin{array}{ccc} X & \xleftarrow{\ \pi'\ } & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[\ \pi\ ]{} & Y' \end{array}$$

Then, for $D, E \in \mathrm{Div}(X) \otimes \mathbb{R}$, $(\pi'^*(D) \cdot \pi'^*(E))_a = (\deg \pi)(D \cdot E)_a$. Moreover, we have $\pi'^*(\omega^a_{X/Y}) = \omega^a_{X'/Y'}$. Thus, $(\omega^a_{X'/Y'} \cdot \omega^a_{X'/Y'})_a = (\deg \pi)(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a$.

Using the above properties, we can express the Neron–Tate height pairing in terms of the admissible pairing. Let C be the generic fiber of f, K the function field of Y, and $L, M \in \mathrm{Pic}^0(C)(\overline{K})$. Then, there exist a base change $Y' \to Y$, a semistable model X' of C over Y', and line bundles $\mathcal{L}$ and $\mathcal{M}$ on X' such that $\mathcal{L}_{\overline{K}} = L$ and $\mathcal{M}_{\overline{K}} = M$. Moreover, we can find vertical $\mathbb{Q}$-divisors V and V' on X' such that $(\mathcal{L} + V \cdot Z) = (\mathcal{M} + V' \cdot Z) = 0$ for all vertical curves Z on X'. Then, it is easy to see that

$$\frac{-1}{[k(Y') : k(Y)]} (\mathcal{L} + V \cdot \mathcal{M} + V')$$

is independent of the choice of V and V'. It is denoted by $(L \cdot M)_{NT}$ and is called the *Neron Tate height pairing*. It is easy to see that $(L \cdot L)_{NT} \geqslant 0$. So $\sqrt{(L \cdot L)_{NT}}$ is denoted by $\|L\|_{NT}$. On the other hand, by the definition of the admissible pairing, we have

$$(\mathcal{L} + V \cdot \mathcal{M} + V') = (\mathcal{L} + V \cdot \mathcal{M} + V')_a.$$

Thus, using the property (2) above, we can see that

$$-[k(Y') : k(Y)](L \cdot M)_{NT} = (\mathcal{L} \cdot \mathcal{M})_a,$$

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which means that the admissible pairing does not depend on the choice of the compactification of L and M, and that of course

$$(L \cdot M)_{NT} = \frac{-(\mathcal{L} \cdot \mathcal{M})_a}{[k(Y') : k(Y)]}.$$

Next, let us consider a height function in terms of the admissible pairing. Let $\mathcal{L}$ be an $\mathbb{R}$-divisor on X and $x \in C(\overline{K})$. Then, taking a suitable base change $\pi\colon Y' \to Y$, there is a semistable model $f'\colon X' \to Y'$ of C such that x is realized as a section B_x of f'. We set

$$h^a_{\mathcal{L}}(x) = \frac{(\pi'^*(\mathcal{L}) \cdot B_x)_a}{\deg \pi},$$

where $\pi' : X' \to X$ is the induced morphism. One can easily see that $h^a_{\mathcal{L}}(x)$ is well-defined by the third property of the above. The following generic lower estimate of the height function is important for our purpose.

THEOREM 2.2 ([9, Theorem 5.3]). *If* $\deg(\mathcal{L}_K) > 0$ *and* $\mathcal{L}$ *is* f*-nef, then, for any* $\varepsilon > 0$, *there is a finite subset* S *of* $C(\overline{K})$ *such that*

$$h^a_{\mathcal{L}}(x) \geqslant \frac{(\mathcal{L} \cdot \mathcal{L})_a}{2\deg(\mathcal{L}_K)} - \varepsilon$$

for all $x \in C(\overline{K}) \setminus S$.

As corollary, we have the following.

COROLLARY 2.3 ([9, Theorem 5.6]). *Let* $D \in \mathrm{Pic}^1(C)(\overline{K})$. *Then, for any* $\varepsilon > 0$, *there is a finite subset* S *of* $C(\overline{K})$ *such that*

$$\|D - x\|^2_{NT} \geqslant \frac{(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a}{4(g-1)} + \frac{\|\omega_C - (2g-2)D\|^2_{NT}}{4g(g-1)} - \varepsilon$$

for all $x \in C(\overline{K}) \setminus S$.

Proof. Let $\pi_1\colon Y_1 \to Y$ be a base change of $f\colon X \to Y$ such that D is defined over the function field $k(Y_1)$ of Y_1. Let $f_1\colon X_1 \to Y_1$ be the semistable model of C over Y_1 (i.e. X_1 is the minimal resolution of singularities of $X \times_Y Y_1$), F a general fiber of f_1, and $\mathcal{D}$ a compactification of D such that $\mathcal{D}$ is a $\mathbb{Q}$-divisor on X_1 and $\mathcal{D}$ is f_1-nef. Using adjuction formula and applying Theorem 2.2 to

$$\mathcal{L} = \omega^a_{X_1/Y_1} + 2\mathcal{D} - (\mathcal{D} \cdot \mathcal{D})_a F,$$

we have our corollary. □

3. Green function of a certain metrized graph

In this section, we will construct a Green function of a certain metrized graph. Let us begin with the following lemma.

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LEMMA 3.1 *Let C be a circle with arc-length l. Fixing a point O on C, let $t : C \to [0, l)$ be a coordinate of C with $t(O) = 0$ coming from an arc-length parameterization of C. We set*

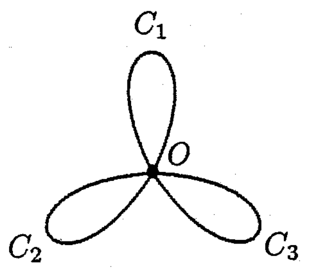
$$\phi(u) = \frac{1}{2l}u^2 - \tfrac{1}{2}|u| (u \in \mathbb{R}) \quad \textit{and} \quad f(x, y) = \phi(t(x) - t(y)).$$

Then, we have the following.

(1) *$f(x, y)$ is symmetric and continuous on $C \times C$.*
(2) *$f(x, y)$ is smooth outside the diagonal.*
(3) *For a fixed $x \in C$, $\Delta_y(f(x, y)) = \delta_x - \frac{\mathrm{d}t}{l}$.*

Proof. We can check these facts by a straightforward calculation. □

Let $C_1, \ldots, C_n$ be circles and G a metrized graph constructed by joining C_i's at a point O. Let l_i be the arc-length of C_i and $t_i : C_i \to [0, l_i)$ a coordinate of C_i with $t_i(O) = 0$.



From now on, we will identify a point on C_i with its coordinate. As in Lemma 3.1, for each i, we set

$$\phi_i(u) = \frac{1}{2l_i}u^2 - \tfrac{1}{2}|u|.$$

We fix a positive integer g. Here we consider a measure μ and a divisor K on G defined by

$$\mu = \frac{g - n}{g}\delta_O + \sum_{i=1}^{n} \frac{\mathrm{d}t_i}{gl_i} \quad \text{and} \quad K = (2g - 2)O.$$

Moreover, let us consider the following function g_μ on $G \times G$.

$$g_\mu(x, y) = \begin{cases} \phi_i(x - y) - \dfrac{g - 1}{g}\left(\phi_i(x) + \phi_i(y)\right) + \dfrac{L}{12g^2} & \text{if } x, y \in C_i \\ \dfrac{1}{g}\left(\phi_i(x) + \phi_j(y)\right) + \dfrac{L}{12g^2} & \text{if } x \in C_i, y \in C_j \text{ and } i \neq j \end{cases},$$

where $L = l_1 + \cdots + l_n$. Then, we can see the following.

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LEMMA 3.2 (1) $\int_G \mu = 1$.
(2) $g_\mu(x,y)$ *is symmetric and continuous on* $G \times G$.
(3) *For a fixed* $x \in G$, $\Delta_y(g_\mu(x,y)) = \delta_x - \mu$.
(4) *For a fixed* $x \in G$, $\int_G g_\mu(x,y)\mu(y) = 0$.
(5) $g_\mu(K,y) + g_\mu(y,y) = \frac{L(2g-1)}{12g^2}$ *for all* $y \in G$.

Proof. (1), (2) These are obvious.
(3) We assume $x \in C_i$. By [9, Lemma a.4, (a)],

$$\Delta_y(g_\mu(x,y)) = \sum_{j=1}^{n} \Delta_y(\, g_\mu(x,y)|_{C_j}).$$

Therefore, using Lemma 3.1, we get

$$\begin{aligned}\Delta_y(g_\mu(x,y)) &= \Delta_y(\, g_\mu(x,y)|_{C_i}) + \sum_{j\neq i}^{n} \Delta_y(\, g_\mu(x,y)|_{C_j}) \\ &= \left(\delta_x - \frac{\mathrm{d}t_i}{l_i} - \frac{g-1}{g}\left(\delta_O - \frac{\mathrm{d}t_i}{l_i}\right)\right) + \sum_{j\neq i}^{n} \frac{1}{g}\left(\delta_O - \frac{\mathrm{d}t_j}{l_j}\right) \\ &= \delta_x - \mu.\end{aligned}$$

(4) We assume $x \in C_i$. Then, by a direct calculation, we can see

$$\int_{C_j} g_\mu(x,t_j)\frac{\mathrm{d}t_j}{gl_j} = \begin{cases} -\dfrac{g-1}{g^2}\phi_i(x) - \dfrac{l_i}{12g^2} + \dfrac{L}{12g^3} & \text{if } j = i \\ \dfrac{1}{g^2}\phi_i(x) - \dfrac{l_j}{12g^2} + \dfrac{L}{12g^3} & \text{if } j \neq i \end{cases}.$$

Therefore,

$$\sum_{j=1}^{n} \int_{C_j} g_\mu(x,t_j)\frac{\mathrm{d}t_j}{gl_j} = \frac{n-g}{g}\left(\frac{1}{g}\phi_i(x) + \frac{L}{12g^2}\right).$$

Hence,

$$\begin{aligned}\int_G g_\mu(x,y)\mu(y) &= \frac{g-n}{g} g_\mu(x,0) + \sum_{j=1}^{n} \int_{C_j} g_\mu(x,t_j)\frac{\mathrm{d}t_j}{gl_j} \\ &= \frac{g-n}{g}\left(\frac{1}{g}\phi_i(x) + \frac{L}{12g^2}\right) \\ &\quad + \frac{n-g}{g}\left(\frac{1}{g}\phi_i(x) + \frac{L}{12g^2}\right) \\ &= 0.\end{aligned}$$

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(5) Since

$$g_\mu(O,x) = \frac{1}{g}\phi_i(x) + \frac{L}{12g^2} \quad \text{and}$$
$$g_\mu(x,x) = \frac{-2(g-1)}{g}\phi_i(x) + \frac{L}{12g^2},$$

(5) follows. □

This lemma says us that μ is the admissible metric with respect to K, g_μ is the Green function of μ, and $c(G,K) = \frac{L(2g-1)}{12g^2}$.

4. Cornalba–Harris–Xiao's inequality over an arbitrary field

In this section, we would like to generalize Cornalba–Harris–Xiao's inequality to fibered algebraic surfaces over an arbitrary field, namely,

THEOREM 4.1 *Let k be a field, X a smooth projective surface over k, Y a smooth projective curve over k, and $f: X \to Y$ a generically smooth morphism with $f_*\mathcal{O}_X = \mathcal{O}_C$. If the genus g of the generic fiber of f is greater than or equal to 2 and $\omega_{X/Y}$ is f-nef, then*

$$(\omega_{X/Y} \cdot \omega_{X/Y}) \geqslant \frac{4(g-1)}{g} \deg(f_*(\omega_{X/Y})).$$

The above was proved in [3] and [8] under the assumption $\text{char}(k) = 0$. Here we prove it using the following result of Bost.

THEOREM 4.2 ([2, Theorem III]). *Let k be a field, Y a smooth projective curve over k, and E a vector bundle on Y. Let*

$$\pi: P = \text{Proj}\left(\bigoplus_{n=0}^{\infty} \text{Sym}^n(E)\right) \longrightarrow Y$$

be the projective bundle of E and $\mathcal{O}_P(1)$ the tautological line bundle on P. If an effective cycle Z of dimension $d \geqslant 1$ on P is Chow semistable on the generic fiber of π, then

$$\frac{\left(\mathcal{O}_P(1)^d \cdot Z\right)}{d \cdot \left(\mathcal{O}_P(1)^{d-1} \cdot Z \cdot F\right)} \geqslant \frac{\deg E}{\text{rk}\, E},$$

where F is a general fiber of π.

First of all, let us begin with the following lemmas.

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LEMMA 4.3 *Let K be a field, C a smooth projective curve over K of genus $g \geqslant 2$, and $\phi: C \to \mathbb{P}^{g-1}$ the morphism given by the complete linear system $|\omega_C|$. Then $\phi_*(C)$ is a Chow semistable cycle on $\mathbb{P}^{g-1}$.*

Proof. Let R be the image of C by ϕ and n an integer given by

$$n = \begin{cases} 1, & \text{if } C \text{ is non-hyperelliptic,} \\ 2, & \text{if } C \text{ is hyperelliptic.} \end{cases}$$

Then, $\phi_*(C) = nR$. Thus a Chow form of $\phi_*(C)$ is the nth power of a Chow form of R. Therefore, $\phi_*(C)$ is Chow semistable if and only if R is Chow semistable. Moreover, Theorem 4.12 in [7] says that Chow semistability of R is derived from linear semistability of R .

Let V be a subspace of $H^0(C, \omega_C)$, $p: \mathbb{P}^{g-1} \dashrightarrow \mathbb{P}^{\dim V - 1}$ the projection defined by the inclusion $V \hookrightarrow H^0(C, \omega_C)$, and $\phi' : C \to \mathbb{P}^{\dim V-1}$ a morphism given by V. Then, $p \cdot \phi = \phi'$. We need to show that

$$\frac{2}{n} = \frac{\deg(R)}{g-1} \leqslant \frac{\deg(p_*(R))}{\dim V - 1} \tag{4.3.1}$$

to see linear semistability of R. Since $\deg(\phi'^*(\mathcal{O}(1))) = n\deg(p_*(R))$, (4.3.1) is equivalent to say

$$2 \leqslant \frac{\deg(\phi'^*(\mathcal{O}(1)))}{\dim V - 1}.$$

On the other hand, if we denote by ω_C^V the image of $V \otimes \mathcal{O}_C \to \omega_C$, then, by Clifford's lemma, we have

$$\dim V - 1 \leqslant \dim |\omega_C^V| \leqslant \frac{\deg(\omega_C^V)}{2}.$$

Thus, we get (4.3.1) because $\deg(\phi'^*(\mathcal{O}(1))) = \deg(\omega_C^V)$. □

Remark 4.4 By [2, Proposition 4.2], $\phi_*(C)$ is actually Chow stable when char(K) $= 0$ or when C is not hyperelliptic. We don't know whether $\phi_*(C)$ is Chow stable if char(K) > 0. Anyway, semistability is enough for our purpose.

Let us start the proof of Theorem 4.1. Let

$$\phi: X \dashrightarrow P = \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} \operatorname{Sym}^n(f_*(\omega_{X/Y}))\right)$$

be the rational map over Y induced by $f^*f_*(\omega_{X/Y}) \to \omega_{X/Y}$. Consider a birational morphism $\mu: X' \to X$ of smooth projective varieties such that $\phi' = \phi \cdot \mu: X' \to P$ is a morphism:

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$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ {\scriptstyle \phi'}\downarrow & & \downarrow{\scriptstyle \phi} \\ P & = & P \end{array}$$

Then, there is an effective vertical divisor D on X' such that $\mu^*(\omega_{X/Y}) = \phi'^*(\mathcal{O}_P(1)) + D$. Let $Z = \phi'_*(X')$. Then, by Lemma 4.3, Z give a Chow semistable cycle on the generic fiber. Thus, by Theorem 4.2, we have

$$\frac{(\phi'^*(\mathcal{O}_P(1)) \cdot \phi'^*(\mathcal{O}_P(1)))}{4(g-1)} \geqslant \frac{\deg(f_*(\omega_{X/Y}))}{g}.$$

On the other hand, since $\omega_{X/Y}$ is f-nef and $(D \cdot D) \leqslant 0$,

$$\begin{aligned} &(\phi'^*(\mathcal{O}_P(1)) \cdot \phi'^*(\mathcal{O}_P(1))) \\ &\quad = (\mu^*(\omega_{X/Y}) - D \cdot \mu^*(\omega_{X/Y}) - D) \\ &\quad = (\omega_{X/Y} \cdot \omega_{X/Y}) - 2\left(\mu^*(\omega_{X/Y}) \cdot D\right) + (D \cdot D) \\ &\quad \leqslant (\omega_{X/Y} \cdot \omega_{X/Y}). \end{aligned}$$

Therefore, we have our desired inequality. □

Remark 4.5 If $\mathrm{char(k)} = 0$, we can give another proof of Theorem 4.1 according to [5]. A rough idea is the following. Since the kernel K of $f^* f_*(\omega_{X/Y}) \to \omega_{X/Y}$ is semistable on the generic fiber of f by virtue of [4], we can apply Bogomolov–Gieseker's inequality to K, which implies Cornalba–Harris–Xiao's inequality by easy calculations.

5. Proof of Theorem 1.2

In this section, we would like to give the proof of Theorem 1.2. First of all, let us fix notations. Let k be a field, X a smooth projective surface over k, Y a smooth projective curve over k, and $f\colon X \to Y$ a generically smooth semistable curve of genus $g \geqslant 2$ over Y. Let K be the function field of Y, $\overline{K}$ the algebraic closure of K, and C the generic fiber of f. We assume that f is non-isotrivial and the stable model of $f : X \to Y$ has only geometrically irreducible fibers. Clearly, for the proof of Theorem 1.2, we may assume that k is algebraically closed. Then, we have the following lower estimate of $(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a$.

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THEOREM 5.1 *Under the above assumptions,* $(\omega^a_{X/Y}\cdot\omega^a_{X/Y})_a$ *is positive. Moreover,*

$$(\omega^a_{X/Y}\cdot\omega^a_{X/Y})_a \geqslant \frac{(g-1)^2}{3g(2g+1)}\delta,$$

where δ *is the number of singularities in singular fibers of* f.

Proof. Let $\mathrm{CV}(f)$ be the set of all critical values of f. For $y\in\mathrm{CV}(f)$, the number of singularities of $f^{-1}(y)$ is denoted by δ_y. Let G_y be the metrized graph of $f^{-1}(y)$ as in Section 2. Then, the total arc-length of G_y is δ_y. Let K_y be the divisor on G_y coming from $\omega_{X/Y}$ as in Section 2, μ_y the admissible metric of K_y, and g_{μ_y} the Green function of μ_y. By the definition of $\omega^a_{X/Y}$ (see Sect. 2), we have

$$\begin{aligned}(\omega^a_{X/Y}\cdot\omega^a_{X/Y})_a &= (\omega_{X/Y}\cdot\omega_{X/Y})\\ &\quad+\sum_{y\in\mathrm{CV}(f)}\{g_{\mu_y}(K_y,K_y)-2(2g-2)c(G_y,K_y)\}.\end{aligned}$$

On the other hand, G_y is isometric to the graph treated in Section 3. Thus, by Lemma 3.2,

$$\begin{aligned}&g_{\mu_y}(K_y,K_y)-2(2g-2)c(G_y,K_y)\\ &\quad=(2g-2)^2\frac{\delta_y}{12g^2}-2(2g-2)\frac{(2g-1)\delta_y}{12g^2}\\ &\quad=-\frac{g-1}{3g}\delta_y.\end{aligned}$$

Thus

$$(\omega^a_{X/Y}\cdot\omega^a_{X/Y})_a=(\omega_{X/Y}\cdot\omega_{X/Y})-\frac{g-1}{3g}\delta.$$

By virtue of Theorem 4.1 and Noether formula

$$\deg(f_*(\omega_{X/Y}))=\frac{(\omega_{X/Y}\cdot\omega_{X/Y})+\delta}{12},$$

we have

$$(\omega_{X/Y}\cdot\omega_{X/Y})\geqslant\frac{g-1}{2g+1}\delta.$$

Therefore, we get

$$(\omega^a_{X/Y}\cdot\omega^a_{X/Y})_a \geqslant \frac{(g-1)^2}{3g(2g+1)}\delta.$$

In particular, $(\omega^a_{X/Y}\cdot\omega^a_{X/Y})_a>0$ if f is not smooth. Further, if f is smooth, then

$$(\omega^a_{X/Y}\cdot\omega^a_{X/Y})_a=(\omega_{X/Y}\cdot\omega_{X/Y})>0$$

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because f is non-isotrivial. □

Let us start the proof of Theorem 1.2. We set

$$A = \sqrt{\frac{(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a}{4(g-1)}}.$$

Then, by Theorem 5.1, A is positive and

$$A \geqslant \sqrt{\frac{g-1}{12g(2g+1)}\delta}.$$

By virtue of Corollary 2.3, for any $D \in \mathrm{Pic}^1(C)(\overline{K})$ and any $P \in \mathrm{Pic}^0(C)(\overline{K})$, there is a finite subset S of $C(\overline{K})$ such that

$$\|x - D - P\|_{NT} > (1-\varepsilon)A$$

for all $x \in C(\overline{K}) \setminus S$. Therefore, we have

$$\{x \in C(\overline{K}) \mid \|j_D(x) - P\|_{NT} \leqslant (1-\varepsilon)A\} \subset S.$$

Thus, we get the second property of A. □

6. Effective lower bound of $(\omega \cdot \omega)$ for arithmetic surfaces

Let K be a number field, O_K the ring of integers, $f : X \to \mathrm{Spec}(O_K)$ a regular semistable arithmetic surface of genus $g \geqslant 2$ over O_K. In [6], we proved the following.

THEOREM 6.1 *If geometric fibers $X_{\overline{P}_1}, \ldots, X_{\overline{P}_n}$ of X at $P_1, \ldots, P_n \in \mathrm{Spec}(O_K)$ are reducible, then*

$$(\omega^{Ar}_{X/O_K} \cdot \omega^{Ar}_{X/O_K}) \geqslant \sum_{i=1}^{n} \frac{\log \#(O_K/P_i)}{6(g-1)}.$$

Using Lemma 3.2, we have the following exact lower estimate for stable curves with only irreducible fibers.

THEOREM 6.2 *Assume that the stable model of $f : X \to \mathrm{Spec}(O_K)$ has only geometric irreducible fibers. If $\{P_1, \ldots, P_n\}$ is the set of critical values of f, then*

$$(\omega^{Ar}_{X/O_K} \cdot \omega^{Ar}_{X/O_K}) \geqslant \sum_{i=1}^{n} \frac{g-1}{3g} \delta_i \log \#(O_K/P_i),$$

where δ_i is the number of singularities of the geometric fiber at P_i. Moreover, equality holds if and only if there is a sequence of distinct points $x_1, x_2, \ldots$ of $X(\overline{\mathbb{Q}})$ such that

$$\lim_{i \to \infty} \|(2g-2)x_i - \omega\|_{NT} = 0.$$

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Proof. In the same way as in the proof of Theorem 5.1, we have

$$(\omega^{a}_{X/O_K}\cdot\omega^{a}_{X/O_K})_a = (\omega^{Ar}_{X/O_K}\cdot\omega^{Ar}_{X/O_K}) - \sum_{i=1}^{n}\frac{g-1}{3g}\delta_i \log\#(O_K/P_i).$$

Therefore, our theorem follows from [9, Corollary 5.7]. □

Combining the above two theorems, we have the following corollary, which is a stronger version of S. Zhang's result [9].

COROLLARY 6.3 *If* $f\colon X\to \operatorname{Spec}(O_K)$ *is not smooth, then*

$$(\omega^{Ar}_{X/O_K}\cdot\omega^{Ar}_{X/O_K}) \geqslant \frac{\log 2}{6(g-1)}.$$

Appendix A. A matrix representation of Laplacian

In this appendix, we will consider a matrix representation of the Laplacian and its easy application.

Let G be a metrized graph and V a set of vertices of G such that $G\setminus V$ is a disjoint union of open segments. Let E be a set of edges of G by V. The length of e in E is denoted by $l(e)$. Recall that $Q(G,V)$ is the set of continuous functions on G whose restriction to each edge in E are quadratic polynomial functions, and $M(G,V)$ is the vector space of measures on G generated by Dirac functions δ_v at $v\in V$ and by Lebesgue measures on edges $e\in E$ arising from the arc-length parameter. First, we define linear maps $p\colon Q(G,V)\to\mathbb{R}^V$ and $q\colon M(G,V)\to\mathbb{R}^V$ in the following ways. If $f\in Q(G,V)$, then $p(f)$ is the restriction to V. If δ_v is a Dirac function at $v\in V$, then

$$q(\delta_v)(v') = \begin{cases} 1 & \text{if } v' = v\\ 0 & \text{if } v'\neq v\end{cases}$$

If dt is a Lebesgue measure on a edge e in E, then

$$q(dt)(v) = \begin{cases} l(e)/2 & \text{if } v \text{ is a vertex of } e\\ 0 & \text{otherwise.}\end{cases}$$

Next let us define a linear map $L\colon\mathbb{R}^V\to\mathbb{R}^V$. For distinct vertices v, v' in V, let $E(v,v')$ be the set of edges in E whose vertices are v and v'. Here we set

$$a(v,v') = \begin{cases} 0 & \text{if } E(v,v')=\emptyset\\ \displaystyle\sum_{e\in E(v,v')}\frac{1}{l(e)} & \text{otherwise}\end{cases}$$

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for $v \neq v'$. Moreover, we set

$$a(v,v) = -\sum_{\substack{v' \in V \\ v' \neq v}} a(v,v').$$

Let $L\colon \mathbb{R}^V \to \mathbb{R}^V$ be the linear map defined by the matrix $(-a(v,v'))_{v,v' \in V}$, i.e., if we denote $q(\delta_v)$ by e_v, then $L(e_v) = -\sum_{v' \in V} a(v,v')e_{v'}$. Thus, we have the following diagram:

$$\begin{array}{ccc} Q(G,V) & \xrightarrow{\ \Delta\ } & M(G,V) \\ \Big\downarrow p & & \Big\downarrow q \\ \mathbb{R}^V & \xrightarrow[\ L\ ]{} & \mathbb{R}^V \end{array}$$

Then, we can see the following proposition as remarked in [1, (5.3)].

PROPOSITION A.1 *The above diagram is commutative, i.e.*, $L \circ p = q \circ \Delta$.

Proof. Let $f \in Q(G,V)$. First, let us consider two special cases of f.

Case 1. A case where f is a linear function on each edge in E. By the definition of Δ, we can see that

$$\Delta(f) = -\sum_{v \in V} \left(\sum_{\substack{v' \in V \setminus \{v\} \\ E(v,v') \neq \emptyset}} \left(\sum_{e \in E(v,v')} \frac{f(v') - f(v)}{l(e)} \right) \right) \delta_v.$$

On the other hand, by the definition of $a(v,v')$,

$$\sum_{\substack{v' \in V \setminus \{v\} \\ E(v,v') \neq \emptyset}} \left(\sum_{e \in E(v,v')} \frac{f(v') - f(v)}{l(e)} \right) = \sum_{v' \in V} a(v,v')f(v').$$

Therefore, we have

$$\Delta(f) = \sum_{v \in V} \left(\sum_{v' \in V} -a(v,v')f(v') \right) \delta_v,$$

which shows us $q(L(f)) = L(p(f))$.

Case 2. A case where there is $e \in E$ such that $f \equiv 0$ on $G \setminus e$. Let v, v' be vertices of e and $\phi\colon [0, l(e)] \to e$ be the arc-length parameterization of e with

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$\phi(0) = v$ and $\phi(l(e)) = v'$. Since $f(v) = f(v') = 0$, f can be written in the form $f(t) = at(t - l(e))$, where t is the arc-length parameter and a is a constant. Thus,

$$\Delta(f) = al(e)\delta_v + al(e)\delta_{v'} - 2a\,\mathrm{d}t.$$

Therefore, $q(\Delta(f)) = 0$, which means that $q(\Delta(f)) = L(p(f))$.

Let us consider a general case. Let f_0 be a continuous function on G such that f_0 is a linear function on each $e \in E$ and $f_0(v) = f(v)$ for all $v \in V$. Then, $f - f_0$ can be written by a sum of functions $f_1, \ldots, f_k$ as in the case 2, i.e.,

$$f = f_0 + f_1 + \cdots + f_k$$

and f_i ($1 \leqslant i \leqslant k$) is zero on the outside of some edge. By the previous observation, we know $q(\Delta(f_i)) = L(p(f_i))$ for all $i = 0, 1, \ldots, k$. Thus, using linearity of each map, we get our lemma. □

As a corollary, we have the following.

COROLLARY A.2 *Let $D = \sum_{v \in V} d_v v$ be a divisor on G, $\mu \in M(G, V)$, and $g \in Q(G, V)$ such that*

$$\int_G \mu = 1 \quad \textit{and} \quad \Delta(g) = \delta_D - (\deg D)\mu.$$

Then, we have

$$d_v + \sum_{v' \in V} a(v, v')g(v') = (\deg D)q(\mu)(v)$$

for all $v \in V$.

Proof. Applying q for $\Delta(g) = \delta_D - (\deg D)\mu$ and using Proposition A.1, we have

$$q(\delta_D) - L(p(g)) = (\deg D)q(\mu).$$

Thus, by the definition of L, we get our corollary. □

Let k be an algebraically closed field, X a smooth projective surface over k, Y a smooth projective curve over k, and $f: X \to Y$ a generically smooth semi-stable curve of genus $g \geqslant 1$ over Y. Let $\mathrm{CV}(f)$ be the set of all critical values of f and $y \in \mathrm{CV}(f)$. Let G_y be the metrized graph of $f^{-1}(y)$ as in Section 2. Let V_y be the set of vertices coming from irreducible curves in $f^{-1}(y)$. For $v \in V_y$, the corresponding irreducible curve is denoted by C_v. Let K_y be the divisor on G_y defined by $K_y = \sum_{v \in V_y} (\omega_{X/Y} \cdot C_v)v$, μ_y the admissible metric of K_y, and g_{μ_y} the Green function of μ_y. In this case, the map $L_y: \mathbb{R}^{V_y} \to \mathbb{R}^{V_y}$ defined in the above

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is given by the matrix $\big(-(C_v \cdot C_{v'})\big)_{v,v' \in V_y}$. Thus, the above corollary implies the following proposition.

PROPOSITION A.3 *Let E be an $\mathbb{R}$-divisor on X and C_v the irreducible curve in $f^{-1}(y)$ corresponding to $v \in V_y$. Then,*

$$(E \cdot C_v)_a = (E \cdot F) q(\mu_y)(v),$$

where F is a general fiber of f. In particular, $(E \cdot C_v)_a$ does not depend on the choice of compactification of E.

Proof. Apply Corollary A.2 to a divisor $D = \sum_{v \in V_y} (E \cdot C_v) v$ on G_y. □

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