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Hinčin's Theorem for Multiplicative Free Convolution

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Abstract. Hinčin proved that any limit law, associated with a triangular array of infinitesimal random variables, is infinitely divisible. The analogous result for additive free convolution was proved earlier by Bercovici and Pata. In this paper we will prove corresponding results for the multiplicative free convolution of measures defined on the unit circle and on the positive half-line.

Introduction

Let us recall that a measure μ on the real line $\mathbb{R}$ is said to be infinitely divisible relative to the classical convolution $*$ if there exist probability measures $\mu_1, \mu_2, \dots$ such that

$$\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ times}}$$

for every natural number n. Hinčin characterized infinitely divisible measures as all the possible weak limits (as $n \to \infty$) of sequences of the form $\delta_{c_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$, where δ_{c_n} is the point mass at $c_n \in \mathbb{R}$, and the probability measures μ_{nj} form an infinitesimal array, in the sense that

$$\lim_{n\to\infty} \min\{\mu_{nj}((-\varepsilon,\varepsilon)) : 1 \le j \le k_n\} = 1$$

for every $\varepsilon > 0$. An analogous result was proved in [2], in which classical convolution $*$ is replaced by additive free convolution $\boxplus$ of measures on the real line.

One can also define the free multiplicative convolution $\mu \boxtimes \nu$ of two measures μ, ν, if these are defined on the unit circle, or on the positive real half-line. These operations are derived from the multiplication of freely independent random variables [7], just as classical convolution is derived from the addition of classically independent random variables. It is our purpose in this paper to show that the analogue of Hinčin's theorem holds for these multiplicative free convolutions. It should be noted that the multiplicative results do not simply follow from the additive one, because the natural change of variables (the exponential) does not turn additive free convolution into multiplicative free convolution. Indeed, it was observed already [5] that there are certain weak limits in the multiplicative free case which do not have a commutative counterpart.

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The case of triangular arrays of measures on the positive half-line, for which the measures μ_{nj} do not depend on j, was studied in [3], where necessary and sufficient conditions for weak convergence of the sequence $\mu_n^{\boxtimes k_n}$ are given.

Even though the two classes of measures we consider are formally studied with the same analytical apparatus, there are technical differences, and they are treated in different sections below.

1 Measures on the Half-Line

The analogue of the Fourier transform for multiplicative free convolution was discovered by Voiculescu [3]. Given a probability measure μ on $\mathbb{R}_+ = [0, \infty)$, one defines the analytic functions $\psi_\mu, \eta_\mu \colon \Omega = \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C}$ by

$$\psi_\mu(z) = \int_0^\infty \frac{zt}{1 - zt}\, d\mu(t), \quad \eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}, \quad z \in \Omega.$$

As observed in [2], the function η_μ actually maps Ω to itself, and

$$\pi > \arg \eta_\mu(z) \geq \arg z, \quad z \in \Omega, \Im z > 0,$$

where the principal value of the argument is indicated. There is an open set $V \subset \mathbb{C}$, containing some interval of the form $(-a, 0)$, where an inverse η_μ^{-1} is defined, that is, $\eta_\mu(\eta_\mu^{-1}(z)) = z$ for $z \in V$. One can then define

$$\Sigma_\mu(z) = \frac{1}{z}\eta_\mu^{-1}(z), \quad z \in V.$$

The fundamental result proved in [7] (see also [5] for measures with unbounded support) is that $\Sigma_{\mu \boxtimes \nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z)$ for z in an open set containing some $(-a, 0)$. In this paper we will consider only measures μ with no mass at zero. For such measures, the function Σ_μ is always defined on an open set containing the entire interval $(-\infty, 0)$.

Weak convergence of probability measures can be translated [5] into convergence properties of the functions Σ. Thus, given probability measures μ and $\mu_n, n \geq 1$ on $\mathbb{R}_+$, the sequence μ_n converges weakly to μ if and only if there is an open set V, intersecting $(-\infty, 0)$, where all the functions $\Sigma_\mu, \Sigma_{\mu_n}$ are defined, and such that Σ_{μ_n} converges to Σ uniformly on V.

Infinite divisibility relative to multiplicative free convolution was characterized in terms of the function Σ [4, 5]. Thus, μ is $\boxtimes$-infinitely divisible if and only if Σ_μ can be represented as $\Sigma_\mu(z) = e^{v(z)}$, where $v \colon \Omega \to \mathbb{C}$ is an analytic function such that $v(\bar{z}) = \overline{v(z)}$ and $\Im v(z) \leq 0$ whenever $\Im z > 0$. This condition amounts to saying that the function η_μ^{-1} has an analytic continuation χ to the entire region Ω, and this analytic continuation satisfies the condition $\arg \chi(z) \leq \arg z$, $\Im z > 0$. In this relation the notation $\arg \chi$ is used for a continuous version of the argument satisfying the requirement that $\arg \chi(z) = \pi$ for $z \in (-\infty, 0)$; thus the function χ does not generally map the upper half-plane into itself.

With these preparations out of the way, we can state the main result of this section.

Theorem 1.1 *Consider a sequence of positive numbers* $(c_n)_{n=1}^{\infty}$, *and array* $\{\mu_{nj} : n \geq 1, 1 \leq j \leq k_n\}$ *of probability measures on* $(0, \infty)$ *such that*

$$\lim_{n\to\infty} \min_{1\leq j\leq k_n} \mu_{nj}((1-\varepsilon, 1+\varepsilon)) = 1$$

for every $\varepsilon > 0$. *If the measures* $\delta_{c_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$ *have a weak limit* μ *which is a probability measure, then* μ *is* $\boxtimes$*-infinitely divisible.*

Proof Consider small positive numbers $\varepsilon, \delta < 1$, and the compact set

$$K_\delta = \{z \in \mathbb{C} : \delta \leq |z| \leq \tfrac{1}{\delta}, \arg z \geq \delta\}.$$

For any probability measure ν on $\mathbb{R}$ we have

$$\psi_\nu(z) - \frac{z}{1-z} = \frac{z}{1-z} \int_0^\infty \frac{t-1}{1-zt}\, d\nu(t),$$

and the inequalities $|1 - zt| \geq 1/\sin\delta$, valid for $z \in K_\delta$, $t > 0$, and $|t-1|/|1-zt| \leq 1/|z|$, valid for $t > 0$ and arbitrary z, show that

$$\left|\psi_\nu(z) - \frac{z}{1-z}\right| \leq \frac{1}{\delta}\left|\frac{z}{1-z}\right| \varepsilon\nu((1-\varepsilon, 1+\varepsilon)) + \frac{1}{|1-z|}(1 - \nu((1-\varepsilon, 1+\varepsilon))$$

for all $z \in K_\delta$. This inequality shows that the functions $\psi_{\mu_{nj}}$ converge uniformly in j and $z \in K_\delta$ to $z/(1-z)$. We conclude that $\eta_{\mu_{nj}}$ converge uniformly in j and $z \in K_\delta$ to z. It follows that there exists N_δ such that for every $n \geq N_\delta$ and every $j = 1, 2, \ldots, k_n$, the function $\eta_{\mu_{nj}}$ is one-to-one on K_δ, and $\eta_{\mu_{nj}}(K_\delta) \supset K_{2\delta}$. For such values of n, j, the inverse $\eta_{\mu_{nj}}^{-1}$ is defined on $K_{2\delta}$ and $\arg \eta_{\mu_{nj}}^{-1}(z) \leq \arg z$, $z \in K_{2\delta}$. Therefore we can find analytic functions v_{nj} defined in the interior of $K_{2\delta}$ such that $\Im v_{nj}(z) \leq 0$ for $\Im z > 0$, $v_{nj}(z)$ is real for $z < 0$, and $\eta_{\mu_{nj}}^{-1}(z) = ze^{v_{nj}(z)}$. Let us now define

$$v_n(z) = -\log c_n + \sum_{j=1}^{k_n} v_{nj}(z), \quad z \in \operatorname{int} K_{2\delta}.$$

If we set $\mu_n = \delta_{c_n} \boxtimes \mu_{n1} \boxtimes \cdots \boxtimes \mu_{nk_n}$, we see that Σ_{μ_n} is continued analytically to the interior of $K_{2\delta}$ by the function e^{v_n}, provided that $n > N_\delta$. We know that there is some open set V, which we may assume is contained in $K_{2\delta}$ and intersects the negative real axis, such that Σ_{μ_n} converges to Σ_μ uniformly on V. On the other hand, the sequence v_n can, upon replacement by a sequence, be assumed to converge uniformly on the compact subsets of $\operatorname{int} K_{2\delta}$ to an analytic function v, or to infinity. This is true because the range of v_n on the intersection of $K_{2\delta}$ with the upper half-plane is contained in the upper half-plane, which is conformally a disk; therefore we can apply the Vitali–Montel theorem. Note now that the case of an infinite limit can be excluded, because $v_n(z) = \log \Sigma_{\mu_n}(z)$ for $z \in \operatorname{int} K_{2\delta} \cap (-\infty, 0)$, and this sequence converges to $\log \Sigma_\mu(z)$ for such values of z. The limit function v will satisfy the condition $\Sigma_\mu(z) = e^{v(z)}$ for $z \in V$, and clearly $\Im v(z) \leq 0$ for $\Im z > 0$. We conclude that Σ_μ has an analytic continuation with the desired properties to the interior of $K_{2\delta}$. The theorem now follows by the characterization of infinite divisibility, because Ω is the union of these open sets. ■

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2 Measures on the Circle

We turn now to probability measures μ defined on the circle $\mathbb{T}$. The functions $\psi_\mu$and η_μ are defined by the same formulas given in the preceding section, but their domain of definition is now the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. The characteristic property of the function η_μ is then $|\eta_\mu(z)| \le |z|$, $z \in \mathbb{D}$; see [2]. The function η_μ will have an inverse in a neighborhood of zero provided that $\eta'_\mu(0) = \int_{\mathbb{T}} \zeta \, d\mu(\zeta) \neq 0$. In this case one defines, as in the preceding section, $\Sigma_\mu(z) = \frac{1}{z}\eta_\mu^{-1}(z)$ in a neighborhood of zero, and the fundamental equation $\Sigma_{\mu \boxtimes \nu} = \Sigma_\mu \Sigma_\nu$ holds in a neighborhood of zero, provided that Σ_μ and Σ_ν can be defined. Let us also note for further use the equality

$$\Sigma_\mu(0) = \frac{1}{\eta'_\mu(0)} = \frac{1}{\int_{\mathbb{T}} \zeta \, d\mu(\zeta)}.$$

Weak convergence of probability measures to a measure μ can again be translated into uniform convergence of the corresponding functions Σ in a neighborhood of zero, provided that the first moment of μ is not zero, so that Σ_μ can be defined. Finally, infinite divisibility was characterized [4] as follows. A measure μ on $\mathbb{T}$ is $\boxtimes$-infinitely divisible if and only if it is either Haar measure on $\mathbb{T}$ (*i.e.*, normalized arclength measure), or the function Σ_μ can be represented as $\Sigma_\mu(z) = e^{v(z)}$, where $v \colon \mathbb{D} \to \mathbb{C}$ is an analytic function such that $\Re v(z) \ge 0$ for $z \in \mathbb{D}$. In other words, μ is $\boxtimes$-infinitely divisible if and only if the function η_μ^{-1} has an analytic continuation χ to $\mathbb{D}$ satisfying the condition $|\chi(z)| \ge |z|$ for $z \in \mathbb{D}$.

We can now prove the analogue of Hinčin's theorem in this context.

Theorem 2.1 *Consider a sequence of numbers $c_n \in \mathbb{T}$, and an array $\{\mu_{nj} : n \ge 1, 1 \le j \le k_n\}$ of probability measures on $\mathbb{T}$ such that*

$$\lim_{n\to\infty} \max_{1 \le j \le k_n} \mu_{nj}(\{z : |z-1| < \varepsilon\}) = 1$$

for every $\varepsilon > 0$. If the measures

$$\delta_{c_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$$

have a weak limit μ, then μ is $\boxtimes$-infinitely divisible.

Proof This time we will consider the compact sets $K_\delta = \{z : |z| \le 1 - \delta\}$. It is then easily verified that for every probability measure ν on $\mathbb{T}$ we have

$$\Bigl|\psi_\nu(z) - \frac{z}{1-z}\Bigr| \le \frac{1}{\delta}\Bigl|\frac{z}{1-z}\Bigr| \varepsilon \nu\{z : |z-1| < \varepsilon\} + \frac{2}{\delta}\Bigl|\frac{z}{1-z}\Bigr| \nu\{z : |z-1| \ge \varepsilon\}$$

for $z \in K_\delta$. This implies, as before, that $\eta_{\mu_{nj}}(z)$ converges to z uniformly in j and $z \in K_\delta$. In particular, it follows that for n sufficiently large, say $n > N_\delta$, the function $\eta_{\mu_{nj}}$ is one-to-one on K_δ, and its range contains $K_{2\delta}$. For such n, the function $\eta_{\mu_{nj}}^{-1}$ is defined on $K_{2\delta}$, and clearly $|\eta_{\mu_{nj}}^{-1}(z)| \ge |z|$ for $z \in K_{2\delta}$. We deduce the existence of an

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analytic function v_{nj} with nonnegative real part, defined in the interior of $K_{2\delta}$, such that $\Sigma_{\mu_{nj}}(z) = e^{v_{nj}(z)}$ for $z \in \mathrm{int} K_{2\delta}$. Observe that the imaginary part of the function v_{nj} is only determined up to an additive constant which is an integral multiple of 2π. Define now functions

$$v_n(z) = -\log c_n + \sum_{j=1}^{k_n} v_{nj}(z), \quad z \in \mathrm{int}\, K_{2\delta},$$

where the (purely imaginary) $\log c_n$ is chosen in such a way that $0 \leq \Im v_n(0) < 2\pi$ for all $n \geq N_\delta$. Passing, if necessary, to a subsequence, we may assume now that the functions v_n converge uniformly on compacts to an analytic function v with nonnegative real part, or to infinity. In the first case, one concludes that Σ_μ is continued by e^v to the interior of $K_{2\delta}$, and therefore μ is infinitely divisible as $\delta > 0$ is arbitrary. We will conclude the proof by showing that, in case the limit is infinite, μ is normalized arclength measure. Observe that in this case, we must have $\lim_{n\to\infty} \Re v_n(0) = \infty$ because $\Im v_n(z)$ remains bounded. Now observe that

$$\prod_{j=1}^{k_n} \Big| \int_{\mathbb{T}} \zeta \, d\mu_{nj}(\zeta) \Big| = \prod_{j=1}^{k_n} \frac{1}{|\Sigma_{\mu_{nj}}(0)|} = e^{-\Re v_n(0)},$$

so that this product has limit zero. We can then choose ℓ_n with the property that both products

$$\prod_{j=1}^{\ell_n} \Big| \int_{\mathbb{T}} \zeta \, d\mu_{nj}(\zeta) \Big|, \qquad \prod_{j=\ell_n+1}^{k_n} \Big| \int_{\mathbb{T}} \zeta \, d\mu_{nj}(\zeta) \Big|$$

converge to zero as $n \to \infty$. By passing, if necessary, to subsequences, we may also assume that the sequences $\delta_{c_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nl_n}$, $\mu_{n,\ell_n+1} \boxtimes \cdots \boxtimes \mu_{k_n}$ converge weakly to probability measures ν, ρ on $\mathbb{T}$. Clearly these measures will satisfy the conditions

$$\mu = \nu \boxtimes \rho, \qquad \int_{\mathbb{T}} \zeta \, d\nu(\zeta) = \int_{\mathbb{T}} \zeta \, d\rho(\zeta) = 0.$$

These conditions imply that μ is normalized arclength measure on $\mathbb{T}$. ∎

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