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Banach Spaces

The foundations of modern Analysis were laid in the early decades of the twentieth century, through the work of Maurice Fréchet, Ivar Fredholm, David Hilbert, Henri Lebesgue, Frigyes Riesz, and many others. These authors realised that it is fruitful to study linear operations in a setting of abstract spaces endowed with further structure to accommodate the notions of convergence and continuity. This led to the introduction of abstract topological and metric spaces and, when combined with linearity, of topological vector spaces, Hilbert spaces, and Banach spaces. Since then, these spaces have played a prominent role in all branches of Analysis.

The main impetus came from the study of ordinary and partial differential equations where linearity is an essential ingredient, as evidenced by the linearity of the main operations involved: point evaluations, integrals, and derivatives. It was discovered that many theorems known at the time, such as existence and uniqueness results for ordinary differential equations and the Fredholm alternative for integral equations, can be conveniently abstracted into general theorems about linear operators in infinite-dimensional spaces of functions.

Stefan Banach, 1898–1945

A second source of inspiration was the discovery, in the 1920s by John von Neumann, that the – at that time brand new – theory of Quantum Mechanics can be put on a solid mathematical foundation by means of the spectral theory of selfadjoint operators on Hilbert spaces. It was not until the 1930s that these two lines of mathematical thinking were brought together in the theory of Banach spaces, named after its creator Stefan Banach (although this class of spaces was also discovered, independently and about the same time, by Norbert Wiener). This theory provides a unified perspective on Hilbert spaces

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and the various spaces of functions encountered in Analysis, including the spaces $C(K)$ of continuous functions and the spaces $L^p(\Omega)$ of Lebesgue integrable functions.

1.1 Banach Spaces

The aim of the present chapter is to introduce the class of Banach spaces and discuss some elementary properties of these spaces. The main classical examples are only briefly mentioned here; a more detailed treatment is deferred to the next two chapters. Much of the general theory applies to both the real and complex scalar field. Whenever this applies, the symbol $\mathbb{K}$ is used to denote the scalar field, which is $\mathbb{R}$ in the case of real vector spaces and $\mathbb{C}$ in the case of complex vector spaces.

1.1.a Definition and General Properties

Definition 1.1 (Norms)**.** A *normed space* is a pair $(X, \|\cdot\|)$, where X is a vector space over $\mathbb{K}$ and $\|\cdot\| : X \to [0,\infty)$ is a *norm*, that is, a mapping with the following properties:

(i) $\|x\| = 0$ implies $x = 0$;
(ii) $\|cx\| = |c|\,\|x\|$ for all $c \in \mathbb{K}$ and $x \in X$;
(iii) $\|x + x'\| \leqslant \|x\| + \|x'\|$ for all $x, x' \in X$.

When the norm $\|\cdot\|$ is understood we simply write X instead of $(X, \|\cdot\|)$. If we wish to emphasise the role of X we write $\|\cdot\|_X$ instead of $\|\cdot\|$.

The properties (ii) and (iii) are referred to as *scalar homogeneity* and the *triangle inequality*. The triangle inequality implies that every normed space is a metric space, with distance function

$$d(x,y) := \|x - y\|.$$

This observation allows us to introduce notions such as openness, closedness, compactness, denseness, limits, convergence, completeness, and continuity in the context of normed spaces by carrying them over from the theory of metric spaces. For instance, a sequence $(x_n)_{n\geqslant 1}$ in X is said to *converge* if there exists an element $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. This element, if it exists, is unique and is called the *limit* of the sequence $(x_n)_{n\geqslant 1}$. We then write $\lim_{n\to\infty} x_n = x$ or simply '$x_n \to x$ as $n \to \infty$'.

The triangle inequality (ii) implies both $\|x\| - \|x'\| \leqslant \|x - x'\|$ and $\|x'\| - \|x\| \leqslant \|x' - x\|$. Since $\|x' - x\| = \|(-1)\cdot(x - x')\| = \|x - x'\|$ by scalar homogeneity, we obtain the *reverse triangle inequality*

$$\big|\|x\| - \|x'\|\big| \leqslant \|x - x'\|.$$

It shows that taking norms $x \mapsto \|x\|$ is a continuous operation.

If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x'_n = x'$ in X and $c \in \mathbb{K}$ is a scalar, then $\|cx_n - cx\| = \|c(x_n - x)\| = |c|\|x_n - x\|$ implies

$$\lim_{n\to\infty} \|cx_n - cx\| = 0.$$

Likewise, $\|(x_n + x'_n) - (x + x')\| = \|(x_n - x) + (x'_n - x')\| \leqslant \|x_n - x\| + \|x'_n - x'\|$ implies

$$\lim_{n\to\infty} \|(x_n + x'_n) - (x + x')\| = 0.$$

This proves sequential continuity, and hence continuity, of the vector space operations.

Throughout this work we use the notation

$$B(x_0; r) := \{x \in X : \|x - x_0\| < r\}$$

for the *open ball* centred at $x_0 \in X$ with radius $r > 0$, and

$$\overline{B}(x_0; r) := \{x \in X : \|x - x_0\| \leqslant r\}$$

for the corresponding *closed ball*. The *open unit ball* and *closed unit ball* are the balls

$$B_X := B(0; 1) = \{x \in X : \|x\| < 1\}, \quad \overline{B}_X := \overline{B}(0; 1) = \{x \in X : \|x\| \leqslant 1\}.$$

Definition 1.2 (Banach spaces)**.** A *Banach space* is a complete normed space.

Thus a Banach space is a normed space X in which every Cauchy sequence is convergent, that is, $\lim_{m,n\to\infty} \|x_n - x_m\| = 0$ implies the existence of an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

The following proposition gives a necessary and sufficient condition for a normed space to be a Banach space. We need the following terminology. Given a sequence $(x_n)_{n\geqslant 1}$ in a normed space X, the sum $\sum_{n\geqslant 1} x_n$ is said to be *convergent* if there exists $x \in X$ such that

$$\lim_{N\to\infty} \Big\| x - \sum_{n=1}^{N} x_n \Big\| = 0.$$

The sum $\sum_{n\geqslant 1} x_n$ is said to be *absolutely convergent* if $\sum_{n\geqslant 1} \|x_n\| < \infty$.

Proposition 1.3. *A normed space X is a Banach space if and only if every absolutely convergent sum in X converges in X.*

Proof 'Only if': Suppose that X is complete and let $\sum_{n\geqslant 1} x_n$ be absolutely convergent. Then the sequence of partial sums $(\sum_{j=1}^{n} x_j)_{n\geqslant 1}$ is a Cauchy sequence, for if $n > m$ the triangle inequality implies

$$\Big\| \sum_{j=1}^{n} x_j - \sum_{j=1}^{m} x_j \Big\| = \Big\| \sum_{j=m+1}^{n} x_j \Big\| \leqslant \sum_{j=m+1}^{n} \|x_j\|,$$

which tends to 0 as $m, n \to \infty$. Hence, by completeness, the sum $\sum_{n\geqslant 1} x_n$ converges.

'If': Suppose that every absolutely convergent sum in X converges in X, and let $(x_n)_{n\geqslant 1}$ be a Cauchy sequence in X. We must prove that $(x_n)_{n\geqslant 1}$ converges in X.

Choose indices $n_1 < n_2 < \ldots$ in such a way that $\|x_i - x_j\| < \frac{1}{2^k}$ for all $i, j \geqslant n_k$, $k = 1, 2, \ldots$ The sum $x_{n_1} + \sum_{k\geqslant 1}(x_{n_{k+1}} - x_{n_k})$ is absolutely convergent since

$$\sum_{k\geqslant 1} \|x_{n_{k+1}} - x_{n_k}\| \leqslant \sum_{k\geqslant 1} \frac{1}{2^k} < \infty.$$

By assumption it converges to some $x \in X$. Then, by cancellation,

$$x = \lim_{m\to\infty} \Big(x_{n_1} + \sum_{k=1}^{m} (x_{n_{k+1}} - x_{n_k})\Big) = \lim_{m\to\infty} x_{n_{m+1}},$$

and therefore the subsequence $(x_{n_m})_{m\geqslant 1}$ is convergent, with limit x. To see that $(x_n)_{n\geqslant 1}$ converges to x, we note that

$$\|x_m - x\| \leqslant \|x_m - x_{n_m}\| + \|x_{n_m} - x\| \to 0$$

as $m \to \infty$ (the first term since we started from a Cauchy sequence and the second term by what we just proved). $\square$

The next theorem asserts that every normed space can be completed to a Banach space. For the rigorous formulation of this result we need the following terminology.

Definition 1.4 (Isometries)**.** A linear mapping T from a normed space X into a normed space Y is said to be an *isometry* if it preserves norms. A normed space X is *isometrically contained* in a normed space Y if there exists an isometry from X into Y.

Theorem 1.5 (Completion)**.** *Let X be a normed space. Then:*

(1) *there exists a Banach space $\overline{X}$ containing X isometrically as a dense subspace;*
(2) *the space $\overline{X}$ is unique up to isometry in the following sense: If X is isometrically contained as a dense subspace in the Banach spaces $\overline{X}$ and $\overline{\overline{X}}$, then the identity mapping on X has a unique extension to an isometry from $\overline{X}$ onto $\overline{\overline{X}}$.*

Proof As a metric space, $X = (X,d)$ has a completion $\overline{X} = (\overline{X}, \overline{d})$ by Theorem D.6. We prove that $\overline{X}$ is a Banach space in a natural way, with a norm $\|\cdot\|_{\overline{X}}$ such that $\overline{d}(x,x') = \|x - x'\|_{\overline{X}}$. The properties (1) and (2) then follow from the corresponding assertions for metric spaces.

Recall that the completion $\overline{X}$ of X, as a metric space, is defined as the set of all equivalence classes of Cauchy sequences in X, declaring the Cauchy sequences $(x_n)_{n\geqslant 1}$ and $(x_n')_{n\geqslant 1}$ to be equivalent if $\lim_{n\to\infty} d(x_n, x_n') = \lim_{n\to\infty} \|x_n - x_n'\| = 0$. The space $\overline{X}$ is a vector space under the scalar multiplication

$$c[(x_n)_{n\geqslant 1}] := [c(x_n)_{n\geqslant 1}]$$

and addition

$$[(x_n)_{n\geqslant 1}] + [(x'_n)_{n\geqslant 1}] := [(x_n + x'_n)_{n\geqslant 1}],$$

where the brackets denote the equivalence class.

If $(x_n)_{n\geqslant 1}$ is a Cauchy sequence in X, the reverse triangle inequality implies that the nonnegative sequence $(\|x_n\|)_{n\geqslant 1}$ is Cauchy, and hence convergent by the completeness of the real numbers. We now define a norm on $\overline{X}$ by

$$\|[(x_n)_{n\geqslant 1}]\|_{\overline{X}} := \lim_{n\to\infty} \|x_n\|.$$

Denoting by $\overline{d}$ the metric on $\overline{X}$ given by $\overline{d}(x,x') := \lim_{n\to\infty} d(x_n, x'_n)$, where $x = (x_n)_{n\geqslant 1}$ and $x' = (x'_n)_{n\geqslant 1}$, it is clear that $\overline{d}(x,x') = \|x - x'\|_{\overline{X}}$. $\square$

1.1.b Subspaces, Quotients, and Direct Sums

Several abstract constructions enable us to create new Banach spaces from given ones. We take a brief look at the three most basic constructions, namely, passing to closed subspaces and quotients and taking direct sums.

Subspaces A subspace Y of a normed space X is a normed space with respect to the norm inherited from X. A subspace Y of a Banach space X is a Banach space with respect to the norm inherited from X if and only if Y is closed in X.

To prove the 'if' part, suppose that $(y_n)_{n\geqslant 1}$ is a Cauchy sequence in the closed subspace Y of a Banach space X. Then it has a limit in X, by the completeness of X, and this limit belongs to Y, by the closedness of Y. The proof of the 'only if' part is equally simple and does not require X to be complete. If $(y_n)_{n\geqslant 1}$ is a sequence in the complete subspace Y such that $y_n \to x$ in X, then $(y_n)_{n\geqslant 1}$ is a Cauchy sequence in X, hence also in Y, and therefore it has a limit y in Y, by the completeness of Y. Since $(y_n)_{n\geqslant 1}$ also converges to y in X, it follows that $y = x$ and therefore $x \in Y$.

Quotients If Y is a closed subspace of a Banach space X, the quotient space X/Y can be endowed with a norm by

$$\|[x]\| := \inf_{y\in Y} \|x - y\|,$$

where for brevity we write $[x] := x + Y$ for the equivalence class of x modulo Y. Let us check that this indeed defines a norm. If $\|[x]\| = 0$, then there is a sequence $(y_n)_{n\geqslant 1}$ in Y such that $\|x - y_n\| < \frac{1}{n}$ for all $n \geqslant 1$. Then

$$\|y_n - y_m\| \leqslant \|y_n - x\| + \|x - y_m\| < \frac{1}{n} + \frac{1}{m},$$

so $(y_n)_{n\geqslant 1}$ is a Cauchy sequence in X. It has a limit $y \in X$ since X is complete, and we have $y \in Y$ since Y is closed. Then $\|x-y\| = \lim_{n\to\infty}\|x-y_n\| = 0$, so $x = y$. This implies that $[x] = [y] = [0]$, the zero element of X/Y. The identity $\|c[x]\| = |c|\|[x]\|$ is trivially verified, and so is the triangle inequality.

To see that the normed space X/Y is complete we use the completeness of X and Proposition 1.3. If $\sum_{n\geqslant 1}\|[x_n]\| < \infty$ and the $y_n \in Y$ are such that $\|x_n - y_n\| \leqslant \|[x_n]\| + \frac{1}{n^2}$, the proposition implies that $\sum_{n\geqslant 1}(y_n - x_n)$ converges in X, say to x. Then, for all $N \geqslant 1$,

$$\Big\|[x] - \sum_{n=1}^{N}[x_n]\Big\| = \Big\|\Big[x - \sum_{n=1}^{N}x_n\Big]\Big\| \leqslant \Big\|x - \sum_{n=1}^{N}x_n + \sum_{n=1}^{N}y_n\Big\| = \Big\|x - \Big(\sum_{n=1}^{N}x_n - y_n\Big)\Big\|.$$

As $N \to \infty$, the right-hand side tends to 0 and therefore $\lim_{N\to\infty}\sum_{n=1}^{N}[x_n] = [x]$ in X/Y.

Direct Sums A *product norm* on a finite cartesian product $X = X_1 \times \cdots \times X_N$ of normed spaces is a norm $\|\cdot\|$ satisfying

$$\|(0,\ldots,0,\underbrace{x_n}_{n-\text{th}},0,\ldots,0)\| = \|x_n\| \leqslant \|(x_1,\ldots,x_N)\|$$

for all $x = (x_1,\ldots,x_N) \in X$ and $n = 1,\ldots,N$. For instance, every norm $|\cdot|$ on $\mathbb{K}^N$ assigning norm one to the standard unit vectors induces a product norm on X by the formula

$$\|(x_1,\ldots,x_N)\| := \big|(\|x_1\|,\ldots,\|x_N\|)\big|. \tag{1.1}$$

As a normed space endowed with a product norm, the cartesian product will be denoted

$$X = X_1 \oplus \cdots \oplus X_N$$

and called a *direct sum* of $X_1,\ldots,X_N$. If every X_n is a Banach space, then the normed space X is a Banach space. Indeed, from

$$\|x\| = \Big\|\sum_{n=1}^{N}(0,\ldots,0,x_n,0,\ldots,0)\Big\| \leqslant \sum_{n=1}^{N}\|x_n\| \leqslant N\|x\| \tag{1.2}$$

we see that a sequence $(x^{(k)})_{k\geqslant 1}$ in X is Cauchy if and only if all its coordinate sequences $(x_n^{(k)})_{k\geqslant 1}$ are Cauchy. If the spaces X_n are complete, these coordinate sequences have limits x_n in X_n, and these limits serve as the coordinates of an element $x = (x_1,\ldots,x_N)$ in X which is the limit of the sequence $(x^{(k)})_{k\geqslant 1}$.

1.1.c First Examples

The purpose of this brief section is to present a first catalogue of Banach spaces. The presentation is not self-contained; the examples will be revisited in more detail in the next chapter, where the relevant terminology is introduced and proofs are given.

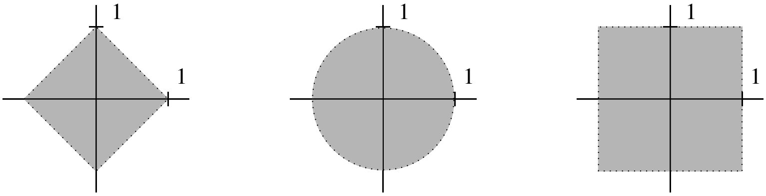

Figure 1.1 The open unit balls of $\mathbb{R}^2$ with respect to the norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$.

Example 1.6 (Euclidean spaces)**.** On $\mathbb{K}^d$ we may consider the euclidean norm

$$\|a\|_2 := \Big(\sum_{j=1}^{d} |a_j|^2\Big)^{1/2},$$

and more generally the p-norms

$$\|a\|_p := \Big(\sum_{j=1}^{d} |a_j|^p\Big)^{1/p}, \quad 1 \leqslant p < \infty,$$

as well as the supremum norm

$$\|a\|_\infty := \sup_{1\leqslant j\leqslant d} |a_j|.$$

It is not immediately obvious that the p-norms are indeed norms; the triangle inequality $\|a+b\|_p \leqslant \|a\|_p + \|b\|_p$ will be proved in the next chapter. It is an easy matter to check that the above norms are all *equivalent* in the sense defined in Section 1.3. *In what follows the euclidean norm of an element $x \in \mathbb{K}^d$ is denoted by* $|x|$ instead of the more cumbersome $\|x\|_2$.

Example 1.7 (Sequence spaces)**.** Thinking of elements of $\mathbb{K}^d$ as finite sequences, the preceding example may be generalised to infinite sequences as follows. For $1 \leqslant p < \infty$ the space ℓ^p is defined as the space of all scalar sequences $a = (a_k)_{k\geqslant 1}$ satisfying

$$\|a\|_p := \Big(\sum_{k\geqslant 1} |a_k|^p\Big)^{1/p} < \infty.$$

The mapping $a \mapsto \|a\|_p$ is a norm which turns ℓ^p into a Banach space. The space ℓ^∞ of all bounded scalar sequences $a = (a_k)_{k\geqslant 1}$ is a Banach space with respect to the norm

$$\|a\|_\infty := \sup_{k\geqslant 1} |a_k| < \infty.$$

The space c_0 consisting of all bounded scalar sequences $a = (a_k)_{k\geqslant 1}$ satisfying

$$\lim_{k\to\infty} a_k = 0$$

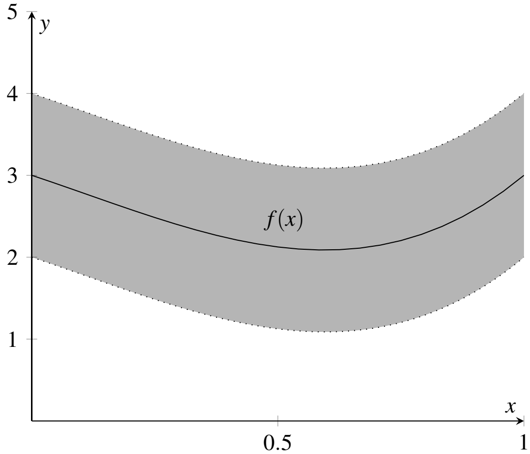

Figure 1.2 The open ball $B(f;1)$ in $C[0,1]$ consists of all functions in $C[0,1]$ whose graph lies inside the shaded area.

is a closed subspace of ℓ^∞. As such it is a Banach space in its own right.

Example 1.8 (Spaces of continuous functions)**.** Let K be a compact topological space. The space $C(K)$ of all continuous functions $f : K \to \mathbb{K}$ is a Banach space with respect to the supremum norm

$$\|f\|_\infty := \sup_{x\in K} |f(x)|.$$

This norm captures the notion of uniform convergence: for functions in $C(K)$ we have $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$ if and only if $\lim_{n\to\infty} f_n = f$ uniformly.

Example 1.9 (Spaces of integrable functions)**.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. For $1 \leqslant p < \infty$, the space $L^p(\Omega)$ consisting of all measurable functions $f : \Omega \to \mathbb{K}$ such that

$$\|f\|_p := \Big(\int_\Omega |f|^p \,\mathrm{d}\mu\Big)^{1/p} < \infty,$$

identifying functions that are equal μ-almost everywhere, is a Banach space with respect to the norm $\|\cdot\|_p$. The space $L^\infty(\Omega)$ consisting of all measurable and μ-essentially bounded functions $f : \Omega \to \mathbb{K}$, identifying functions that are equal μ-almost everywhere, is a Banach space with respect to the norm given by the μ-essential supremum

$$\|f\|_\infty := \mu\text{-}\operatorname*{ess\,sup}_{\omega\in\Omega} |f(\omega)| := \inf\big\{r > 0 : |f| \leqslant r \ \mu\text{-almost everywhere}\big\}.$$

Example 1.10 (Spaces of measures)**.** Let $(\Omega, \mathscr{F})$ be a measurable space. The space

$M(\Omega)$ consisting of all $\mathbb{K}$-valued measures of bounded variation on $(\Omega, \mathscr{F})$ is a Banach space with respect to the variation norm

$$\|\mu\| := |\mu|(\Omega) := \sup_{\mathscr{A} \in \mathbb{F}} \sum_{A \in \mathscr{A}} |\mu(A)|,$$

where $\mathbb{F}$ denotes the set of all finite collections of pairwise disjoint sets in $\mathscr{F}$.

Example 1.11 (Hilbert spaces)**.** A *Hilbert space* is an inner product space $(H, (\cdot|\cdot))$ that is complete with respect to the norm

$$\|h\| := (h|h)^{1/2}.$$

Examples include the spaces $\mathbb{K}^d$ with the euclidean norm, ℓ^2, and the spaces $L^2(\Omega)$. Further examples will be given in later chapters.

1.1.d Separability

Most Banach spaces of interest in Analysis are *infinite-dimensional* in the sense that they do not have a finite spanning set. In this context the following definition is often useful.

Definition 1.12 (Separability)**.** A normed space is called *separable* if it contains a countable set whose linear span is dense.

Proposition 1.13. *A normed space X is separable if and only if X contains a countable dense set.*

Proof The 'if' part is trivial. To prove the 'only if' part, let $(x_n)_{n\geqslant 1}$ have dense span in X. Let Q be a countable dense set in $\mathbb{K}$ (for example, one could take $Q = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and $Q = \mathbb{Q} + i\mathbb{Q}$ if $K = \mathbb{C}$). Then the set of all Q-linear combinations of the x_n, that is, all linear combinations involving coefficients from Q, is dense in X. □

Finite-dimensional spaces, the sequence spaces c_0 and ℓ^p with $1 \leqslant p < \infty$, the spaces $C(K)$ with K compact metric, and $L^p(D)$ with $1 \leqslant p < \infty$ and $D \subseteq \mathbb{R}^d$ open, are separable. The separability of $C(K)$ and $L^p(D)$ follows from the results proved in the next chapter.

1.2 Bounded Operators

Having introduced normed spaces and Banach spaces, we now introduce a class of linear operators acting between them which interact with the norm in a meaningful way.

1.2.a Definition and General Properties

Let X and Y be normed spaces.

Definition 1.14 (Bounded operators)**.** A linear operator $T : X \to Y$ is *bounded* if there exists a finite constant $C \geqslant 0$ such that

$$\|Tx\| \leqslant C\|x\|, \quad x \in X.$$

Here, and in the rest of this work, we write Tx instead of the more cumbersome $T(x)$. A *bounded operator* is a linear operator that is bounded.

The infimum C_T of all admissible constants C in Definition 1.14 is itself admissible. Thus C_T is the least admissible constant. We claim that it equals the number

$$\|T\| := \sup_{\|x\| \leqslant 1} \|Tx\|.$$

To see this, let C be an admissible constant in Definition 1.14, that is, we assume that $\|Tx\| \leqslant C\|x\|$ for all $x \in X$. Then $\|T\| = \sup_{\|x\| \leqslant 1} \|Tx\| \leqslant C$. This being true for all admissible constants C, it follows that $\|T\| \leqslant C_T$. The opposite inequality $C_T \leqslant \|T\|$ follows by observing that for all $x \in X$ we have

$$\|Tx\| \leqslant \|T\|\|x\|,$$

which means that $\|T\|$ an admissible constant. This inequality is trivial for $x = 0$, and for $x \neq 0$ it follows from scalar homogeneity, the linearity of T and the definition of the number $\|T\|$:

$$\|Tx\| = \Big\|\frac{1}{\|x\|}Tx\Big\|\|x\| = \Big\|T\frac{x}{\|x\|}\Big\|\|x\| \leqslant \|T\|\|x\|.$$

Proposition 1.15. *For a linear operator $T : X \to Y$ the following assertions are equivalent:*

(1) *T is bounded;*
(2) *T is continuous;*
(3) *T is continuous at some point $x_0 \in X$.*

Proof The implication (1)⇒(2) follows from

$$\|Tx - Tx'\| = \|T(x - x')\| \leqslant \|T\|\|x - x'\|$$

and the implication (2)⇒(3) is trivial. To prove the implication (3)⇒(1), suppose that T is continuous at x_0. Then there exists a $\delta > 0$ such that $\|x_0 - y\| < \delta$ implies $\|Tx_0 - Ty\| < 1$. Since every $x \in X$ with $\|x\| < \delta$ is of the form $x = x_0 - y$ with $\|x_0 - y\| < \delta$ (take $y = x_0 - x$) and T is linear, it follows that $\|x\| < \delta$ implies $\|Tx\| < 1$. By scalar homogeneity and the linearity of T we may scale both sides with a factor δ, and obtain that $\|x\| < 1$ implies $\|Tx\| < 1/\delta$. From this, and the continuity of $x \mapsto \|x\|$, it follows that $\|x\| \leqslant 1$ implies $\|Tx\| \leqslant 1/\delta$, that is, T is bounded and $\|T\| \leqslant 1/\delta$. □

Easy manipulations involving the properties of norms and linear operators, such as those used in the above proofs, will henceforth be omitted.

The set of all bounded operators from X to Y is a vector space in a natural way with respect to pointwise scalar multiplication and addition by putting

$$(cT)x := c(Tx), \quad (T+T')x := Tx + Tx'.$$

This vector space will be denoted by $\mathscr{L}(X,Y)$. We further write $\mathscr{L}(X) := \mathscr{L}(X,X)$.

For all $T, T' \in \mathscr{L}(X,Y)$ and $c \in \mathbb{K}$ we have

$$\|cT\| = |c|\|T\|, \quad \|T+T\| \leqslant \|T\| + \|T'\|.$$

Let us prove the second assertion; the proof of the first is similar. For all $x \in X$, the triangle inequality gives

$$\|(T+T)x\| \leqslant \|Tx\| + \|T'x\| \leqslant (\|T\| + \|T'\|)\|x\|,$$

and the result follows by taking the supremum over all $x \in X$ with $\|x\| \leqslant 1$.

Noting that $\|T\| = 0$ implies $T = 0$, it follows that $T \mapsto \|T\|$ is a norm on $\mathscr{L}(X,Y)$. Endowed with this norm, $\mathscr{L}(X,Y)$ is a normed space. If $T : X \to Y$ and $S : Y \to Z$ are bounded, then so is their composition ST and we have

$$\|ST\| \leqslant \|S\|\|T\|.$$

Indeed, for all $x \in X$ we have

$$\|STx\| \leqslant \|S\|\|Tx\| \leqslant \|S\|\|T\|\|x\|$$

and the result follows by taking the supremum over all $x \in X$.

Proposition 1.16. *If Y is complete, then $\mathscr{L}(X,Y)$ is complete.*

Proof Let $(T_n)_{n\geqslant 1}$ be a Cauchy sequence in $\mathscr{L}(X,Y)$. From $\|T_nx - T_mx\| \leqslant \|T_n - T_m\|\|x\|$ we see that $(T_nx)_{n\geqslant 1}$ is a Cauchy sequence in Y for every $x \in X$. Let Tx denote its limit. The linearity of each of the operators T_n implies that the mapping $T : x \mapsto Tx$ is linear and we have $\|Tx\| = \lim_{n\to\infty} \|T_nx\| \leqslant M\|x\|$, where $M := \sup_{n\geqslant 1} \|T_n\|$ is finite since Cauchy sequences in normed spaces are bounded. This shows that the linear operator T is bounded, so it is an element of $\mathscr{L}(X,Y)$. To prove that $\lim_{n\to\infty} \|T_n - T\| = 0$, fix $\varepsilon > 0$ and let $N \geqslant 1$ be so large that $\|T_n - T_m\| < \varepsilon$ for all $m, n \geqslant N$. Then, for $m, n \geqslant N$, from

$$\|T_nx - T_mx\| \leqslant \varepsilon\|x\|$$

it follows, upon letting $m \to \infty$, that

$$\|T_n x - Tx\| \leqslant \varepsilon \|x\|.$$

This being true for all $x \in X$ and $n \geqslant N$, it follows that $\|T_n - T\| \leqslant \varepsilon$ for all $n \geqslant N$. $\square$

The important special case $Y = \mathbb{K}$ leads to the following definition.

Definition 1.17. The *dual space* of a normed space X is the Banach space

$$X^* := \mathscr{L}(X, \mathbb{K}).$$

For $x \in X$ and $x^* \in X^*$ one usually writes $\langle x, x^* \rangle := x^*(x)$. The elements of the dual space X^* are often referred to as *bounded functionals* or simply *functionals*. Duality is a subject in its own right which will be taken up in Chapter 4. In that chapter, explicit representations of duals of several classical Banach spaces are given. For Hilbert spaces this duality takes a particularly simple form, described by the Riesz representation theorem, to be proved in Chapter 3.

It often happens that a linear operator can be shown to be well defined and bounded on a dense subspace. In such cases, a *density argument* can be used to extend the operator to the whole space.

Proposition 1.18 (Density argument – extending operators)**.** *Let X be a normed space and Y be a Banach space, and let X_0 be a dense subspace of X. If $T_0 : X_0 \to Y$ is a bounded operator, there exists a unique bounded operator $T : X \to Y$ extending T_0. The norm of this extension satisfies $\|T\| = \|T_0\|$.*

Proof Fix $x \in X$, and suppose that $\lim_{n\to\infty} x_n = x$ with $x_n \in X_0$ for all $n \geqslant 1$. The boundedness of T_0 implies that $\|T_0 x_n - T_0 x_m\| \leqslant \|T_0\| \|x_n - x_m\| \to 0$ as $m, n \to \infty$, so $(T_0 x_n)_{n\geqslant 1}$ is a Cauchy sequence in Y. Since Y is complete, we have $T_0 x_n \to y$ for some $y \in Y$.

If also $x'_n \to x$, the same argument shows that $T_0 x'_n \to y'$ for some (possibly different) $y' \in Y$. From

$$\|T_0 x'_n - T_0 x_n\| \leqslant \|T_0\| \|x'_n - x_n\| \leqslant \|T_0\| (\|x'_n - x\| + \|x - x_n\|)$$

it follows that

$$\|y' - y\| = \lim_{n\to\infty} \|T_0 x'_n - T_0 x_n\| = 0$$

and therefore $y' = y$.

Denoting the common limit $y = y'$ by Tx, we thus obtain a well-defined mapping $x \mapsto Tx$. It is evident that this mapping extends T_0, for if $x \in X_0$ we may take $x_n = x$ and then $Tx = \lim_{n\to\infty} T_0 x_n = T_0 x$.

It is easily checked that T is linear. To show that it is bounded, with $\|T\| \leqslant \|T_0\|$, we just note that

$$\|Tx\| = \lim_{n\to\infty} \|T_0 x_n\| \leqslant \|T_0\| \lim_{n\to\infty} \|x_n\| = \|T_0\|\|x\|.$$

The converse inequality $\|T\| \geqslant \|T_0\|$ trivially holds since T extends T_0.

Finally, if the bounded operators T and T' both extend T_0, then the bounded operator $T - T'$ equals 0 on the dense subspace X_0 and hence, by continuity, on all of X. $\square$

Under a uniform boundedness assumption, a similar density argument can be used to extend the existence of limits from a dense subspace to the whole space.

Proposition 1.19 (Density argument – extending convergence of operators)**.** *Let X be a normed space and Y a Banach space, and let X_0 be a dense subspace of X. Let $(T_n)_{n\geqslant 1}$ be a sequence of operators in $\mathscr{L}(X,Y)$ satisfying $\sup_{n\geqslant 1}\|T_n\| < \infty$. If the limit $\lim_{n\to\infty} T_n x_0$ exists in Y for all $x_0 \in X_0$, then the limit $Tx := \lim_{n\to\infty} T_n x$ exists in Y for all $x \in X$. Moreover, the operator $T : x \mapsto Tx$ is linear and bounded from X to Y, and*

$$\|T\| \leqslant \liminf_{n\to\infty} \|T_n\|.$$

Proof We will show that the sequence $(T_n x)_{n\geqslant 1}$ is Cauchy for every $x \in X$. Fix arbitrary $x \in X$ and $\varepsilon > 0$ and choose $x_0 \in X_0$ in such a way that $\|x - x_0\| < \varepsilon/M$, where $M := \sup_{n\geqslant 1}\|T_n\|$. Since $(T_n x_0)_{n\geqslant 1}$ is a Cauchy sequence, there is an $N \geqslant 1$ such that $\|T_n x_0 - T_m x_0\| < \varepsilon$ for all $m,n \geqslant N$. Then, for all $m,n \geqslant N$,

$$\begin{aligned}\|T_n x - T_m x\| &\leqslant \|T_n x - T_n x_0\| + \|T_n x_0 - T_m x_0\| + \|T_m x_0 - T_m x\| \\ &\leqslant M\|x - x_0\| + \varepsilon + M\|x_0 - x\| < 3\varepsilon.\end{aligned}$$

The sequence $(T_n x)_{n\geqslant 1}$ is thus Cauchy. Since Y is complete this sequence has a limit, which we denote by Tx. Linearity of $T : x \mapsto Tx$ is clear, and boundedness along with the estimate for the norm follow from

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| = \liminf_{n\to\infty} \|T_n x\| \leqslant \liminf_{n\to\infty} \|T_n\|\|x\|.$$

$\square$

This proposition should be compared with Proposition 5.3, which provides the following partial converse: if X is a Banach space, Y is a normed space, and $(T_n)_{n\geqslant 1}$ is a sequence in $\mathscr{L}(X,Y)$ such that $Tx := \lim_{n\to\infty} T_n x$ exists in Y for all $x \in X$, then $\sup_{n\geqslant 1}\|T_n\| < \infty$.

Definition 1.20 (Null space and range)**.** The *null space* of a bounded operator $T \in \mathscr{L}(X,Y)$ is the subspace

$$\mathsf{N}(T) := \{x \in X : \; Tx = 0\}.$$

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The *range* of T is the subspace

$$\mathsf{R}(T) := \{Tx : x \in X\}.$$

By linearity, both the null space $\mathsf{N}(T)$ and the range $\mathsf{R}(T)$ are subspaces. By continuity, the null space of a bounded operator is closed. The following result gives a useful sufficient criterion for the range of a bounded operator to be closed.

Proposition 1.21. *Let X be a Banach space and Y be a normed space. If $T \in \mathscr{L}(X,Y)$ satisfies $\|Tx\| \geqslant C\|x\|$ for some $C > 0$ and all $x \in X$, then T is injective and has closed range.*

Proof Injectivity is clear. Suppose that $Tx_n \to y$ in Y; we must prove that $y \in \mathsf{R}(T)$. From $\|x_n - x_m\| \leqslant C^{-1}\|Tx_n - Tx_m\|$ it follows that $(x_n)_{n\geqslant 1}$ is a Cauchy sequence in X and therefore converges to some $x \in X$. Then $y = \lim_{n\to\infty} Tx_n = Tx$. □

We conclude by introducing some terminology that will be used throughout this work. In the next four definitions, X and Y are normed spaces.

Definition 1.22 (Isomorphisms)**.** An *isomorphism* is a bijective operator $T \in \mathscr{L}(X,Y)$ whose inverse is bounded as well. An *isometric isomorphism* is an isomorphism that is also isometric. The spaces X and Y are called (isometrically) isomorphic if there exists an (isometric) isomorphism from X to Y.

Definition 1.23 (Contractions)**.** A *contraction* is an operator $T \in \mathscr{L}(X,Y)$ satisfying $\|T\| \leqslant 1$.

Definition 1.24 (Uniform boundedness)**.** A subset $\mathscr{T}$ of $\mathscr{L}(X,Y)$ is said to be *uniformly bounded* if it is a bounded subset of $\mathscr{L}(X,Y)$, i.e., if $\sup_{T\in\mathscr{T}} \|T\| < \infty$.

Definition 1.25 (Uniform, strong, and weak convergence of operators)**.** A sequence $(T_n)_{n\geqslant 1}$ in $\mathscr{L}(X,Y)$ is said to:

(1) *converge uniformly* to an operator $T \in \mathscr{L}(X,Y)$ if

$$\lim_{n\to\infty} \|T_n - T\| = 0;$$

(2) *converge strongly* to an operator $T \in \mathscr{L}(X,Y)$ if

$$\lim_{n\to\infty} \|T_n x - Tx\| = 0, \quad x \in X;$$

(3) *converge weakly* to an operator $T \in \mathscr{L}(X,Y)$ if

$$\lim_{n\to\infty} \langle T_n x - Tx, y^* \rangle = 0, \quad x \in X,\ y^* \in Y^*,$$

where Y^* is the dual of Y and $\langle y, y^* \rangle := y^*(y)$ for $y \in Y$. In these situations we call T the *uniform limit*, respectively the *strong limit*, respectively the *weak limit*, of the sequence $(T_n)_{n \geqslant 1}$. Uniqueness of weak limits is assured by the Hahn–Banach theorem (see Corollary 4.11).

Uniform convergence implies strong convergence and strong convergence implies weak convergence, but the converses generally fail. For instance, the projections onto the first n coordinates in ℓ^p, $1 \leqslant p < \infty$, converge strongly to the identity operator, but not uniformly; and the operators T^n, where T is the right shift in ℓ^p, $1 < p < \infty$, converges weakly to the zero operator but not strongly (for the case $p = 1$ see Problem 4.33).

1.2.b Subspaces, Quotients, and Direct Sums

Restrictions If T is a bounded operator from a normed space X into a normed space Y, then the restriction of T to a subspace X_0 of X defines a bounded operator $T|_{X_0}$ from X_0 into Y of norm $\|T|_{X_0}\| \leqslant \|T\|$.

Quotients Let Y be a closed subspace of a Banach space X. By the definition of the quotient norm, the *quotient map* $q : x \mapsto x + Y$ is bounded from X to X/Y of norm $\|q\| \leqslant 1$.

Let Z be a normed space and let $T \in \mathscr{L}(X, Z)$ be a bounded operator with the property that Y is contained in the null space $\mathsf{N}(T)$. We claim that

$$T_{/Y}(x + Y) := Tx, \quad x \in X,$$

defines a well-defined and bounded *quotient operator* $T_{/Y} : X/Y \to Z$ of norm $\|T_{/Y}\| = \|T\|$. Well-definedness of $T_{/Y}$ is clear, and for all $x \in X$ and $y \in Y$ we have $\|Tx\| = \|T(x + y)\| \leqslant \|T\| \|x + y\|$. Taking the infimum over all $y \in Y$ gives the bound

$$\|T_{/Y}(x + Y)\| = \|Tx\| \leqslant \|T\| \inf_{y \in Y} \|x + y\| = \|T\| \|x + Y\|.$$

Hence $T_{/Y}$ is bounded and $\|T_{/Y}\| \leqslant \|T\|$. For the converse inequality we note that

$$\|Tx\| = \|T_{/Y}(x + Y)\| \leqslant \|T_{/Y}\| \|x + Y\| = \|T_{/Y}\| \inf_{y \in Y} \|x - y\| \leqslant \|T_{/Y}\| \|x\|.$$

Direct Sums If X_n is a normed space and $T_n \in \mathscr{L}(X_n)$ for $n = 1, \ldots, N$, then the *direct sum operator*

$$T = \bigoplus_{n=1}^{N} T_n : (x_1, \ldots, x_N) \mapsto (T_1 x_1, \ldots, T_N x_N)$$

is bounded on $X = \bigoplus_{n=1}^{N} X_n$ with respect to any product norm; this follows from (1.2). If the product norm is of the form (1.1), then $\|T\| = \max_{1 \leqslant n \leqslant N} \|T_n\|$.

1.2.c First Examples

We revisit the examples of Section 1.1.c and discuss how various natural operations used in Analysis give rise to bounded operators.

Example 1.26 (Matrices)**.** Every $m \times n$ matrix $A = (a_{ij})_{i,j=1}^{m,n}$ defines a bounded operator in $\mathscr{L}(\mathbb{K}^n, \mathbb{K}^m)$ and its norm satisfies

$$\|A\|^2 = \sup_{|x|\leqslant 1} |Ax|^2 = \sup_{|x|\leqslant 1} \sum_{i=1}^{m} \Big| \sum_{j=1}^{n} a_{ij}x_j \Big|^2 \leqslant \sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2, \tag{1.3}$$

where the last step follows from the Cauchy–Schwarz inequality. More generally, every linear operator from a finite-dimensional normed space X into a normed space Y is bounded; this will be shown in Corollary 1.37.

The upper bound (1.3) for the norm of a matrix A is not sharp. An explicit method to determine the operator norm of a matrix is described in Problem 4.14.

Example 1.27 (Point evaluations)**.** Let K be a compact topological space. For each $x_0 \in K$ the point evaluation $E_{x_0} : f \mapsto f(x_0)$ is bounded as an operator from $C(K)$ into $\mathbb{K}$ with norm $\|E_{x_0}\| = 1$. Boundedness with norm $\|E_{x_0}\| \leqslant 1$ follows from

$$|E_{x_0} f| = |f(x_0)| \leqslant \sup_{x\in K} |f(x)| = \|f\|_\infty.$$

By considering $f = \mathbf{1}$, the constant-one function on K, it is seen that $\|E_{x_0}\| = 1$.

Example 1.28 (Integration)**.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. The mapping $I_\mu : f \mapsto \int_\Omega f \,\mathrm{d}\mu$ is bounded from $L^1(\Omega)$ to $\mathbb{K}$ with norm $\|I_\mu\| = 1$. Boundedness with norm $\|I_\mu\| \leqslant 1$ follows from

$$|I_\mu f| = \Big| \int_\Omega f \,\mathrm{d}\mu \Big| \leqslant \int_\Omega |f| \,\mathrm{d}\mu = \|f\|_1.$$

By considering nonnegative functions it is seen that $\|I_\mu\| = 1$.

Example 1.29 (Pointwise multipliers)**.** Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and fix $1 \leqslant p \leqslant \infty$. For any $m \in L^\infty(\Omega)$, the pointwise multiplier $T_m : f \mapsto mf$ defines a bounded operator on $L^p(\Omega)$ with norm $\|T_m\| = \|m\|_\infty$. Indeed, for μ-almost all $\omega \in \Omega$ we have

$$|(mf)(\omega)| = |m(\omega)||f(\omega)| \leqslant \|m\|_\infty |f(\omega)|.$$

For $1 \leqslant p < \infty$, upon integration we obtain

$$\|T_m f\|_p^p = \int_\Omega |mf|^p \,\mathrm{d}\mu \leqslant \|m\|_\infty^p \int_\Omega |f|^p \,\mathrm{d}\mu = \|m\|_\infty^p \|f\|_p^p,$$

T_m is bounded on $L^p(\Omega)$ and $\|T_m\| \leqslant \|m\|_\infty$. For $p = \infty$ the analogous bound follows by taking essential suprema. Equality $\|T_m\| = \|m\|_\infty$ is obtained by considering, for

$0 < \varepsilon < 1$, functions supported on measurable sets $F_\varepsilon \in \mathscr{F}$ where $|m| \geqslant (1-\varepsilon)\|m\|_\infty$ μ-almost everywhere.

Example 1.30 (Integral operators)**.** Let μ be a finite Borel measure on a compact metric space K. With respect to the *product metric* $d((s,t),(s',t')) := d(s,t) + d(s',t')$, $K \times K$ is a compact metric space (see Proposition D.13). Let $k \in C(K \times K)$ and define, for $f \in C(K)$, the function $Tf : K \to \mathbb{K}$ by

$$Tf(s) := \int_K k(s,t) f(t)\,\mathrm{d}\mu(t), \quad s \in K.$$

Using the uniform continuity of k (see Theorem D.12), it is easy to see that Tf is a continuous function. Indeed, given $\varepsilon > 0$, choose $\delta > 0$ so small that $d((s,t),(s',t')) < \delta$ implies $|k(s,t) - k(s',t')| < \varepsilon$. Then $d(s,s') < \delta$ implies

$$|Tf(s) - Tf(s')| \leqslant \varepsilon \int_K |f(t)|\,\mathrm{d}\mu(t) \leqslant \varepsilon\mu(K)\|f\|_\infty.$$

As a result, T acts as a linear operator on $C(K)$. To prove boundedness, we estimate

$$|Tf(s)| \leqslant \int_K |k(s,t)||f(t)|\,\mathrm{d}\mu(t) \leqslant \mu(K)\|k\|_\infty\|f\|_\infty.$$

Taking the supremum over $s \in K$, this results in

$$\|Tf\|_\infty \leqslant \mu(K)\|k\|_\infty\|f\|_\infty.$$

It follows that T is bounded and $\|T\| \leqslant \mu(K)\|f\|_\infty$.

For kernels $k \in L^\infty(K \times K, \mu \times \mu)$ the same prescription defines a bounded operator on $L^\infty(K,\mu)$ satisfying the same estimate. If one takes $k \in L^2(K \times K, \mu \times \mu)$, this prescription gives a bounded operator T on $L^2(K,\mu)$ satisfying

$$\|T\| \leqslant \|k\|_2. \tag{1.4}$$

Indeed, by the Cauchy–Schwarz inequality (its abstract version for Hilbert spaces will be proved in Chapter 3) and Fubini's theorem we obtain

$$\begin{aligned}&\int_K \Big| \int_K k(s,t) f(t)\,\mathrm{d}\mu(t) \Big|^2 \mathrm{d}\mu(s) \\ &\qquad \leqslant \int_K \Big(\int_K |k(s,t)|^2\,\mathrm{d}\mu(t) \Big) \Big(\int_K |f(t)|^2\,\mathrm{d}\mu(t) \Big)\,\mathrm{d}\mu(s) = \|k\|_2^2\|f\|_2^2\end{aligned}$$

and the claim follows. This inequality generalises the one of Example 1.26.

Example 1.31 (Volterra operator)**.** For all $f \in L^2(0,1)$, the Cauchy–Schwarz inequality implies that the indefinite integral

$$Tf(s) := \int_0^s f(t)\,\mathrm{d}t, \quad s \in [0,1],$$

is well defined and that $|Tf(s) - Tf(s')| \leqslant |s-s'|^{1/2}\|f\|_2$ for all $s,s' \in [0,1]$. From this we infer that $Tf \in C[0,1]$ and, by taking $s' = 0$, that $\|Tf\|_\infty \leqslant \|f\|_2$. This implies that T is bounded from $L^2(0,1)$ into $C[0,1]$ with norm $\|T\| \leqslant 1$.

Composing T with the natural inclusion mapping from $C[0,1]$ into $L^2(0,1)$, the indefinite integral can be viewed as a bounded operator on $L^2(0,1)$ of norm at most 1. A sharper bound is obtained by applying the last part of the preceding example (with $k(s,t) = \mathbf{1}_{(0,s)}(t)$). This gives that T is bounded as an operator on $L^2(0,1)$ with norm

$$\|T\| \leqslant \|k\|_2 = 1/\sqrt{2} \approx 0.7071\ldots.$$

Interestingly, this norm bound is not sharp; it can be shown that the norm of this operator equals

$$\|T\| = 2/\pi \approx 0.6366\ldots$$

This will be proved using the spectral theory of selfadjoint operators in Chapter 8.

As this brief list of examples already shows, operators occurring naturally in Analysis have a tendency to be bounded. This raises the natural question whether linear operators acting between Banach spaces X and Y are always bounded. If one is willing to accept the Axiom of Choice the answer is negative, even for separable Hilbert spaces X and $Y = \mathbb{K}$ (see Problem 3.23). In Zermelo–Fraenkel Set Theory without the Axiom of Choice, it is consistent that every linear operator acting between Banach spaces is bounded. The reader is referred to the Notes to Chapter 3 for a further discussion of this topic.

1.3 Finite-Dimensional Spaces

The aim of this section is to prove that every finite-dimensional normed space is a Banach space. This will be deduced as an easy consequence of the fact that every two norms on a finite-dimensional normed space are equivalent, in the sense made precise in the next definition.

Definition 1.32 (Equivalent norms)**.** Two norms $\|\cdot\|$ and $|||\cdot|||$ on a vector space X are *equivalent* if there exist constants $0 < c \leqslant C < \infty$ such that for all $x \in X$ we have

$$c\|x\| \leqslant |||x||| \leqslant C\|x\|.$$

Example 1.33. Any two product norms on the product $X = X_1 \times \cdots \times X_N$ of normed spaces are equivalent. Indeed, (1.2) shows that every product norm on X is equivalent to the product norm $\|x\|_1 := \sum_{n=1}^N \|x_n\|$ on X.

In the above situation we have the inclusions of open balls

$$B_{\|\cdot\|}(x;r/C) \subseteq B_{|||\cdot|||}(x;r) \subseteq B_{\|\cdot\|}(x;r/c).$$

Hence if two norms on a given vector space are equivalent the resulting normed spaces have the same open sets. This implies that topological notions such as openness, closedness, compactness, convergence, and so forth, are preserved under passing to an equivalent norm.

Theorem 1.34 (Equivalence of norms in finite dimensions)**.** *Every two norms on a finite-dimensional vector space are equivalent.*

Proof Let $(X, \|\cdot\|)$ be a finite-dimensional normed space, say of dimension d, and let $(x_j)_{j=1}^d$ be a basis for X. Relative to this basis, every $x \in X$ admits a unique representation $x = \sum_{j=1}^d c_j x_j$. We may use this to define a norm $\|\cdot\|_2$ on X by

$$\Big\| \sum_{j=1}^d c_j x_j \Big\|_2 := \Big(\sum_{j=1}^d |c_j|^2 \Big)^{1/2}.$$

The theorem follows once we have shown that the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent.

Let $M = \max_{1 \leqslant j \leqslant d} \|x_j\|$. By the triangle inequality and the Cauchy–Schwarz inequality, for all $x = \sum_{j=1}^d c_j x_j$ we have

$$\|x\| \leqslant \sum_{j=1}^d |c_j| \|x_j\| \leqslant M \sum_{j=1}^d |c_j| \leqslant M d^{1/2} \Big(\sum_{j=1}^d |c_j|^2 \Big)^{1/2} = M d^{1/2} \|x\|_2. \tag{1.5}$$

This gives one of the two inequalities in the definition of equivalence of norms.

To prove that a similar inequality holds in the opposite direction, let S_2 denote the unit sphere in $(X, \|\cdot\|_2)$. Since $(c_1, \dots, c_d) \mapsto \sum_{j=1}^d c_j x_j$ maps the unit sphere of $\mathbb{K}^d$ isometrically (hence continuously) onto S_2, S_2 is compact. Consider the identity mapping $I : x \mapsto x$, viewed as a mapping from $(X, \|\cdot\|_2)$ to $(X, \|\cdot\|)$. The inequality (1.5) implies that I is bounded and therefore continuous. Since taking norms is continuous as well and S_2 is compact, the mapping $x \mapsto \|Ix\|$ is continuous from S_2 to $[0, \infty)$ and takes a minimum at some point $x_0 \in S_2$.

Denoting this minimum by m, we claim that $m > 0$. It is clear that $m \geqslant 0$. Reasoning by contradiction, if we had $m = \|Ix_0\| = 0$, then $Ix_0 = 0$ in X, hence $x_0 = 0$ as an element of S_2. Then $\|x_0\|_2 = 0$, while at the same time $\|x_0\|_2 = 1$ because $x_0 \in S_2$. This contradiction proves the claim.

For any nonzero $x \in X$ we have $\frac{x}{\|x\|_2} \in S_2$ and therefore $\|I \frac{x}{\|x\|_2}\| \geqslant m$. This gives the estimate

$$m \|x\|_2 \leqslant \|Ix\| = \|x\|$$

for nonzero $x \in X$; for trivial reasons it also holds for $x = 0$. □

Corollary 1.35. *Every d-dimensional normed space is isomorphic to $\mathbb{K}^d$. In particular, every finite-dimensional normed space is a Banach space.*

Proof The first assertion has been proved in the course of the proof of Theorem 1.34, and the second assertion follows from it since $\mathbb{K}^d$ is complete. $\square$

Corollary 1.36. *Every finite-dimensional subspace of a normed space is closed.*

Proof By Corollary 1.35, every a finite-dimensional subspace of a normed space is complete, and it has been shown in the first paragraph of Section 1.1.b that every complete subspace of a normed space is closed. $\square$

Corollary 1.37. *Every linear operator from a finite-dimensional normed space X into a normed space Y is bounded.*

Proof Let$(x_j)_{j=1}^d$ be a basis for X. If $T : X \to Y$ is linear, for $x = \sum_{j=1}^d c_j x_j$ we obtain, by the Cauchy–Schwarz inequality,

$$\|Tx\| = \Big\| \sum_{j=1}^d c_j T x_j \Big\| \leqslant \sum_{j=1}^d |c_j| \|Tx_j\| \leqslant M d^{1/2} \|x\|_2,$$

where $\|x\|_2 := (\sum_{j=1}^d |c_j|^2)^{1/2}$ as in Theorem 1.34 and $M := \max_{1 \leqslant n \leqslant d} \|Tx_n\|$. By Theorem 1.34 there exists a constant $K \geqslant 0$ such that $\|x\|_2 \leqslant K\|x\|$ for all $x \in X$. Combining this with the preceding estimate we obtain

$$\|Tx\| \leqslant M d^{1/2} \|x\|_2 \leqslant K M d^{1/2} \|x\|.$$

This means that T is bounded with norm at most $KMd^{1/2}$. $\square$

Every bounded subset of a finite-dimensional normed space X is relatively compact; this follows from the corresponding result for $\mathbb{K}^d$ and the fact that X is isomorphic to $\mathbb{K}^d$ for some $d \geqslant 1$ by Corollary 1.35. Conversely, a normed space with the property that every bounded subset is relatively compact is finite-dimensional:

Theorem 1.38 (Finite-dimensional Banach spaces)**.** *The unit ball of a normed space X is relatively compact if and only if X is finite-dimensional.*

The proof depends on the following lemma:

Lemma 1.39 (Riesz)**.** *If Y is a proper closed subspace of a normed space X, then for every $\varepsilon > 0$ there exists a norm one vector $x \in X$ with $d(x,Y) \geqslant 1 - \varepsilon$.*

Here, $d(x,Y) = \inf_{y \in Y} \|x - y\|$ is the distance from x to Y.

Proof Fix any $x_0 \in X \setminus Y$; such x_0 exists since Y is a proper subspace of X. Fix $\varepsilon > 0$ and choose $y_0 \in Y$ such that $\|x_0 - y_0\| \leqslant (1+\varepsilon) d(x_0,Y)$. The vector $(x_0 - y_0)/\|x_0 - y_0\|$ has norm one, and for all $y \in Y$ we have

$$\Big\| \frac{x_0 - y_0}{\|x_0 - y_0\|} - y \Big\| = \Big\| \frac{x_0 - y_0 - y\|x_0 - y_0\|}{\|x_0 - y_0\|} \Big\| \geqslant \frac{d(x_0,Y)}{(1+\varepsilon) d(x_0,Y)} = \frac{1}{1+\varepsilon}.$$

It follows that

$$d\Big(\frac{x_0 - y_0}{\|x_0 - y_0\|}, Y\Big) \geqslant \frac{1}{1+\varepsilon}.$$

Since $(1+\varepsilon)^{-1} \to 1$ as $\varepsilon \downarrow 0$, this completes the proof. □

Proof of Theorem 1.38 It remains to prove the 'only if' part. Suppose that X is infinite-dimensional and pick an arbitrary norm one vector $x_1 \in X$. Proceeding by induction, suppose that norm one vectors $x_1, \dots, x_n \in X$ have been chosen such that $\|x_k - x_j\| \geqslant \frac{1}{2}$ for all $1 \leqslant j \neq k \leqslant n$. Choose a norm one vector $x_{n+1} \in X$ by applying Riesz's lemma to the proper closed subspace $Y_n = \mathrm{span}\{x_1, \dots, x_n\}$ and $\varepsilon = \frac{1}{2}$ (that Y_n is closed follows from Corollary 1.36). Then $\|x_{n+1} - x_j\| \geqslant \frac{1}{2}$ for all $1 \leqslant j \leqslant n$.

The resulting sequence $(x_n)_{n\geqslant 1}$ is contained in the closed unit ball of X and satisfies $\|x_j - x_k\| \geqslant \frac{1}{2}$ for all $j \neq k \geqslant 1$, so $(x_n)_{n\geqslant 1}$ has no convergent subsequence. It follows that the closed unit ball of X is not compact. □

1.4 Compactness

Let X be a normed space. By Theorem 1.38, the collections of bounded subsets of X and relatively compact subsets of X coincide if and only if X is finite-dimensional. Thus, in infinite-dimensional spaces, relative compactness is a stronger property than boundedness. The purpose of the present section is to record some easy but useful general results on compactness that will be frequently used. Compactness in the spaces $C(K)$ and $L^p(\Omega)$ will be studied in the next chapter, and *compact operators*, that is, operators which map bounded sets into relatively compact sets, are studied in Chapter 7.

By a general result in the theory of metric spaces (Theorem D.10), every relatively compact set in a normed space is totally bounded, and the converse holds in Banach spaces. This fact is used in the proof of the following necessary and sufficient condition for compactness. For sets A and B in a vector space V we write

$$A + B := \{u + v : u \in A, v \in B\}.$$

Proposition 1.40. *A subset S of a Banach space X is relatively compact if and only if for all $\varepsilon > 0$ there exists a relatively compact set $K_\varepsilon \subseteq X$ such that $S \subseteq K_\varepsilon + B(0;\varepsilon)$.*

Proof 'If': The existence of the sets K_ε implies that S is totally bounded and hence relatively compact, for if the balls $B(x_{1,\varepsilon};\varepsilon), \dots, B(x_{n_\varepsilon,\varepsilon};\varepsilon)$ cover K_ε, then the balls $B(x_{1,\varepsilon};2\varepsilon), \dots, B(x_{n_\varepsilon,\varepsilon};2\varepsilon)$ cover S.

'Only if': This is trivial (take $K_\varepsilon = S$ for all $\varepsilon > 0$). □

The *convex hull* of a subset F of a vector space V is the smallest convex set in V

containing F. This set is denoted by $\mathrm{co}(F)$. When F is a subset of a normed space, the closure of $\mathrm{co}(F)$ is denoted by $\overline{\mathrm{co}}(F)$ and is referred to as the *closed convex hull* of F.

As a first application of Proposition 1.40 we have the following result.

Proposition 1.41. *The closed convex hull of a compact set in a Banach space is compact.*

Proof Let K be a compact subset of the Banach space X. For every $N \geqslant 1$ the set

$$\mathrm{co}_N(K) := \Big\{ \sum_{n=1}^{N} \lambda_n x_n : \ x_n \in K \text{ and } 0 \leqslant \lambda_n \leqslant 1 \text{ for all } n = 1, \dots, N, \ \sum_{n=1}^{N} \lambda_n = 1 \Big\}$$

is contained in the image of the compact set $[0,1]^N \times K^N$ under the continuous mapping that sends $((\lambda_1, \dots, \lambda_N), (x_1, \dots, x_N))$ to $\sum_{n=1}^N \lambda_n x_n$.

Let $\varepsilon > 0$ be arbitrary, let the open balls $B(\xi_1; \varepsilon), \dots, B(\xi_M; \varepsilon)$ cover K, and consider an element $x \in \mathrm{co}(K)$, say $\sum_{j=1}^k \lambda_j x_j$. For each $j = 1, \dots, k$ let $1 \leqslant m_j \leqslant M$ be an index such that

$$\|x_j - \xi_{m_j}\| = \min_{m=1,\dots,M} \|x_j - \xi_m\|.$$

Then

$$\Big\| x - \sum_{j=1}^{k} \lambda_j \xi_{m_j} \Big\| \leqslant \sum_{j=1}^{k} \lambda_j \|x_j - \xi_{m_j}\| < \sum_{j=1}^{k} \lambda_j \varepsilon = \varepsilon.$$

Since $\sum_{j=1}^k \lambda_j \xi_{m_j} = \sum_{m=1}^M (\sum_{j: m_j = m} \lambda_j) \xi_m \in \mathrm{co}_M(K)$, this implies that $x \in \mathrm{co}_M(K) + B(0; \varepsilon)$. This shows that $\mathrm{co}(K) \subseteq \mathrm{co}_M(K) + B(0; \varepsilon)$. It now follows from Proposition 1.40 that $\mathrm{co}(K)$ is relatively compact. $\square$

The second result asserts that strong convergence implies uniform convergence on relatively compact sets.

Proposition 1.42. *Let X and Y be normed spaces, let the operators $T_n \in \mathscr{L}(X,Y)$, $n \geqslant 1$, be uniformly bounded, and let $T \in \mathscr{L}(X,Y)$. If $\lim_{n\to\infty} T_n = T$ strongly, then for all relatively compact subsets K of X we have*

$$\lim_{n\to\infty} \sup_{x \in K} \|T_n x - Tx\| = 0.$$

It will be shown in Proposition 5.3 that if X is a Banach space, strong convergence $T_n \to T$ already implies uniform boundedness of the operators T_n.

Proof Let K be a relatively compact subset of X, let $\varepsilon > 0$ be arbitrary, and select finitely many open balls $B(x_1; \varepsilon), \dots, B(x_k; \varepsilon)$ covering K. Choose $N \geqslant 1$ so large that $\|T_n x_j - T x_j\| < \varepsilon$ for all $n \geqslant N$ and $j = 1, \dots, k$. Let $M := \sup_{n \geqslant 1} \|T_n\|$; this number is

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finite by assumption. Fixing an arbitrary $x \in K$, choose $1 \leqslant j_0 \leqslant k$ such that $\|x - x_{j_0}\| < \varepsilon$. Then, for $n \geqslant N$,

$$\begin{aligned}\|T_n x - Tx\| &\leqslant \|T_n x - T_n x_{j_0}\| + \|T_n x_{j_0} - Tx_{j_0}\| + \|Tx_{j_0} - Tx\| \\ &\leqslant M\varepsilon + \varepsilon + M\varepsilon = (2M+1)\varepsilon.\end{aligned}$$

Taking the supremum over $x \in K$, it follows that if $n \geqslant N$, then

$$\sup_{x \in K} \|T_n x - Tx\| \leqslant (2M+1)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this proves the final assertion. □

1.5 Integration in Banach Spaces

In a variety of circumstances, some of which will be encountered in later chapters, one wishes to integrate X-valued functions, where X is a Banach space. In order to have the tools available when they are needed, we insert a brief discussion of the X-valued counterparts of the Riemann and Lebesgue integrals.

1.5.a The Riemann Integral

Let K be a compact metric space and let μ be a finite Borel measure on K. We will set up the Riemann integral with respect to μ for continuous functions $f : K \to X$. To this end we need the following terminology. A *partition* of K is a finite collection of pairwise disjoint Borel subsets of K whose union equals K. The *mesh* of a partition is the diameter of the largest subset in the partition.

Bernhard Riemann, 1826–1866

Proposition 1.43 (Riemann integral). *Let μ be a finite Borel measure on a compact metric space K, let X be a Banach space, and let $f : K \to X$ be a continuous function. There exists a unique element in X, denoted by $\int_K f \, d\mu$, with the following property: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $(K_n)_{n=1}^N$ is a partition of K of mesh less than δ and $(t_n)_{n=1}^N$ is a collection of points in K with $t_n \in K_n$ for all $n = 1, \ldots, N$, then*

$$\Big\| \int_K f \, d\mu - \sum_{n=1}^N \mu(K_n) f(t_n) \Big\| < \varepsilon.$$

The proof of this theorem follows the undergraduate construction of the Riemann integral for continuous functions $f:[0,1]\to\mathbb{K}$ step-by-step and is therefore omitted. The element $\int_K f\,\mathrm{d}\mu$ is called the *Riemann integral of f with respect to μ*. Whenever this is convenient we use the more elaborate notation $\int_K f(t)\,\mathrm{d}\mu(t)$.

Proposition 1.44. *Let μ be a finite Borel measure on a compact metric space K, let X be a Banach space, and let $f:K\to X$ be a continuous function. Then*

$$\Big\|\int_K f\,\mathrm{d}\mu\Big\| \leqslant \int_K \|f\|\,\mathrm{d}\mu.$$

Proof For any partition $(K_n)_{n=1}^N$ of K and any collection of points $(t_n)_{n=1}^N$ in K with $t_n\in K_n$ for all $n=1,\dots,N$ we have

$$\Big\|\sum_{n=1}^N \mu(K_n)f(t_n)\Big\| \leqslant \sum_{n=1}^N \mu(K_n)\|f(t_n)\|$$

by the triangle inequality. The result follows by taking the limit along any sequence of partitions whose meshes tend to zero. □

In the special case where $K=[0,1]$ and μ is the Lebesgue measure, the usual calculus rules apply (defining differentiability of an X-valued function in the obvious way):

Proposition 1.45. *Let X be a Banach space and let $f:[0,1]\to X$ be a function. Then:*

(1) *if f is differentiable at the point $t_0\in[0,1]$, then f is continuous at t_0;*
(2) *if f is differentiable on $(0,1)$ and $f'\equiv 0$ on $(0,1)$, then f is constant on $(0,1)$;*
(3) *if f is continuously differentiable on $[0,1]$, then*

$$\int_0^1 f'(t)\,\mathrm{d}t = f(1)-f(0).$$

Proof (1): Fix an arbitrary $\varepsilon>0$. The assumption implies there exists $\delta>0$ such that if $t\in[0,1]$ with $|t-t_0|<\delta$, then

$$\Big\|\frac{f(t)-f(t_0)}{t-t_0}-f'(t_0)\Big\|<\varepsilon.$$

Then $\|f(t)-f(t_0)\|<(\varepsilon+\|f'(t_0)\|)|t-t_0|$ and continuity at t_0 follows.

(2): The usual calculus proof via Rolle's theorem does not extend to the present setting, as it uses the order structure of the real numbers.

Fix an arbitrary $\varepsilon>0$. For each $t\in(0,1)$, the assumption $f'(t)=0$ implies that there exists $h(t)>0$ such that the interval $I_t:=(t-h(t),t+h(t))$ is contained in $(0,1)$ and

$$\|f(t)-f(s)\|\leqslant\varepsilon|t-s|,\quad s\in I_t.$$

Fix a closed subinterval $[a,b]\subseteq(0,1)$. The intervals I_t, $t\in[a,b]$, cover the compact set

$[a,b]$ and therefore this set is contained in the union of finitely many intervals $I_{t_1},\dots,I_{t_N}$. By adding the intervals I_a and I_b and relabelling (and perhaps discarding some of the intervals), we may assume that $a=t_1$, $b=t_N$, and $I_{t_n}\cap I_{t_{n+1}}\neq\varnothing$ for $n=1,\dots,N-1$. Choosing $s_n\in I_{t_n}\cap I_{t_{n+1}}$ we have

$$\begin{aligned}\|f(t_{n+1})-f(t_n)\| &\leqslant \|f(t_{n+1})-f(s_n)\|+\|f(s_n)-f(t_n)\|\\ &\leqslant \varepsilon(t_{n+1}-s_n)+\varepsilon(s_n-t_n)=\varepsilon(t_{n+1}-t_n).\end{aligned}$$

Now let $t\in[a,b]$, say $t\in I_{t_k}$. Then

$$\begin{aligned}\|f(t)-f(a)\| &\leqslant \|f(t)-f(t_k)\|+\|f(t_k)-f(t_{k-1})\|+\cdots+\|f(t_2)-f(t_1)\|\\ &\leqslant \varepsilon(t-t_k)+\varepsilon(t_k-t_{k-1})+\cdots+\varepsilon(t_2-t_1)=\varepsilon(t-a).\end{aligned}$$

This being true for all $\varepsilon>0$ it follows that $f(t)=f(a)$ for all $t\in[a,b]$. This proves that f is constant on every subinterval $[a,b]\subseteq(0,1)$ and therefore on $(0,1)$.

(3): For the function $g:[0,1]\to X$, $g(t):=f(t)-\int_0^t f'(s)\,\mathrm{d}s$, we have

$$\lim_{h\to 0}\frac{1}{h}(g(t+h)-g(t))=f'(t)-\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}f'(s)\,\mathrm{d}s=0$$

by continuity, and therefore g is continuously differentiable on $[0,1]$ with derivative $g'=0$. It follows from (2) that g is constant on $(0,1)$, hence on $[0,1]$ by continuity, and then $g(0)=f(0)$ implies

$$f(t)-\int_0^t f'(s)\,\mathrm{d}s=g(t)=g(0)=f(0),\quad t\in[0,1].$$

Taking $t=1$ gives the result. □

In Chapter 4 we will sketch a different proof using duality.

1.5.b The Bochner Integral

We turn next to the more delicate problem of generalising the Lebesgue integral to functions taking values in a Banach space X. The results of this section will be needed only in Chapter 13.

In what follows we fix a measure space $(\Omega,\mathscr{F})$. It is a matter of experience that if one attempts to define the measurability of a function $f:\Omega\to X$ by imposing that $f^{-1}(B)$ be in $\mathscr{F}$ for all Borel (equivalently, for all open) subsets of X, one arrives at a notion of measurability that is not very practical, the problem being that it does not connect well with approximation theorems such as the dominated convergence theorem. It turns out that it is better to start from the following necessary and sufficient condition for measurability in the scalar-valued setting: *A scalar-valued function is measurable if and only if it is the pointwise limit of a sequence of simple functions.*

For a function $f : \Omega \to \mathbb{K}$ and $x \in X$ we define $f \otimes x : \Omega \to X$ by

$$(f \otimes x)(\omega) := f(\omega)x. \tag{1.6}$$

Definition 1.46 (Simple functions, strong measurability)**.** A function $f : \Omega \to X$ is called *simple* if it is a finite linear combination of functions of the form $\mathbf{1}_F \otimes x$ with $F \in \mathscr{F}$ and $x \in X$, and *strongly measurable* if it is the pointwise limit of a sequence of simple functions.

A scalar-valued function is strongly measurable if and only if it is measurable, and for such functions we omit the adjective 'strongly'.

Theorem 1.47 (Pettis measurability theorem, first version)**.** *A function $f : \Omega \to X$ is strongly measurable if and only if f takes its values in a separable closed subspace X_0 of X and the nonnegative functions $\|f(\cdot) - x_0\|$ are measurable for all $x_0 \in X_0$.*

A second version of this theorem will be proved in Chapter 4 (see Theorem 4.19).

Proof 'If': Let $(x_n)_{n\geqslant 1}$ be dense in X_0 and define the functions $\phi_n : X_0 \to \{x_1, \dots, x_n\}$ as follows. For each $y \in X_0$ let $k(n,y)$ be the least integer $1 \leqslant k \leqslant n$ such that

$$\|y - x_k\| = \min_{1\leqslant j\leqslant n} \|y - x_j\|,$$

and put $\phi_n(y) := x_{k(n,y)}$. Since $(x_n)_{n\geqslant 1}$ is dense in X_0 we have

$$\lim_{n\to\infty} \|\phi_n(y) - y\| = 0, \quad y \in X_0.$$

Now define $\psi_n : \Omega \to X$ by

$$\psi_n(\omega) := \phi_n(f(\omega)), \quad \omega \in \Omega.$$

We have

$$\{\omega \in \Omega : \ \psi_n(\omega) = x_1\} = \Big\{\omega \in \Omega : \ \|f(\omega) - x_1\| = \min_{1\leqslant j\leqslant n} \|f(\omega) - x_j\|\Big\}$$

and, for $2 \leqslant k \leqslant n$,

$$\begin{aligned}&\{\omega \in \Omega : \ \psi_n(\omega) = x_k\} \\ &\quad = \Big\{\omega \in \Omega : \ \|f(\omega) - x_k\| = \min_{1\leqslant j\leqslant n} \|f(\omega) - x_j\| < \min_{1\leqslant j<k-1} \|f(\omega) - x_j\|\Big\}.\end{aligned}$$

In both identities, the set on the right-hand side is in $\mathscr{F}$. Hence each ψ_n is simple, takes values in X_0, and for all $\omega \in \Omega$ we have

$$\lim_{n\to\infty} \|\psi_n(\omega) - f(\omega)\| = \lim_{n\to\infty} \|\phi_n(f(\omega)) - f(\omega)\| = 0.$$

'Only if': Let $f_n \to f$ pointwise with each f_n simple. Let X_0 be the closed linear span of the ranges of the functions f_n. Then X_0 is separable and f takes its values in X_0. Moreover, $\omega \mapsto \|f(\omega) - x_0\| = \lim_{n\to\infty} \|f_n(\omega) - x_0\|$ is measurable. □

Corollary 1.48. *If* $\lim_{n\to\infty} f_n = f$ *pointwise, with each* f_n *strongly measurable, then* f *is strongly measurable.*

Proof We check the conditions of the Pettis measurability theorem. Every function $f_n : \Omega \to X$ is the pointwise limit of a sequence of simple functions $f_{nm} : \Omega \to X$, and every f_{nm} takes at most finitely many different values. It follows that f takes its values in the closed linear span of these countably many finite sets, which is a separable subspace of X. The measurability of the functions $\|f_n - x_0\|$ implies that $\|f - x_0\|$ is measurable. □

Definition 1.49 (μ-Simple functions)**.** A simple function $f = \sum_{n=1}^{N} \mathbf{1}_{F_n} \otimes x_n$ is called μ*-simple* if $\mu(F_n) < \infty$ for all $n = 1, \dots, N$. For such functions we define

$$\int_\Omega f \, \mathrm{d}\mu := \sum_{n=1}^{N} \mu(F_n) x_n.$$

We leave it as a simple exercise to verify that $\int_\Omega f \, \mathrm{d}\mu$ is well defined in the sense that it does not depend on the representation of f as a linear combination of functions $\mathbf{1}_{F_n} \otimes x_n$ with $\mu(F_n) < \infty$. If f is μ-simple, the triangle inequality implies

$$\Big\| \int_\Omega f \, \mathrm{d}\mu \Big\| \leqslant \int_\Omega \|f\| \, \mathrm{d}\mu. \tag{1.7}$$

Definition 1.50 (Bochner integral)**.** A strongly measurable function $f : \Omega \to X$ is said to be *Bochner integrable* with respect to μ if there is a sequence of μ-simple functions $f_n : \Omega \to X$ such that

$$\lim_{n\to\infty} \int_\Omega \|f - f_n\| \, \mathrm{d}\mu = 0. \tag{1.8}$$

In that case we define the *Bochner integral* of f by

$$\int_\Omega f \, \mathrm{d}\mu := \lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu. \tag{1.9}$$

The nonnegative functions $\|f - f_n\|$ are measurable by the Pettis measurability theorem, so the integral in (1.8) is well defined. The limit in (1.9) exists since the assumption together with (1.7) (applied to $f_n - f_m$) implies that $(\int_\Omega f_n \, \mathrm{d}\mu)_{n\geqslant 1}$ is a Cauchy sequence in X. We leave it as another simple exercise to verify that $\int_\Omega f \, \mathrm{d}\mu$ is well defined in the sense that it does not depend on the sequence of approximating functions f_n. It is equally elementary to verify that if $\Omega = K$ is a compact metric space and $\mathscr{F}$ is its Borel σ-algebra, then every continuous function $f : K \to X$ is Bochner integrable with respect to μ and the Bochner integral coincides with the Riemann integral.

Proposition 1.51. *A strongly measurable function* $f : \Omega \to X$ *is Bochner integrable with respect to* μ *if and only if*

$$\int_\Omega \|f\| \, \mathrm{d}\mu < \infty.$$

In this situation we have

$$\Big\| \int_\Omega f \,\mathrm{d}\mu \Big\| \leqslant \int_\Omega \|f\| \,\mathrm{d}\mu.$$

Proof ‘If’: Let f be a strongly measurable function satisfying $\int_\Omega \|f\| \,\mathrm{d}\mu < \infty$. Let g_n be simple functions such that $\lim_{n\to\infty} g_n = f$ pointwise and define

$$f_n := \mathbf{1}_{\{\|g_n\| \leqslant 2\|f\|\}} g_n.$$

Then each f_n is simple and we have $\lim_{n\to\infty} f_n = f$ pointwise. Since $\|f_n\| \leqslant 2\|f\|$ pointwise and $\int_\Omega \|f\| \,\mathrm{d}\mu < \infty$, each f_n is μ-simple and by dominated convergence we obtain

$$\lim_{n\to\infty} \int_\Omega \|f - f_n\| \,\mathrm{d}\mu = 0.$$

‘Only if’: If f is Bochner integrable and the μ-simple function $g : \Omega \to X$ is such that $\int_\Omega \|f - g\| \,\mathrm{d}\mu \leqslant 1$, then

$$\int_\Omega \|f\| \,\mathrm{d}\mu \leqslant 1 + \int_\Omega \|g\| \,\mathrm{d}\mu < \infty.$$

The final assertion follows from (1.7) by approximation. □

Problems

1.1 Show that in any normed space X, for all $x_0 \in X$ and $r > 0$ the following assertions hold:

(a) $B(x_0; r) = \{x \in X : \|x - x_0\| < r\}$ is an open set.
(b) $\overline{B}(x_0; r) = \{x \in X : \|x - x_0\| \leqslant r\}$ is a closed set.
(c) $\overline{B}(x_0; r) = \overline{B(x_0; r)}$, that is, $\overline{B}(x_0; r)$ is the closure of $B(x_0; r)$.

1.2 Let X be a normed space.

(a) Show that if $x, y \in X$ satisfy $\|x - y\| < \varepsilon$ with $0 < \varepsilon < \|x\|$, then $y \neq 0$ and

$$\Big\| x - \frac{\|x\|}{\|y\|} y \Big\| < 2\varepsilon.$$

(b) Show that the constant 2 in part (a) is the best possible.

1.3 Show that a norm $\|\cdot\|$ on the product $X = X_1 \times \cdots \times X_N$ of normed spaces is a product norm if and only if $\|x\|_\infty \leqslant \|x\| \leqslant \|x\|_1$ for all $x = (x_1, \ldots, x_N) \in X$, where

$$\|x\|_\infty := \max_{1 \leqslant n \leqslant N} \|x_n\|, \quad \|x\|_1 := \sum_{n=1}^N \|x_n\|.$$

1.4 Show that if $X = X_1 \oplus \cdots \oplus X_N$ is a direct sum of normed spaces, then each summand X_n is closed as a subspace of X.

1.5 Prove that if $T \in \mathscr{L}(X,Y)$ is bounded, then

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|<1} \|Tx\|.$$

1.6 Let X and Y be normed spaces and let $T \in \mathscr{L}(X,Y)$. Prove that for all $x \in X$ and $r > 0$ we have

$$\sup_{y \in B(x;r)} \|Ty\| \geqslant r\|T\|.$$

1.7 Let $X_0 := C_c^1(0,1)$ be the vector space of all C^1-functions $f : (0,1) \to \mathbb{K}$ with compact support in $(0,1)$.

(a) Show that $X := \{f \in C[0,1] : f(0) = f(1) = 0\}$ is a Banach space and that X_0 can be naturally identified with a dense subspace in X.

(b) Show that for each $f \in X_0$ the limit $\lim_{n\to\infty} T_n f$ exists with respect to the norm of X and equals f', where

$$T_n f(t) = \frac{f(t+1/n) - f(t)}{1/n}.$$

(c) Show that there are functions $f \in X$ for which the limit $\lim_{n\to\infty} T_n f$ does not exist in X.

This example shows that the uniform boundedness assumption cannot be omitted in Proposition 1.19.

1.8 Show that if two norms $\|\cdot\|$ and $\|\cdot\|'$ on a normed space X are equivalent, then the norms of the completions of $(X, \|\cdot\|)$ and $(X, \|\cdot\|')$ are equivalent.

1.9 Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on a vector space X. Show that the following assertions are equivalent:

(1) there exists a constant $C \geqslant 0$ such that $\|x\| \leqslant C\|x\|'$ for all $x \in X$;
(2) every open set in $(X, \|\cdot\|)$ is open in $(X, \|\cdot\|')$;
(3) every convergent sequence in $(X, \|\cdot\|')$ is convergent in $(X, \|\cdot\|)$;
(4) every Cauchy sequence in $(X, \|\cdot\|')$ is Cauchy in $(X, \|\cdot\|)$.

1.10 Let X be a Banach space with respect to the norms $\|\cdot\|$ and $\|\cdot\|'$. Suppose that $\|\cdot\|$ and $\|\cdot\|'$ agree on a subspace Y that is dense in X with respect to both norms. We ask whether the norms agree on all of X.

(a) Comment on the following attempt to prove this: Apply Proposition 1.18 to the identity mapping on Y, viewed as a mapping from the normed space $(Y, \|\cdot\|)$ to the normed $(Y, \|\cdot\|')$ and as a mapping in the opposite direction.

(b) Comment on the following attempt to prove this: Let $x \in X$ be fixed and, using density, choose a sequence $x_n \to x$ with $x_n \in Y$ for all $n \geqslant 1$. Then $\|x\| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|x_n\|' = \|x\|'$.

(c) Comment on Problem 2.8 as an attempt to disprove this.

(d) Prove that the answer is affirmative if we make the additional assumption that $\|\cdot\| \leqslant C\|\cdot\|'$ for some constant $0 < C < \infty$.

1.11 Provide the details to the ‘if’ part of the proof of Proposition 1.13.

1.12 Let X be a normed space.

(a) Show that if X is separable, then the completion of X is separable.
(b) Show that if X is a Banach space and Y is a closed subspace of X, then X is separable if and only if both Y and X/Y are separable.

1.13 Determine whether the following sets are open and/or closed in $C[0,1]$:

(a) $\{f \in C[0,1] : f(t) \geqslant 0 \text{ for all } t \in [0,1]\}$;
(b) $\{f \in C[0,1] : f(t) > 0 \text{ for all } t \in [0,1]\}$.

Consider the set $S := \{f \in L^1(0,1) : f(t) \geqslant 0 \text{ for almost all } t \in (0,1)\}$.

(c) Determine whether S is a closed subset of $L^1(0,1)$.
(d) Characterise the functions belonging to the interior S°.

1.14 This problem gives an example of a bounded operator that does not attain its norm. Let X be the space of continuous functions $f : [0,1] \to \mathbb{K}$ satisfying $f(0) = 0$.

(a) Show that X is a closed subspace of $C[0,1]$.

Thus, with the norm inherited from $C[0,1]$, X is a Banach space.

(b) Show that the operator $T : X \to \mathbb{K}$,

$$Tf := \int_0^1 f(t)\,\mathrm{d}t,$$

is bounded and has norm $\|T\| = 1$.
(c) Prove that $|Tf| < 1$ for all $f \in X$ with $\|f\|_\infty \leqslant 1$.

1.15 This problem gives an example of a bounded operator whose range is not closed. Consider the linear operator T on $C[0,1]$ given by the indefinite integral

$$Tf(t) = \int_0^t f(s)\,\mathrm{d}s, \quad t \in [0,1].$$

(a) Show that T is bounded and compute its norm.
(b) Show that $\mathsf{R}(T)$ is not closed in $C[0,1]$.

1.16 Let X be a Banach space and Y be a normed space. Show that if $T : X \to Y$ is a bounded operator satisfying $\|Tx\| \geqslant C\|x\|$ for some $C > 0$ and all $x \in X$, then its range $\mathsf{R}(T)$ is complete and T is an isomorphism from X to $\mathsf{R}(T)$.

1.17 Let X and Y be finite-dimensional normed spaces. Prove that if $T_n, T \in \mathscr{L}(X,Y)$, then the following assertions are equivalent:

(1) $\lim_{n\to\infty} T_n = T$ uniformly;
(2) $\lim_{n\to\infty} T_n = T$ strongly;

(3) $\lim_{n\to\infty} T_n = T$ weakly.

1.18 Let $1 \leqslant p < \infty$.

(a) Show that ℓ^p is a dense subspace of c_0.
(b) Show that the inclusion mapping of ℓ^p into c_0 is bounded.

1.19 Show that a normed space X and its completion $\overline{X}$ have the same dual, that is, the restriction mapping $\overline{x}^* \mapsto \overline{x}^*|_X$ is an isometric isomorphism from $\overline{X}^*$ onto X^*.

1.20 Let X be a real vector space. The product $X \times X$ can be given the structure of a complex vector space by introducing a complex scalar multiplication as follows:

$$(a+ib)(x,y) := (ax-by, bx+ay).$$

The idea is to think of the pair $(x,y) \in X \times X$ as "$x+iy$".

(a) Check that this formula for the scalar multiplication does indeed turn $X \times X$ into a complex vector space.

The resulting complex vector space is denoted by $X_{\mathbb{C}}$.
Suppose now that X is a real normed space.

(b) Prove that the formula

$$\|(x,y)\| := \sup_{\theta\in[0,2\pi]} \|(\cos\theta)x + (\sin\theta)y\|$$

defines a norm on $X_{\mathbb{C}}$ which turns it into a complex normed space. Show that $X_{\mathbb{C}}$ is a Banach space if and only if X is a Banach space.
(c) Show that this norm on $X_{\mathbb{C}}$ extends the norm of X in the sense that $\|(x,0)\| = \|(0,x)\| = \|x\|$ for all $x \in X$.
(d) Show that $\|(x,y)\| = \|(x,-y)\|$ for all $x,y \in X$.
(e) Show that any two norms on $X_{\mathbb{C}}$ which satisfy the identities in parts (c) and (d) are equivalent.

1.21 Let X be a real Banach space and let $X_{\mathbb{C}}$ be the complex Banach space constructed in Problem 1.20.

(a) Show that if T is a (real-)linear bounded operator on X, then T extends to a bounded (complex-)linear operator $T_{\mathbb{C}}$ on $X_{\mathbb{C}}$ by putting $T_{\mathbb{C}}(x,y) := (Tx, Ty)$.
(b) Show that $\|T_{\mathbb{C}}\| = \|T\|$.

1.22 As a variation on Proposition 1.40, show that a bounded subset S of a Banach space X is relatively compact if and only if for every $\varepsilon > 0$ there exists a finite-dimensional subspace X_ε of X such that $S \subseteq X_\varepsilon + B(0;\varepsilon)$.

1.23 Show that a subset K of a Banach space X is relatively compact if and only if K is contained in the closed convex hull of a sequence $(x_n)_{n\geqslant 1}$ in X satisfying $\lim_{n\to\infty} x_n = 0$.

Hint: For the 'only if' part, cover K with finitely many balls of radius 3^{-n} and let C_n be the set of their centres; $n = 1,2,\dots$ Let $D_1 := C_1$ and, for $n \geqslant 2$,

$$D_n := \{c_n - c_{n-1} : \; c_n \in C_n, \; c_{n-1} \in C_{n-1}, \; \|c_n - c_{n-1}\| < 3^{-n+1}\}.$$

Check that each $x \in K$ can be represented as an absolutely convergent sum $x = \sum_{n\geqslant 1} d_n$ with $d_n \in D_n$. Consider the sequence $(x_n)_{n\geqslant 1}$ given by $x_n := 2^n d_n$.

1.24 Let $(\Omega, \mathscr{F})$ be a measurable space. Adapting the proof of Theorem 1.47 show that if $f : \Omega \to X$ is strongly measurable, there are simple functions $f_n : \Omega \to X$ such that $f_n \to f$ and $\|f_n\| \leqslant \|f\|$ pointwise.

1.25 Let K be a compact metric space, let μ a finite Borel measure on K, and let X be a Banach space. Prove that every continuous function $f : K \to X$ is Bochner integrable with respect to μ and that its Bochner integral equals its Riemann integral.

1.26 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and let X_0 be a closed subspace of the Banach space X. Let $f : \Omega \to X$ satisfy $f(\omega) \in X_0$ for all $\omega \in \Omega$.

(a) Show that if f is strongly measurable as an X-valued function, then f is strongly measurable as an X_0-valued function.

Assume now that $f : \Omega \to X$ satisfies $f(\omega) \in X_0$ for almost all $\omega \in \Omega$.

(b) Show that if f is strongly measurable as an X-valued function, then f is strongly measurable as an X_0-valued function.
(c) Show that if f is Bochner integrable as an X-valued function, then f is Bochner integrable as an X_0-valued function.

1.27 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Show that if $T : X \to Y$ is a bounded operator and $f : \Omega \to X$ is Bochner integrable with respect to μ, then $Tf : \Omega \to Y$ is Bochner integrable with respect to μ and

$$T \int_\Omega f \, \mathrm{d}\mu = \int_\Omega Tf \, \mathrm{d}\mu.$$

1.28 Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and let $(\Omega', \mathscr{F}')$ be a measurable space. Let $\phi : \Omega \to \Omega'$ be measurable and let $f : \Omega' \to X$ be strongly measurable. Let $\nu = \mu \circ \phi^{-1}$ be the image measure of μ under ϕ.

(a) Show that $f \circ \phi$ is strongly measurable.
(b) Show that $f \circ \phi$ is Bochner integrable with respect to μ if and only if f is Bochner integrable with respect to ν, and that in this situation we have

$$\int_\Omega f \circ \phi \, \mathrm{d}\mu = \int_{\Omega'} f \, \mathrm{d}\nu.$$

1.29 Let $(\Omega, \mathscr{F}, \mu)$ be a probability space. Prove that if $f : \Omega \to X$ is Bochner integrable, then $\int_\Omega f \, \mathrm{d}\mu$ is contained in the closed convex hull of $\{f(\omega) : \; \omega \in \Omega\}$.