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COMPRESSIBLE ENDS OF LEAVES IN FOLIATED 3-MANIFOLDS

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Abstract

In this paper we study the asymptotic behavior of cylindrical ends in compact foliated 3-manifolds and give a sufficient condition for these ends to spiral onto a toral leaf.

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1. Introduction and preliminaries

In this paper we shall be concerned with orientable foliations $\mathscr{F}$ of dimension two of compact three dimensional orientable manifolds. If the boundary of a manifold M is not empty, we always assume the components of ∂M are leaves of the foliation, and all foliations and maps we consider are assumed to be at least of class C^2.

The study of the limit set of an end and the comprehension of the asymptotic behavior of a leaf following this end, is a difficult problem in its full generality. Many authors have studied the limit set of ends under certain conditions (see for example the introduction in [C-C1]). In particular J. Cantwell and L. Conlon have proven a theorem similar to Theorem 4 below for totally proper ends in a codimension one foliation [C-C1] as well as for isolated planar ends of proper leaves with non-exponential growth (see [C-C2, Cor. 3.4]). More precisely, they proved that totally proper leaves spiral on leaves at lower level (see in [C-C1, section 6]). Nishimori [Ni] has studied the asymptotic behavior of isolated ends whose limit set is a compact leaf. Hector [H] has classified foliations for which all leaves are cylinders, furnishing us with models where cylindrical ends appear. In the present work, for the first time, we get results without the hypothesis that L is proper and without any assumptions on the growth type of the leaves. Instead of these assumptions we use the compressibility of the cylindrical end and the hypothesis that the end is of trivial linking type (see definitions 1 and 3 below). Roughly speaking, the idea of the proof is to consider and study the foliation $\mathscr{F}_D$ induced by $\mathscr{F}$ on a disc D associated to a compressible cylindrical end e.

Finally, it is not known if there are cylindrical compressible ends of non-trivial linking type. Contrarily, it seems plausible that the following must have an affirmative answer:

QUESTION. Is each cylindrical compressible end in M of trivial linking type?

We will need the concept of the end of an open leaf of $\mathscr{F}$. For details concerning this concept see [C-C1, C-C2]. Let L be an open leaf of $\mathscr{F}$. An end e of L is defined by a decreasing sequence $\cdots \supset K_n \supset K_{n+1} \supset \cdots$ such that:

(a) Each K_n is a closed 2-dimensional submanifold of L with boundary.
(b) The boundary ∂K_n is diffeomorphic to the unit sphere $\mathbb{S}^1$.
(c) $\cap_n K_n = \emptyset$.

Such a sequence $(K_n)_{n=1,2,\ldots}$ defining an end e is called a *fundamental neighborhood system of* e and we write $e = (K_n)_{n=1,2,\ldots}$.

The *e-limit set* of L is defined by e-lim$(L) = \cap_i \overline{K_i}$; as usual, $\overline{K_i}$ denotes the closure of K_i in M. It is elementary that e-lim(L) is compact and $\mathscr{F}$-satured.

An end $e = (K_n)_{n=1,2\ldots}$ is called *cylindrical* if K_1 (and hence each K_i) is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

The ends of a leaf L often become apparent in terms of simpler leaves on which they wind around. These results are a kind of Poincaré-Bendixson theory for foliations of codimension one. In the following we prove a theorem of this type for cylindrical compressible ends of trivial linking type. We need the following definitions.

DEFINITION 1. Let $e = (K_n)_{n=1,2\ldots}$ be a cylindrical end of a leaf L. The end e will be called *compressible* if there exists a 2-dimensional disc D embedded in M such that: $\partial D \subset K_1$, ∂D is a non-contractible curve in L and D is transverse to L along ∂D. Such a disc D, will be called a disc *associated* to e.

A simple closed curve of K_1 which is non-contractible in K_1 will be called a *generator* of K_1. Obviously any two generators of K_1 are freely homotopic in K_1.

DEFINITION 2. We will say that two disjoint generators γ_1, γ_2 of K_1 form a *trivial link* in M if for any two embedded discs D_1, D_2 in M with $\partial D_1 = \gamma_1$, $\partial D_2 = \gamma_2$ there is an isotopie $h_t : D_1 \to M, 0 \leq t \leq 1$ such that:

(1) $h_0(D_1) = D_1$, $h_1(D_1) = D_1'$ and $D_1' \cap D_2 = \emptyset$;
(2) $h_t(\gamma_1) = \gamma_1$ for each $t \in [0, 1]$.

DEFINITION 3. We will say that a compressible end $e = (K_n)_{n=1,2,\ldots}$ is of *trivial linking type* if any two disjoint generators of K_1 form a trivial link in M.

REMARK 1. It is not difficult to prove that if two generators of K_1 form a trivial link then any two generators of K_1 form a trivial link in M

We may use a weaker (and more standard) definition for the trivial link between two generators of K_1 (see for example [Ro]). If we do so, the hypothesis $\pi_2(M) = 0$ must be added in the theorem below. In any case we will not discuss this alternative as it is beyond the scope of this work.

The aim of this paper is to prove:

THEOREM 4. *Let $\mathscr{F}$ be a 2-dimensional orientable foliation on a compact orientable 3-manifold M. Assume that each embedded 2-sphere in M separates M (that is, the complement of the embedded sphere in M, consists of exactly two components). Let $e = (K_n)_{n=1,2\dots}$ be a cylindrical compressible end of trivial linking type of a leaf $L \in \mathscr{F}$, where $L \neq \mathbb{R}^2$. Then there exists a smooth embedding $f : \mathbb{S}^1 \times [0, \varepsilon) \to M$ and a decreasing sequence (t_n), $0 < t_n < \varepsilon$, converging to zero such that*

(i) *For all $p \in \mathbb{S}^1$, $f(\{p\} \times [0, \varepsilon))$ is an arc transverse to $\mathscr{F}$.*

(ii) *For all $n \in \mathbb{N}$, $f(\mathbb{S}^1 \times \{t_n\}) = c_n \subset K_1$ and c_n is non-contractible in L.*

(iii) *The curve $f(\mathbb{S}^1 \times \{0\}) = c_0$ belongs to a toral leaf L_0 and c_0 is non-contractible in L_0. Moreover, e-$\lim(L) = L_0$.*

2. Proof of the theorem

Choose a point $y_0 \in$ e-$\lim(L)$ such that $y_0 \notin L$. Such a point exists since, by compactness of e-$\lim(L)$, one has $L \neq$ e-$\lim(L)$. Therefore there exists a sequence of points $(y_n)_{n=1,2,\dots}$ such that:

(i) $y_n \in K_n$, $n = 1, 2, \dots$

(ii) y_n converges to y_0 along an arc S transverse to $\mathscr{F}$.

For each $n = 1, 2, \dots$ we consider a simple closed curve β_n in K_n which contains y_n and is not contractible in L. Moreover we choose the curves β_n to be pairwise disjoint.

Since $h : \mathbb{D}^2 \to M$ is a C^r-map we may assume that h is in general position with respect to $\mathscr{F}$. That is to say, h is transverse to $\mathscr{F}$ in a neighborhood of $\partial\mathbb{D}^2$ and the foliation $h^*\mathscr{F}$, induced on $\mathbb{D}^2$ from $\mathscr{F}$ by the map h, contains singularities of saddle or central type. Fix a transverse orientation on $\mathscr{F}$ and let η (respectively $\eta_{h^*\mathscr{F}}$) be the unit vector field normal to $\mathscr{F}$ (respectively $h^*\mathscr{F}$). Let $[y_1, y_0] = J \subset S$ be the segment between the points y_1 and y_0. Then we have

LEMMA 5. *There exists an embedding $h : \mathbb{D}^2 \to M$ in general position with respect to $\mathscr{F}$ such that $h(\partial\mathbb{D}^2) = \beta_1$ and $J \subset h(\mathbb{D}^2)$. Moreover, if l'_k is the leaf of $h^*\mathscr{F}$ passing through $h^{-1}(y_k)$, then $h(l'_k)$ cannot be a closed curve contractible in L for any $k \in \mathbb{N}$.*

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PROOF. Since the end e is compressible, there exists an embedding $h_0 : \mathbb{D}^2 \to M$ in general position with respect to $\mathscr{F}$ such that $h_0(\partial \mathbb{D}^2) = \beta_1$. If $J \subset h_0(\mathbb{D}^2)$, set $h = h_0$. If not, then we may assume that J intersects $h_0(\mathbb{D}^2)$ transversely in a finite number of points. Therefore, by a deformation of $h_0(\mathbb{D}^2)$ in M, we may assume that $h_0 : \mathbb{D}^2 \to M$ is an embedding such that:

(i) $h_0(\partial \mathbb{D}^2) = \beta_1'$ where $\beta_1' \subset K_1$;
(ii) $\beta_1' \cap \beta_1 = \emptyset$ and β_1' is freely homotopic to β_1 in K_1;
(iii) $J \cap h_0(\mathbb{D}^2) = \emptyset$.

In the following we consider an embedding $h_1 : \mathbb{S}^1 \times [0, 1] \to M$ which satisfies:

(i) $h_1(\mathbb{S}^1 \times \{0\}) = \beta_1$ and $J \subset h_1(\mathbb{S}^1 \times [0, 1/2])$;
(ii) h_1 is transverse to $\mathscr{F}$ in a neighborhood of $\beta_1 \cup J$;
(iii) $h_1(\mathbb{S}^1 \times [0, 1]) \cap h_0(\mathbb{D}^2) = \emptyset$.

The curve $h_1(\mathbb{S}^1 \times \{1\}) = \beta_2'$ is homotopic to β_1' and, obviously there exists an embedding $h_2 : \mathbb{S}^1 \times [0, 1] \to M$ such that:

(i) $h_2(\mathbb{S}^1 \times \{0\}) = \beta_2'$, $h_2(\mathbb{S}^1 \times \{1\}) = \beta_1'$;
(ii) $h_2(\mathbb{S}^1 \times (0, 1)) \cap h_0(\mathbb{D}^2) = \emptyset$;
(iii) $h_2(\mathbb{S}^1 \times (0, 1)) \cap h_1(\mathbb{S}^1 \times [0, 1]) = \emptyset$.

By gluing the maps h_2, h_1 along β_2' and h_1, h_0 along β_1' we get a continuous injection $f : \mathbb{D}^2 \to M$ with $f(\partial \mathbb{D}^2) = \beta_1$ and such that:

(i) f is differentiable in a neighborhood V of β_1 with $J \subset V$;
(ii) f is transverse to $\mathscr{F}$ in a neighborhood U of $\beta_1 \cup J$ with $U \subset V$.

By rounding the corners, we can replace f by a (C^r-) embedding $f_0 : \mathbb{D}^2 \to M$ which coincides with f in a neighborhood of $\partial \mathbb{D}^2$ and such that:

(i) J is contained in $f_0(\mathbb{D}^2)$;
(ii) f_0 is transverse to $\mathscr{F}$ in a neighborhood of $\beta_1 \cup J$.

Now it is well known (see [G, Ch. IV, Lemma 1.6]) that for every $\varepsilon > 0$ we can find a C^r-map $h : \mathbb{D}^2 \to M$, in general position with respect to $\mathscr{F}$, ε-near to f_0, which coincides with f_0 in a neighborhood of $\partial \mathbb{D}^2 \cup f_0^{-1}(J)$. This map h is an embedding since ε can be chosen arbitrarily small and it satisfies the conditions of Lemma 5.

In the following, the theorem of Novikov concerning the existence of Reeb components [N] easily implies that if some curve l_k' is closed then its image $h(l_k')$ cannot be homotopic to zero in L. In fact: suppose that $h(l_k')$ is homotopic to zero in L. Then we can find a differentiable map $G : \mathbb{S}^1 \times [0, 1] \to M$ such that:

(a) $G(\mathbb{S}^1 \times [0, 1]) \subset h(\mathbb{D}^2)$ and for every $t \in [0, 1]$, the curve $g_t : \mathbb{S}^1 \to M$, defined by $g_t(x) = G(t, x)$, is contained in a leaf L_t of $\mathscr{F}$. Moreover $g_0(\mathbb{S}^1) = h(l_k')$.

(b) For every $x \in \mathbb{S}^1$, the curve $t \to G(t, x)$ is transverse to $\mathscr{F}$ and the orientation of this curve is opposite to the transverse orientation of $\mathscr{F}$ induced by the normal field η.

(c) For every $t \in [0, 1)$, the curve g_t is homotopic to a constant in L_t and g_1 is not homotopic to a constant in L_1.

From Novikov's theorem mentioned above, the leaf L_1 must be a toral leaf which bounds a solid torus T in M and all leaves L_t are diffeomorphic to planes contained in the interior of T. Therefore, the leaf L must be a plane contained in the interior of T. Obviously, this is impossible since there are points of L outside of T.

Next, we will prove the following:

PROPOSITION 6. *There exists an embedding $g : \mathbb{D}^2 \to M$ in general position with respect to $\mathcal{F}$ with $g(\partial \mathbb{D}^2) = \beta_1$ and such that:*

(1) *There is an arc J' transverse to $g^*\mathcal{F}$ with $g(J') = J$, and a sequence y'_n in J' with $g(y'_n) = y_n$, such that all the leaves of $g^*\mathcal{F}$ passing through y'_n, $n = 2, 3, \dots$, are simple closed curves. We denote these curves by a'_n. The points y'_n converge to a point $y'_0 \in J'$.*

(2) *If $a_n = g(a'_n)$ then $a_n \subset L$ for $n = 1, 2, \dots$ and they are non-contractible curves in L. For each n, there is a component V_n of $L - a_n$ such that $K_1 \supset \mathrm{cl}\, V_1 \supset \mathrm{cl}\, V_2 \supset \cdots \supset \mathrm{cl}\, V_n \supset \cdots$ and each $\mathrm{cl}\, V_n$ is diffeomorphic to $\mathbb{S}^1 \times [0, \infty)$; here by $\mathrm{cl}\, V_n$ we denote the closure of V_n in L.*

REMARK 2. We may assume that such an embedding $g : \mathbb{D}^2 \to M$ intersects L transversely. In fact, $g^*\mathcal{F}$ has a finite number of singularities. So if the leaves l induced by L on $g(\mathbb{D}^2)$ contain saddle points we displace them to leaves next to l by a small perturbation of $g(\mathbb{D}^2)$ in M.

Assuming Proposition 6 is not true we obtain a contradiction with the aid of Lemmas 7, 8 and 9 which follow. In fact, suppose that Proposition 6 is not true. Then we have

LEMMA 7. *If Proposition 6 is not true, there is an embedding $g : \mathbb{D}^2 \to M$ in general position with respect to F such that:*

(1) *There is an arc J' transverse to $g^*\mathcal{F}$ with $g(J') = J$ and a sequence of points y'_n in J' with $g(y'_n) = y_n$ converging to y'_0 following the positive direction of J'.*

(2) *There is an $i \in \{2, 3, \dots\}$ such that the leaves a'_m of $g^*\mathcal{F}$ passing through y'_m for $m < i$, are simple closed curves and their images $a_m = g(a'_m)$ are simple and non-contractible in L. Otherwise the leaf l'_i passing through y'_i is homeomorphic to $\mathbb{R}$.*

(3) *There is a component V_n of $L - a_n$, $n = 1, 2, \dots, i-1$ such that $K_1 \supset \mathrm{cl}\, V_1 \supset \mathrm{cl}\, V_2 \supset \cdots \supset \mathrm{cl}\, V_{i-1}$, and each $\mathrm{cl}\, V_n$, $n = 1, 2, \dots, i-1$ is diffeomorphic to $\mathbb{S}^1 \times [0, \infty)$.*

(4) *The natural number i posited in (2) above is the smallest natural for which there exists an embedding $g : \mathbb{D}^2 \to M$ in general position with respect to $\mathcal{F}$ which satisfies the conditions (1)–(3) of the lemma.*

PROOF. If the conditions (1)–(3) of Lemma 7 were satisfied for each $i \in \{2, 3, \dots\}$ then Proposition 6 would be valid. On the other hand Lemma 5 guarantees that for $i = 2$ there exists a map $g : \mathbb{D}^2 \to M$ in general position with respect to $\mathscr{F}$ which satisfies conditions (1)–(3). Therefore if Proposition 6 is not true there exists an $i \geq 2$ for which conditions (1)–(4) are satisfied.

LEMMA 8. *Let $l_i = g(l_i')$ and let $r(t), t \in (-\infty, +\infty)$, be a parametrization of l_i. Set $l_i^+ = r([0, +\infty)), l_i^- = r((-\infty, 0])$. Then l_i^+, l_i^- intersect all curves β_n of L for $n > i$.*

PROOF. The curve l_i does not intersect β_1 since $\mathscr{F}$ is tangent to $g(\partial\mathbb{D}^2) = \beta_1$. Suppose now that l_i^+ does not intersect all the curves β_n for $n > i$. Then there is an annulus C in L ($C \approx \mathbb{S}^1 \times [0, 1]$) such that

$$l_i^+ \cap C \neq \emptyset \quad \text{and} \quad l_i^+ \cap \partial C = \emptyset.$$

In fact, there exists a natural number $k > i$ such that $l_i^+ \cap \beta_k = \emptyset$. Note also that the point y_i of l_i^+ belongs to $K_1 \cap \beta_i$. Thus, if we consider the annulus C in K_1 with $\partial C = \beta_1 \cup \beta_k$, we have that $l_i^+ \subset \operatorname{int} C$.

On the other hand, as we mentioned in Remark 2, g intersects L transversely. Therefore $g(\mathbb{D}^2) \cap C$ is a compact submanifold of C. This implies that l_i^+ must intersect the boundary ∂C of C, which gives a contradiction; similarly if we consider l_i^-.

LEMMA 9. *There is an arc $[z', w']$ transverse to $g^*\mathscr{F}$ such that the points $z = g(z')$, $w = g(w')$ belong to β_j for some $j > i$.*

PROOF. Lemma 8 asserts that $l_i = g(l_i')$ intersects all curves β_n for $n > i$. Fix $j > i$. Let $w \in l_i^+ \cap \beta_j$, $z \in l_i^- \cap \beta_j$ and let $z', w' \in l_i'$ be such that $g(z') = z$, $g(w') = w$.

Since l_i' is a non-compact and non-singular leaf of $g^*\mathscr{F}$, we can choose a neighborhood V in $\mathbb{D}^2$ diffeomorphic to $[0, 1] \times [0, 1]$ and points $x', y' \in l_i'$ (see Figure 1) such that:

(i) $(g^*\mathscr{F})_V$ is the product foliation $[0, 1] \times \{t\}, t \in [0, 1]$;

(ii) if $(l_i')^+$ is the subset of l_i' with $g((l_i')^+) = l_i^+$, then $(l_i')^+ \cap V$ has an infinite number of connected components;

(iii) x', y' lie in different components of $(l_i')^+ \cap V$;

(iv) if $r'(t), t \in (-\infty, +\infty)$ is a parametrization of l_i' and $r'(t_1) = z', r'(t_2) = w'$, $r'(t_3) = x', r'(t_4) = y'$, then $t_1 < t_2 < t_3 < t_4$.

Let $l_i'[z', x']$ (respectively $l_i'[w', y']$) denotes the subarc of l_i' with endpoints z', x' (respectively w', y'). It is well known that there are tubular neighborhoods Z of

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$l_i'[z', x']$ and W of $l_i'[w', y']$ in $\mathbb{D}^2$ such that $(g^*\mathscr{F})_Z$ (respectively $(g^*\mathscr{F})_W$) is the product foliation.

Consider an arc $[x', y']$ transverse to $(g^*\mathscr{F})_V$. Now we can find disjoint subarcs $[x', x'']$, $[y'', y']$ of $[x', y']$, as well as arcs $[z', x'']$ and $[w', y'']$ transverse to $(g^*\mathscr{F})_Z$ and $(g^*\mathscr{F})_W$ respectively such that if $[x'', y''] \subset [x', y']$, then the arcs $[x'', y'']$, $[z', x'']$ and $[w', y'']$ are pairwise disjoint (see for example Figure 1). The union $[z', x''] \cup [x'', y''] \cup [y'', w']$ is an arc in $\mathbb{D}^2$ transverse to $g^*\mathscr{F}$.

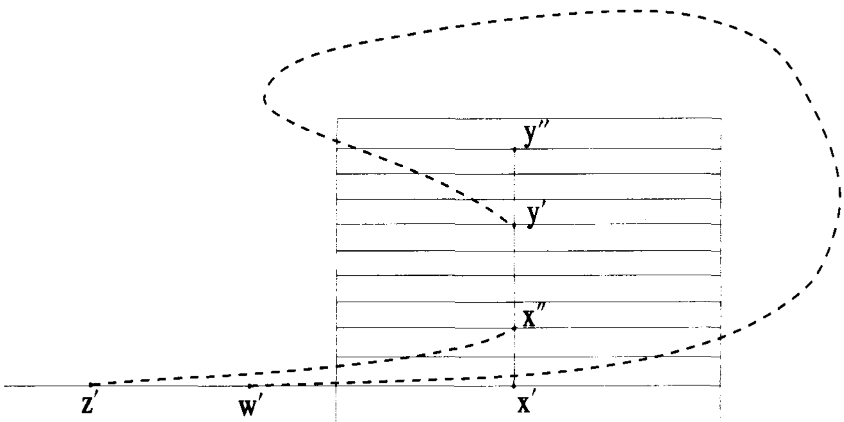

FIGURE 1

Now there exists a disc D embedded in M with $\partial D = \beta_j$ and $D \cap g(\mathbb{D}^2) = \emptyset$. This follows from the fact that $g(\partial\mathbb{D}^2) = \beta_1$ and β_1, β_j form a trivial link. In what follows we can choose disjoint generators β_j', β_j'' of K_1 sufficiently near to β_j such that if L_j is the annulus in K_1 with $\partial L_j = \beta_j' \cup \beta_j''$ then $\beta_j \subset \operatorname{int} L_j$ and $L_j \cap D = \beta_j$. We have the following:

CLAIM. We can choose discs D_j', D_j'' embedded in M such that:
(i) $\partial D_j' = \beta_j'$, $\partial D_j'' = \beta_j''$;
(ii) $D_j' \cap g(\mathbb{D}^2) = \emptyset$, $D_j'' \cap g(\mathbb{D}^2) = \emptyset$;
(iii) $S = L_j \cup D_j' \cup D_j''$ is a topological 2-sphere which bounds a 3-ball B in M.

PROOF OF CLAIM. We consider a tubular neighbourhood V of L_j in M diffeomorphic to $L_j \times (-1, +1)$, where $L_j \approx L_j \times \{0\}$ under this diffeomorphism. We consider also a neighborhood W of D diffeomorphic to $D \times (-1, +1)$ such that $\partial D \times (-1, +1) \subset L_j$ and $W \cap g(\mathbb{D}^2) = \emptyset$. Therefore, we can construct the discs

D'_j, D''_j of the claim within the union $V \cup W$. In Figure 2(a) the discs D'_j, D''_j are designated by thick arcs.

From the previous claim we have that S separates M into two connected components. Now we examine the relative position of $g([z', w'])$ with respect to S. All vectors of η on L_j are pointing either inwards or outward of the component B and $g([z', w'])$ is transverse to $\mathscr{F}$. Therefore there exists an $\varepsilon > 0$ such that $g([z', z' + \varepsilon])$ is contained in B and $g((w', w' - \varepsilon])$ is contained in the complement of B. Since $g([z', w'])$ is transverse to $\mathscr{F}$ we may assume that $g([z', w']) \cap D''_j \neq \emptyset$ or $g([z', w']) \cap D'_j \neq \emptyset$. However, by construction of the arc $g([z', w'])$ and of the discs D'_j, D''_j, $g([z', w'])$ intersects neither D'_j nor D''_j; this gives the promised contradiction.

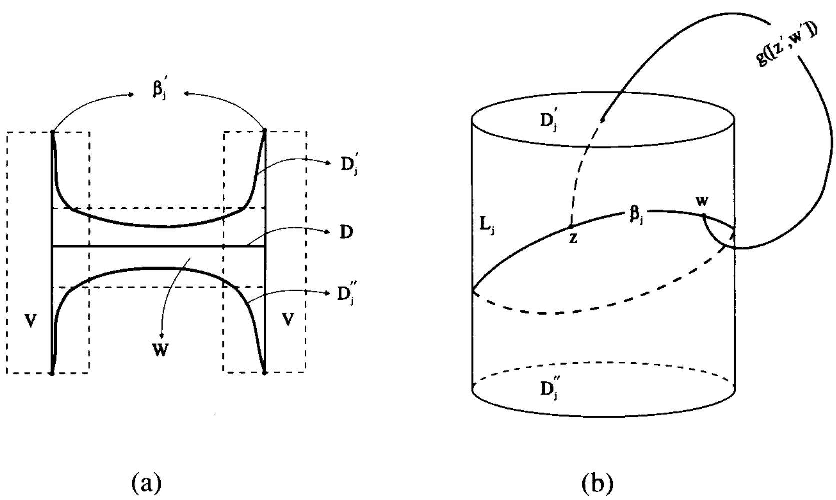

FIGURE 2

Therefore, Proposition 6 is valid. So we deduce immediately the following:

LEMMA 10. *There exists an embedding* $f : \mathbb{S}^1 \times [0, 1) \to M$ *and a sequence* $t_n \in [0, 1)$ *decreasing to* 0 *such that:*

(1) *If* $f_t = f(\cdot, t)$, $t \in [0, 1)$ *then* $f_{t_n}(\mathbb{S}^1) = a_n \subset L$ *and* $f_0(\mathbb{S}^1) = a_0 \subset L_0$.

(2) *For each* $p \in \mathbb{S}^1$ *the curve* $f_t(p)$, $t \in [0, 1)$, *is transverse to* $\mathscr{F}$.

PROOF. From proposition 6 we have that the sequence of closed curves a_n of L converges uniformly to a closed curve a_0 of a leaf L_0. Obviously a_0 is non-contractible in L_0 since all the curves a_n are non-contractible in L. Therefore we can obtain easily the family of embeddings $f_{t_n}(\mathbb{S}^1)$ which satisfies the properties 1, 2 of the lemma. Note that here we are using the hypothesis that $\mathscr{F}$ is transversely orientable.

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Now we will prove that L_0 is a torus in M. In order to prove this we work similarly to the appendix of [R-R], where it is proved that the existence of a vanishing cycle implies the existence of a Reeb component. For this proof we omit some details, and we refer to [R-R] for them.

LEMMA 11. *The leaf L_0 is homeomorphic to a torus.*

PROOF. In order to prove that L_0 is compact it suffices to show that there is no closed transversal curve intersecting L_0. Suppose γ is a closed transversal curve which intersects L_0 at a point p. We may assume that γ coincides with the segment $\{f_t(p), 0 \le t \le \varepsilon\}$ where $\varepsilon < 1$ and we orient γ in the positive sense of the foliation $\mathscr{F}$. Choose $t_m, t_n : 0 < t_m < t_n < \varepsilon$. Let $L(m, n) \subset L$ be the annulus limited by a_m, a_n and let $D(m, n) \subset g(\mathbb{D}^2)$ be the annulus limited by the same curves. Then $T(m, n) = L(m, n) \cup D(m, n)$ is a torus and we will prove that $T(m, n)$ separates M in two connected components. In fact, consider a disc D embedded in M such that $D \cap T(m, n) = d$ is a generator of $L(m, n)$. Such a disc exists since the end e is compressible and of trivial linking type. Let now D' be a disc parallel to D such that:

(i) $D' \cap L(m, n) = d'$ and d' is a generator of $L(m, n)$ disjoint from d.

(ii) If A is the annulus in $L(m, n)$ with $\partial A = d \cup d'$ then $A \cup D \cup D'$ is a 2-sphere which bounds a 3-ball in M.

If $A' = L(m, n) - A$ then $A' \cup D \cup D'$ is a 2-sphere which separates M by hypothesis. Therefore $T(m, n)$ separates M.

Denote by $S(m, n)$ the component of M-$T(m, n)$ such that all the positive normals point into $S(m, n)$ along $L(m, n)$. This implies that $\gamma(t) \in S(m, n)$ for $t > t_n$ and therefore L_0 is compact. Now, since the curve a_0 is non-contractible in L_0 it follows that L_0 is a leaf of genus greater than 0.

Finally as in [R-R, Lemma 5] we define a diffeomorphism $\Phi(m, n) : L_0 - f_0(S^1) \to L(m, n)$ which proves that L_0 is a torus.

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