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Small non-Leighton two-complexes†

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Abstract

How many 2-cells must two finite CW-complexes have to admit a common, but not finite common, covering? Leighton's theorem says that both complexes must have 2-cells. We construct an almost (?) minimal example with two 2-cells in each complex.

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0. *Introduction*

Leighton Theorem [10] If two finite graphs have a common covering, then they have a common finite covering.

Alternative proofs and various generalisations of this result can be found, e.g., in [2], [4], [14], [15], [19], and references therein.

Does a similar result hold for any CW-complexes, i.e.

> *is it true that, if, for finite CW-complexes K_1 and K_2, there exist a CW-complex K and cellular coverings $K_1 \leftarrow K \to K_2$, then there exists a finite CW-complex K with this property?*

This natural question was posed (in other terms) in [1] and [16]. Notice the cellularity requirement. Surely, we would obtain an equivalent question if we replace this condition with a formally stronger *combinatorialness* one: the image of each cell is a cell. However, without the cellularity condition, the answer would be negative: indeed, the torus and the genus-two surface have no finite common coverings (as the fundamental group the genus-two orientable surface $\langle x, y, z, t \mid [x, y][z, t] = 1\rangle$ contains no abelian subgroups of finite index), while the universal coverings of these surfaces are homeomorphic, because they are the plane. The cellularity condition rules out such examples: if we take, e.g., the standard one-vertex cell structures on the torus and genus-two surface, then, on the covering plane, we obtain:

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(i) the usual square lattice on the (Euclidean) plane (in the torus-case);

(ii) and an octagonal lattice on the (Lobachevskii) plane (in the genus-two case);

(i.e. though the universal coverings are homeomorphic, the cell structure on them are principally different). This example cannot be saved by a complication of the cell structures on the torus and genus-two surface (as was noted in [1] and [16]; the authors of [1] even conjectured that the answer to the (cellular version of) the question is positive).

Nevertheless, the answer turned out to be negative as was shown in [18] (and actually, much earlier in [17]); the complexes K_1, K_2 forming such a *non-Leighton* pair from [18] contain as few as six 2-cells each. In [8], this number was reduced to four:[1]

> *there exist two two-complexes containing four 2-cells each that have a common covering but have not finite common coverings.*

(Henceforth, we omit the prefix "CW-" and word "cellular": a *complex* means a CW-complex, and all mapping between complexes are assumed to be cellular in this paper.) The non-Leighton complexes K_1 and K_2 from [8] are the *standard complexes* of the following group presentations Γ_i, i.e. one-vertex complexes with edges corresponding to the generators and 2-cells attached by the relators:

$$\Gamma_1 = F_2 \times F_2 = \langle a, b, x, y \mid [a,x],\ [a,y],\ [b,x],\ [b,y]\rangle \quad \text{and}$$

$$\Gamma_2 = \left\langle a, b, x, y \;\middle|\; axay,\ ax^{-1}by^{-1},\ ay^{-1}b^{-1}x^{-1},\ bxb^{-1}y^{-1} \right\rangle.$$

Both of these complex are covered by the Cartesian product of two trees (Cayley graphs of the free group F_2); and no finite common cover exists, because the fundamental group of such hypothetical covering complex would embed in both groups Γ_i as finite-index subgroups, but, in Γ_1, any finite-index subgroup contains a finite-index subgroup which is the direct product of free groups, while Γ_2 has no such finite-index subgroups [8] (Γ_2 is not even residually finite [3], [5]). The results of [8] imply also a minimality of this example in the sense that:

> *if we restrict ourselves to complexes K_i covered by products of trees, then four two-dimensional cells is the minimum among all non-Leighton pairs.*

If we do not restrict ourselves, then smaller non-Leighton pairs arise.

MAIN THEOREM (a simplified version). *There exist two finite two-complexes containing two 2-cells each that have a common covering, but have not finite common coverings.*

(Explicit forms of these two two-2-cell two-complexes can be found at the very end of this paper.) Thus, the only question remaining open concerns complexes with a single 2-cell. This question seems to be difficult (although it is closely related to the well-developed theory one-relator groups). The point is that a classification of one-relator groups up to commensurability is not an easy task even for the *Baumslag–Solitar groups* $\mathrm{BS}(n,m) \stackrel{\text{def}}{=} \langle c, d \mid c^{nd} = c^m \rangle$ (though, in this special case, it was recently obtained [6]). Henceforth, $x^{ky} \stackrel{\text{def}}{=} y^{-1}x^k y$, where x and y are elements of a group and $k \in \mathbb{Z}$.

1 although the authors of [8] did not pursue this purpose; it was a byproduct of their results.

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In conclusion, note that results on coverings of two-complexes can imply nontrivial facts about graphs, because one can "model" 2-cells in graphs by means of additional vertices and edges, see [4]. Higher dimensional complexes are of little sense here: if complexes K_1 and K_2 form a non-Leighton pair, then their two-skeleta also form such a pair, as is easy to verify. A detailed exposition of the general theory of coverings and CW-complexes can be found, e.g., in [7].

1. *Algebraic lemmata*

The following fact is well known [13], we give a short proof for the reader's convenience.

COMMUTATOR LEMMA. *In the group* $H = \mathrm{BS}(3,5) = \langle c, d \mid c^{3d} = c^5 \rangle$, *the commutator* $h = [c^d, c]$ *belongs to any finite-index subgroup.*

Proof. Each finite-index subgroup contains a normal finite-index subgroup (see, e.g., [9]) Therefore, it suffices to show that h lies in the kernel of any homomorphism $\varphi : H \to K$ to any finite group K.

The elements $\varphi(c^3)$ and $\varphi(c^5)$ have the same order (because they are conjugate); hence, the order of $\varphi(c)$ is not divisible by three. Therefore, $\varphi(c) \in \langle \varphi(c^3) \rangle$. Thus, $\varphi(c)^{\varphi(d)} \in \langle \varphi(c) \rangle$ and $h = [c^d, c]$ belongs to the kernel of φ. This completes the proof.

BOTTLE LEMMA. *If a group G has a subgroup* $\langle a, b \rangle = \langle a, b \mid a^b = a^{-1} \rangle \simeq \mathrm{BS}(1,-1)$, *and the element b lies in all finite-index subgroups of G, then any finite-index subgroup of G contains a subgroup isomorphic to the Klein-bottle group* $\mathrm{BS}(1,-1)$.

Proof. Any finite-index subgroup contains all elements conjugate to b, because the intersection R of all finite-index subgroups is normal. Therefore, $a^2 = b^{-1}b^a \in R$ and $\langle a^2, b \rangle \subseteq R$ It remains to note that $a^{2b} = a^{-2}$, and the groups $\langle a^2 \rangle$ and $\langle b \rangle$ are infinite; hence, the subgroup $\langle a^2, b \rangle$ is isomorphic to $\mathrm{BS}(1,-1)$, because,

in any group, infinite-order elements x and y such that $x^y = x^{-1}$
generate a subgroup isomorphic to the Klein-bottle group. (1)

Indeed, there is obvious epimorphism

$$\varphi : \mathrm{BS}(1,-1) = \langle a, b \rangle \longrightarrow \langle x, y \rangle .$$

Any element $g \in \mathrm{BS}(1,-1)$ can be written in the form $g = a^k b^l$. If $g = a^k b^l \in \ker \varphi$, then $\ker \varphi \ni [b, g] = b^{-1} b^{-l} a^{-k} b a^k b^l = a^{\pm 2k}$. Therefore, $k = 0$ (because $|\langle x \rangle| = \infty$). But then $l = 0$ too, because $1 = \varphi(g) = \varphi(b^l) = y^l$, and $|\langle y \rangle| = \infty$. Thus, $\ker \varphi = \{1\}$ and φ is an isomorphism. This completes the proof.

NO-BOTTLE LEMMA. *The amalgamated free product*

$$G = \left\langle a, c, d \;\middle|\; [a, [c^d, c]] = 1,\ c^{3d} = c^5 \right\rangle = \langle a, b \mid [a, b] = 1 \rangle \underset{b = [c^d, c]}{*} \left\langle c, d \;\middle|\; c^{3d} = c^5 \right\rangle$$

of the free abelian group and the Baumslag–Solitar group $\mathrm{BS}(3,5)$ *contains no subgroups isomorphic to the Klein-bottle group* $K = \mathrm{BS}(1,-1)$.

Proof. The group $\mathrm{BS}(3,5)$ does not contain subgroups isomorphic to K [11] and is torsion-free. Therefore, applying once again (1), we obtain that the quotient

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$$G/\langle\langle [a,G]\rangle\rangle = \langle a\rangle_\infty \times \mathrm{BS}(3,5)$$

by the normal closure $\langle\langle [a,G]\rangle\rangle$ of the set $[a,G]$ of commutators of a and all elements of G has no nonidentity elements conjugate their inverse. Therefore, any element of G conjugate to its inverse lies in $N = \langle\langle [a,G]\rangle\rangle$. This subgroup intersects trivially the free factors (and their conjugates). Thus, all elements of N have length at least two, and the following conjugation criterion (see, e.g., [12]) applies:

> *two cyclically reduced words of length $\geqslant 2$ in an amalgamated free product $U \underset{W}{*} V$ are conjugate if and only if one of them can be obtained from the other by a cyclic permutation and subsequent conjugation by an element of W.*

In $U \underset{W}{*} V$, an equality of reduced words $u_1v_1\cdots = u_1'v_1'\cdots$ implies the equalities of the double cosets $Wu_1W = Wu_1'W$, $Wv_1W = Wv_1'W, \ldots$ $Wv_1W = Wv_1'W$Therefore, if a cyclically reduced word $x \in N \lhd \langle a,b \mid [a,b]=1\rangle \underset{b=[c^d,c]}{*} \left\langle c,d \,\middle|\, c^{3d}=c^5\right\rangle$ is conjugate to its inverse, then, for a letter x_1 of x, we obtain the equality $x_1 = b^k x_1^{-1} b^l$ (because the map $f_{k,n}: i \mapsto k-i \pmod n$ from the set (of subscripts) $\{1,\ldots,n\}$ to itself has either a fixed point, or an almost fixed point: $f_{k,n}(i) = i+1 \pmod n$ for some i; the latter case would imply that $x_1 = b^k x_2^{-1} b^l$ for some adjacent letters x_1 and x_2 of the reduced word x, which is impossible). Substituting $x_1 = b^k\widehat{x}_1$, we obtain $\widehat{x}_1^2 = b^{l-k}$; thus:

(i) either $\widehat{x}_1^2 \in \langle b\rangle$ for some $\widehat{x}_1 \in (\langle a\rangle_\infty \times \langle b\rangle_\infty) \setminus \langle b\rangle$;
(ii) or $\widehat{x}_1^2 \in \left\langle [c^d,c]\right\rangle$ for some $\widehat{x}_1 \in \left\langle c,d \,\middle|\, c^{3d}=c^5\right\rangle \setminus \left\langle [c^d,c]\right\rangle$.

The first is impossible of course. The impossibility of the second case can be verified, e.g., as follows:

(i) the quotient group $Q = \left\langle c,d \,\middle|\, c^{3d}=c^5\right\rangle / \left\langle\left\langle [c^d,c]\right\rangle\right\rangle$ is torsion-free; indeed, Q is the HNN-extension $Q = \left\langle c,e,d \mid [e,c]=1,\ e^3=c^5,\ c^d=e\right\rangle$ of the abelian group $A = \left\langle c,e \mid [e,c]=1,\ e^3=c^5\right\rangle$, which is torsion-free (moreover, it is easy to verify that $A \simeq \mathbb{Z}$ and $Q \simeq \mathrm{BS}(3,5)$);
(ii) therefore, $\widehat{x}_1$ lies in the normal closure $F = \left\langle\left\langle [c^d,c]\right\rangle\right\rangle$, which is a free group, because, by the Karrass–Solitar theorem (see, e.g., [12]), any subgroup of an HNN-extension is free if it intersects conjugates of the base trivially. It remains to show that $[c^d,c]$ is not a square in F (because in a free group an inclusion $\alpha^2 \in \langle\beta\rangle$ implies that $\langle\alpha,\beta\rangle$ is cyclic by the Nielsen–Schreier theorem and, hence, $\alpha \in \langle\beta\rangle$ if β is not a square). The commutator $[c^d,c]$ is not a square in F, because, assuming the contrary and noting that automorphic images of squares are squares too, we obtain $F = \left\langle\left\langle [c^d,c]\right\rangle\right\rangle = \left\langle\left\langle \widehat{x}^2\right\rangle\right\rangle \subseteq \left\langle \{f^2 \mid f \in F\}\right\rangle$, which cannot hold in a nontrivial free group F. This completes the proof.

2. *Proof of the main theorem*

Take the fundamental groups of the torus and the Klein bottle:

$$G_1 = \mathrm{BS}(1,1) = \langle a,b \mid [a,b]=1\rangle \text{ and } G_{-1} = \mathrm{BS}(1,-1) = \left\langle a,b \,\middle|\, a^b = a^{-1}\right\rangle$$

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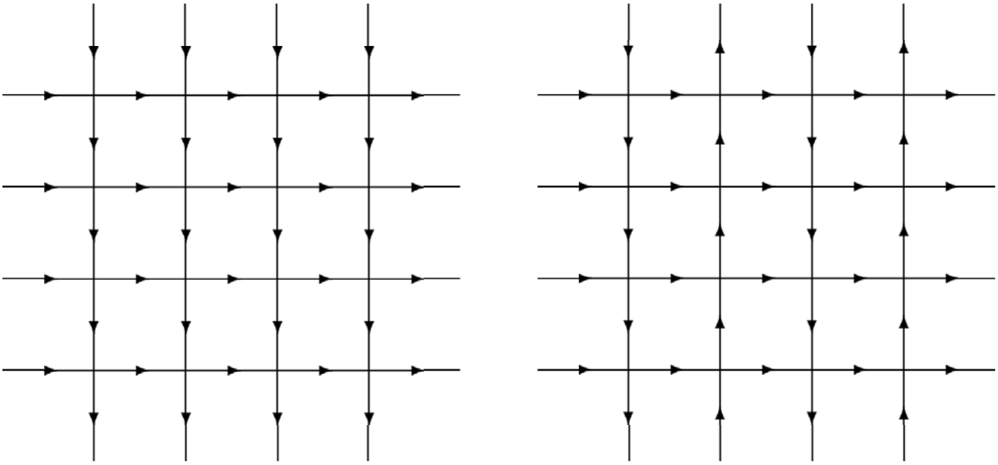

Fig. 1. Universal coverings of the standard complexes of presentations G_1 (left) and G_{-1} (right); vertical/horizontal edge are labelled by a and b, respectively; each small square is filled with a 2-cell.

and consider the amalgamated free products $H_\varepsilon = G_\varepsilon \underset{b=h}{*} H$ of G_ε and a group

$$H = \langle X \mid R \rangle \supseteq \langle h \rangle_\infty$$

(henceforth $\varepsilon = \pm 1$). Let K_ε be the standard complex of the (standard) presentation of H_ε:

$$H_\varepsilon = \left\langle \{a\} \sqcup X \;\middle|\; \{a^{\widehat{h}} a^{-\varepsilon}\} \sqcup R \right\rangle,$$

where $\widehat{h}$ is a word in the alphabet $X^{\pm 1}$ representing the element $h \in H$.

The Cayley graphs of G_ε are isomorphic surely (as abstract undirected graphs), the same is true for the universal coverings of the standard complexes of presentations of the groups G_ε (these covering complexes are planes partitioned on squares, Figure 1).

A slightly less trivial observation is that, for groups H_ε, the universal coverings are isomorphic too:

for any infinite-order element h of any group H, the universal coverings of complexes K_ε are isomorphic. $(*)$

In what follows, we explain this simple fact in details; the readers who regard this fact as obvious, can skip to Observation (**).

It suffices to show that some coverings $\widehat{K}_\varepsilon \to K_\varepsilon$ have isomorphic $\widehat{K}_\varepsilon$; we prefer to take the coverings corresponding to the normal closure $\langle\langle a \rangle\rangle$ of $a \in H_\varepsilon$. In explicit form, these complexes $\widehat{K}_\varepsilon$ are the following ones:

(i) the vertices are elements of H;

(ii) the edges with labels from X are drawn as in the Cayley graph of the group H: an edge with label $x \in X$ go from each vertex $h' \in H$ to the vertex $h'x \in H$;

(iii) in addition, to each vertex $h' \in H$, a directed loop (edge) $a_{h'}$ labelled by a is attached;

(iv) to each cycle whose label is a relator from R, an oriented 2-cell is attached;

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(v) to each cycle with label $a^{\widehat{h}}a^{-\varepsilon}$, an oriented 2-cell (a *special* cell) is attached; thus, going along the boundary of a special cell in the positive direction, we meet two edges labelled by a, namely, $a_{h'}$ and $a_{h'h}^{-\varepsilon}$, where, as usual, $a_{h'h}^{-1}$ means that the edge $a_{h'h}$ is traversed against its direction.

The isomorphism $\Phi : \widehat{K}_1 \to \widehat{K}_{-1}$ is the following:

(i) the vertices, edges with labels from X and nonspecial 2-cells (corresponding to relators from R) are mapped identically;

(ii) to define the mapping Φ on edges labelled by a and special 2-cells, we choose a set T of left-coset representatives of $\langle h\rangle$ in H, and put $\Phi(a_{th^k}) = a_{th^k}^{(-1)^k}$ for all $t \in T$ and $k \in \mathbb{Z}$ (i.e., in each coset, each second loop labelled by a is inverted); then the mapping of singular cells are defined naturally: a cell of $\widehat{K}_1$ with edges $a_{h'}$ and $a_{h'h}^{-1}$ on its boundary is mapped to the cell of $\widehat{K}_{-1}$ containing $a_{h'}$ and $a_{h'h}$ on its boundary.

The next simple observation is that:

if $h \in H$ belongs to all finite-index subgroups of H, and the complexes K_ε have a finite common covering, then the group H_1 contains a subgroup isomorphic to the Klein-bottle group BS$(1, -1)$. $(**)$

Indeed, in H_{-1}, the element $b = h$ is contained in all subgroups of finite index (because the intersection of each such subgroup with H is of finite index in H and, therefore, contains h). By the bottle lemma (applied to $G = H_{-1}$), we obtain that each finite-index subgroup contains a subgroup isomorphic to the Klein-bottle group. It remains to note that, if a finite complex $\widehat{K}$ covers K_1 and K_{-1}, then its fundamental group $\pi_1(\widehat{K})$ embeds into $\pi_1(K_\varepsilon) = H_\varepsilon$ as a finite-index subgroup.

Now, we take a particular group H, namely, let H be the Baumslag–Solitar group: $H = \mathrm{BS}(3, 5) = \langle c, d \mid c^{3d} = c^5\rangle$, and let $h \in H$ be the commutator: $h = [c^d, c]$. This element h is contained in any finite-index subgroup of H by the commutator lemma. According to (**), this means that, if complexes K_ε would have a common finite covering, then $H_1 = \langle a, c, d \mid [a, [c^d, c]] = 1, c^{3d} = c^5\rangle$ would contain the Klein-bottle group as a subgroup, which contradicts the no-bottle lemma. Therefore, there are no finite common coverings for complexes K_ε; while an infinite common covering exists according to $(*)$. Thus, the following fact is proven.

MAIN THEOREM. *The standard complexes of presentations*

$$H_\varepsilon = \left\langle a, c, d \;\middle|\; a^{[c^d,c]} = a^\varepsilon,\ c^{3d} = c^5 \right\rangle,$$

where $\varepsilon = \pm 1$, containing two 2-cells and one vertex, and three edges have a common covering, but have no finite common coverings.

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