J. Austral. Math. Soc. Ser. B 38(1996), 140–148

ON GENERALISED CONVEX MULTI-OBJECTIVE NONSMOOTH PROGRAMMING

S. K. MISHRA and R. N. MUKHERJEE[1]

(Received 15 February 1994; revised 21 July 1994)

Abstract

We extend the concept of V-pseudo-invexity and V-quasi-invexity of multi-objective programming to the case of nonsmooth multi-objective programming problems. The generalised subgradient Kuhn-Tucker conditions are shown to be sufficient for a weak minimum of a multi-objective programming problem under certain assumptions. Duality results are also obtained.

1. Introduction

In the differentiable case, Jeyakumar and Mond [3] defined a vector invexity that avoids the major difficulty of verifying that the inequality holds for the same function $\eta(\cdot, \cdot)$ for invex functions. Jeyakumar and Mond [3] established sufficient optimality criteria under V-pseudo-invexity and V-quasi-invexity and obtained duality results under these assumptions. This relaxation allows us to treat nonlinear fractional programming problems also. Egudo and Hanson [2] used the concept of Zhao [4] to generalise the concept of V-invexity of Jeyakumar and Mond [3] to the nonsmooth case by replacing the gradients with the gradients of Clarke [1].

In this paper we extend the concept of V-pseudo-invexity and V-quasi-invexity of Jeyakumar and Mond [3] to the nonsmooth case. Further sufficient optimality conditions and duality results have been derived for such nonsmooth multi-objective programming.

[1]Dept of Applied Math, Inst. of Technology, Banaras Hindu University, Varanasi 221005, India.

2. Preliminaries

Egudo and Hanson [2] considered the nonlinear multi-objective programming problem:

$$\begin{aligned} &\text{Minimise} \quad (f_i(x); i = 1, 2, \dots, p) &&\text{(P)}\\ &\text{subject to} \quad g_j(x) \le 0,\ j = 1, 2, \dots, m \end{aligned}$$

where $f_i : R^n \to R$, $i = 1, 2, \dots, p$ and $g_j : R^n \to R$, $j = 1, 2, \dots, m$ are locally Lipschitz functions.

The generalised directional derivative of a Lipschitz function f at x in the direction d denoted by $f^0(x; d)$ (see, for example, Clarke [1]) is :

$$f^0(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1} (f(y + td) - f(y)).$$

The Clarke generalised subgradient of f at x is denoted by

$$\partial f(x) = \left\{\xi : f^0(x; d) \ge \xi^T d, \forall d \in R^n\right\}.$$

Egudo and Hanson [2] defined invexity for locally Lipschitz functions as follows. A locally Lipschitz function $f(x)$ is invex on $X_0 \subset R^n$ if for $x, u \in X_0$ there exists a function $\eta(x, u) : X_0 \times X_0 \to R$ such that $f(x) - f(u) \ge \xi^T \eta(x, u), \forall \xi \in \partial f(u)$.

The following example is from [2].

$$f(x) = \begin{cases} 20 - x & \text{if} \quad x \le -15 \\ 5 - 2x & \text{if} \quad -15 \le x \le 0 \\ 5 + 2x & \text{if} \quad 0 \le x \le 15 \\ 20 + x & \text{if} \quad x \ge 15. \end{cases}$$

The function $f(x)$ is regular in the sense of Clarke [1] in that $f^0(x; d) = f'(x; d)$, where $f'(x; d)$ is the directional derivative

$$f'(x; d) = \lim_{t \downarrow 0} t^{-1} (f(x + td) - f(x)).$$

It was shown in [2] that $f(x)$ is invex.

A locally Lipschitz $f(x)$ is pseudo-invex on $X_0 \subset R^n$ if for $x, u \in X_0$ there exists a function $\eta(x, u) : X_0 \times X_0 \to R$ such that $\xi^T \eta(x, u) \ge 0 \Rightarrow f(x) \ge f(u)$, $\forall \xi \in \partial f(u)$.

A locally Lipschitz $f(x)$ is quasi-invex on $X_0 \subset R^n$ if for $x, u \in X_0$ there exists a function $\eta(x, u) : X_0 \times X_0 \to R$ such that $f(x) \le f(u) \Rightarrow \xi^T \eta(x, u) \le 0$, $\forall \xi \in \partial f(u)$.

It is clear from the definitions that every locally Lipschitz invex function is locally Lipschitz pseudo-invex and locally Lipschitz quasi-invex. Examples can be constructed easily.

3. Generalised invex vector functions

In the differentiable case Jeyakumar and Mond [3] defined vector invexity thus: (P) is said to be V-invex if there exist $\eta : X_0 \times X_0 \to R^n$ and $\alpha_i, \beta_j : X_0 \times X_0 \to R^+\backslash\{0\}$ such that

$$f_i(x) - f_i(u) - \alpha_i(x,u)\nabla f_i(u)\eta(x,u) \geq 0,$$
$$g_j(x) - g_j(u) - \beta_j(x,u)\nabla g_j(u)\eta(x,u) \geq 0.$$

Jeyakumar and Mond [3] further extended V-invexity to V-pseudo-invexity and V-quasi-invexity.

Using the results of Zhao [4], Egudo and Hanson [2] generalised the V-invexity concept of Jeyakumar and Mond [3] to the nonsmooth case by replacing the gradients ∇f_i and ∇g_j with the generalised gradients of Clarke [1]. Hence (P) is said to be V-invex if there exist $\eta : X_0 \times X_0 \to R^n$ and $\alpha_i, \beta_j : X_0 \times X_0 \to R^+\backslash\{0\}$ such that

$$f_i(x) - f_i(u) - \alpha_i(x,u)\xi_i\eta(x,u) \geq 0, \quad \forall \xi_i \in \partial f_i(u),$$
$$g_j(x) - g_j(u) - \beta_j(x,u)\zeta_j\eta(x,u) \geq 0, \quad \forall \zeta_j \in \partial g_j(u).$$

The following example is a V-invex nonsmooth multi-objective programming problem. Consider the multi-objective problem

$$V\text{-minimise} \quad \left(\left| \frac{2x_1 - x_2}{x_1 + x_2} \right|, \frac{x_1 + 2x_2}{x_1 + x_2} \right)$$

subject to $x_1 - x_2 \leq 0$, $1 - x_1 \leq 0$, $1 - x_2 \leq 0$, $\alpha_i(x,u) = 1$ for $i = 1,2$, $\beta_j(x,u) = (x_1 + x_2)/3$ for $j = 1,2$ and

$$\eta_i(x,u) = \left(\frac{3(x_1 - 1)}{x_1 + x_2}, \frac{2(x_2 - 2)}{x_1 + x_2} \right)^{\mathsf{T}}.$$

As we can see the generalised directional derivative of $f_1(x) = \left|\frac{2x_1 - x_2}{x_1 + x_2}\right|$ is

$$\begin{aligned} f^0(x;d) &= \limsup_{\substack{y_1 \to x_1 \\ t \downarrow 0}} t^{-1} \left[\left| \frac{2(y_1 + td) - x_2}{y_1 + td + x_2} \right| - \left| \frac{2y_1 - x_2}{y_1 + x_2} \right| \right] \\ &= \limsup_{\substack{y_1 \to x_1 \\ t \downarrow 0}} t^{-1} \left[\frac{3tdx_2}{(y_1 + x_2 + td)(y_1 + x_2)} \right] \quad \left(\text{if } \frac{2x_1 - x_2}{x_1 + x_2} \geq 0 \right) \\ &= \frac{3dx_2}{(x_1 + x_2)^2}. \end{aligned}$$

https://doi.org/10.1017/S0334270000000515 Published online by Cambridge University Press

If we take $x_1 = 1$ and $x_2 = 2$ (that is, for an efficient solution $(1, 2)$) then $f^0(x; d) = 2d/3$.

If $y_2 \to x_2$, then $f^0(x; d) = -d/3$. Thus $(2d/3, -d/3) \in \partial f_1(u)$. It is easy to see that $(-2/9, 1/9) \in \partial f_2(u)$. At these particular points we can easily see that the above program is V-invex for the nonsmooth case.

We now extend V-invexity as in Egudo and Hanson [2] to V-pseudo-invexity and V-quasi-invexity.

A vector function $f : X_0 \to R^p$ is said to be V-pseudo-invex if there exist functions $\eta : X_0 \times X_0 \to R^p$ and $\alpha_i : X_0 \times X_0 \to R_+\backslash\{0\}$ such that for each $x, u \in X_0$,

$$\sum_{i=1}^{p} \xi_i \eta(x, u) \geq 0 \Rightarrow \sum_{i=1}^{p} \alpha_i(x, u) f_i(x) \geq \sum_{i=1}^{p} \alpha_i(x, u) f(u), \quad \forall \xi_i \in \partial f_i(u).$$

The vector function f is said to be V-quasi-invex if there exist functions $\eta : X_0 \times X_0 \to R^p$ and $\beta_i : X_0 \times X_0 \to R_+\backslash\{0\}$ such that for each $x, u \in X_0$,

$$\sum_{i=1}^{p} \beta_i(x, u) f_i(x) \leq \sum_{i=1}^{p} \beta_i(x, u) f_i(u)$$
$$\Rightarrow \sum_{i=1}^{p} \zeta_i \eta(x, u) \leq 0, \quad \forall \zeta_i \in \partial f_i(u).$$

It is apparent from the definitions that every V-invex function of Egudo and Hanson [2] is V-pseudo-invex and V-quasi-invex as defined above.

Recall from Jeyakumar and Mond [3] that $u \in X_0$ is said to be a (global) weak minimum of a vector function $f : X_0 \to R^p$ if there exists no $x \in X^0$ for which $f_i(x) < f_i(u), i = 1, \ldots, p$.

4. Sufficiency and duality

In this section we show that the subgradient Kuhn-Tucker conditions are sufficient for a weak minimum in (P) when generalised V-invexity is present.

THEOREM 4.1. *Let (u, τ, λ) satisfy the Kuhn-Tucker conditions that*

$$0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u), \quad \lambda_j g_j(u) = 0, \quad j = 1, 2, \ldots, m,$$
$$\tau_i \geq 0, \quad \tau^T e > 0, \quad y_i \geq 0.$$

If $(\tau_1 f_1, \ldots, \tau_p f_p)$ is V-pseudo-invex and $(\lambda_1 g_1, \ldots, \lambda_m g_m)$ is V-quasi-invex in nonsmooth sense, and u is feasible in (P), then u is a global weak minimum of (P).

PROOF. Since $0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u)$, there exist $\xi_i \in \partial f_i(u)$ and $\zeta_j \in \partial g_j(u)$ such that

$$\sum_{i=1}^{p} \tau_i \xi_i + \sum_{j=1}^{m} \lambda_j \zeta_j = 0.$$

Suppose that u is not a global weak minimum point. Then, following the lines of proof of Theorem 3.1 of Jeyakumar and Mond [3], the V-pseudo-invexity conditions yield $\sum_{i=1}^{p} \tau_i \xi_i \eta(x_0, u) < 0$. Thus, we have $\sum_{j=1}^{m} \lambda_j \zeta_j \eta(x_0, u) > 0$. Then, V-quasi-invexity yields $\sum_{j=1}^{m} \beta_j(x_0, u) \lambda_j g_j(x_0) > \sum_{j=1}^{m} \beta_j(x_0, u) \lambda_j g_j(u)$. Since x_0 is feasible for (P), that is, $\lambda_j g_j(x_0) \le 0$, and $\lambda_j g_j(u) = 0$, $j = 1, 2, \dots, \lambda_j > 0$, $\beta_j > 0$. This contradicts the previous inequality.

For the problem (P), consider a corresponding Mond-Weir dual problem.

$$\text{Maximise} \quad (f_i(u) : i = 1, 2, \dots, p) \tag{D}$$

$$\text{subject to} \quad 0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u), \quad \lambda_j g_j(u) \ge 0, \quad j = 1, \dots, m.$$

$$\tau_i \ge 0, \quad \sum_{i=1}^{p} \tau_i = 1, \quad \lambda_j \ge 0.$$

THEOREM 4.2 (Weak Duality). *Let X be feasible in* (P) *and let* (u, τ, λ) *be feasible in* (D). *If* $(\tau_1 f_1, \dots, \tau_p f_p)$ *is V-pseudo-invex and* $(\lambda_1 g_1, \dots, \lambda_m g_m)$ *is V-quasi-invex as in Theorem* 4.1, *then* $(f_1(x), \dots, f_p(x))^{\mathrm{T}} - (f_1(u), \dots, f_p(u))^{\mathrm{T}} \notin -\operatorname{int} R_+^p$.

PROOF. From the feasibility conditions, and $\beta_j(x, u) > 0$, we have

$$\sum_{j=1}^{m} \beta_j(x, u) \lambda_j g_j(x) \le \sum_{j=1}^{m} \beta_j(x, u) \lambda_j g_j(u).$$

Then, by V-quasi-invexity, we have $\sum_{j=1}^{m} \zeta_j \eta(x, n) \le 0$, $\forall \zeta_j \in \partial g_j(u)$. Since

$$0 \in \sum_{i=1}^{p} \tau_i \partial f_i(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u),$$

there exist $\xi_i \in \partial f_i(u)$ and $\zeta_j \in \partial g_j(u)$ such that $\sum_{i=1}^{p} \tau_i \xi_i + \sum_{j=1}^{m} \lambda_j \zeta_j(u) = 0$. This implies that

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x, n) + \sum_{j=1}^{m} \lambda_j \zeta_j \eta(x, u) = 0.,$$

https://doi.org/10.1017/S0334270000000515 Published online by Cambridge University Press

Thus,

$$\sum_{i=1}^{p} \tau_i \xi_i \eta(x, u) \geq 0, \quad \forall \xi_i \in \partial f_i(u).$$

The conclusion now follows from the V-pseudo-invexity condition since $\tau e = 1$ and $\alpha(x, u) > 0$.

THEOREM 4.3 (Strong Duality). *Let x^0 be a weak minimum of* (P) *at which a constraint qualification is satisfied. Then there exist $\tau^0 \in R^p$, $\lambda^0 \in R^m$ such that (x^0, τ^0, λ^0) is feasible in* (D). *If weak duality holds between* (P) *and* (D), *then (x^0, τ^0, λ^0) is a weak minimum of* (D).

PROOF. From Kuhn-Tucker necessary conditions (see, for example, Theorem 6.1.3 of Clarke [1]), there exist $\tau \in R^p$, $\lambda \in R^m$ such that

$$0 \in \sum_{i=1}^{p} \tau_i \partial f_i(x^0) + \sum_{j=1}^{m} \lambda_j \partial g_j(x^0),$$

$\tau_i \geq 0$, $\tau \neq 0$, $\lambda_j \geq 0$, $\lambda_j g_j(x^0) = 0$, $j = 1, 2, \ldots, m$. Now since $\tau_i \geq 0$, $\tau \neq 0$ we can scale the τ_i's and λ_j's as

$$\tau_i^0 = \tau_i \Big/ \left(\sum_{i=1}^{p} \tau_i\right) \quad \text{and} \quad \lambda_j^0 = \lambda_j \Big/ \left(\sum_{i=1}^{p} \tau_i\right).$$

Now we have (x^0, τ^0, λ^0) that is feasible in (D).

If (x^0, τ^0, λ^0) is not a weak maximum of (D), then there exists a feasible (u, τ, λ) for (D) such that

$$\left(f_1(u), \ldots, f_p(u)\right)^{\mathrm{T}} - \left(f_1(x^0), \ldots, f_p(x^0)\right)^{\mathrm{T}} \in \operatorname{int} R_+^p.$$

Since x^0 is feasible in (P), this contradicts weak duality (Theorem 4.2).

5. Nonsmooth multi-objective fractional programming

In this section we apply the results of the previous section to study nonsmooth fractional multi-objective problems.

In the differentiable case, Jeyakumar and Mond [3] considered the fractional programming problem,

$$V\text{-minimize} \quad \left(\frac{p_1(x)}{q_1(x)}, \ldots, \frac{p_r(x)}{q_r(x)}\right) \tag{FI}$$

subject to $x \in X_0$, $g(x) \leq 0$, where $p_i : X_0 \to R$, $q_i : X_0 \to R$ and $g : X_0 \to R^m$. It is assumed that $p_i(x) \geq 0$, for each x on the feasible set $\Delta = \{x \in X_0 : g(x) \leq 0\}$, $q_i(x) > 0$, for each $x \in \Delta$. The problem (FI) is said to be a V-invex fractional problem if the functions p, q and g satisfy

$$x, u \in \Delta \Rightarrow \begin{cases} p_i(x) - p_i(u) & \geq \gamma_i(x, u) p_i'(u) \eta(x, u) \\ q_i(x) - q_i(u) & \geq \gamma_i(x, u) q_j'(u) \eta(x, u) \\ g_j(x) - g_j(u) & \geq \beta_j(x, u) g_j'(u) \eta(x, u) \end{cases}$$

with $\eta : X_0 \times X_0 \to R^n$, $\gamma_i, \beta_j : X_0 \times X_0 \to R_+ \backslash \{0\}$.

Following Egudo and Hanson [2] we can generalise (FI) to the nonsmooth case by replacing p_i', q_i' and g_j' with the generalised gradients of Clarke. Hence (FI) is said to be V-invex nonsmooth fractional if there exists $\eta : X_0 \times X_0 \to R^n$ and $\gamma_i, \beta_j : X_0 \times X_0 \to R_+ \backslash \{0\}$ such that for all $x, u \in \Delta$

$$\begin{aligned} p_i(x) - p_i(u) &\geq \gamma_i(x, u) \xi_i \eta(x, u), & \forall \xi_i \in \partial p_i(u), \\ q_i(x) - q_i(u) &\leq \gamma_i(x, u) \zeta_i \eta(x, u), & \forall \zeta_i \in \partial q_i(u), \\ g_j(x) - g_j(u) &\geq \beta_j(x, u) \mu_j \eta(x, u), & \forall \mu_j \in \partial g_j(u). \end{aligned} \tag{FI$'$}$$

We need the following proposition from Clarke [1] in order to prove the main Theorem of this section.

PROPOSITION 5.1. (Clarke [1]). *Let f_1, f_2 be Lipschitz near x, and suppose $f_2(x) \neq 0$. Then f_1/f_2 is Lipschitz near x, and*

$$\partial \left(\frac{f_1}{f_2} \right)(x) \subset \frac{f_2(x) \partial f_1(x) - f_1(x) \partial f_2(x)}{(f_2(x))^2}.$$

If in addition $f_1(x) \geq 0$, $f_2(x) > 0$ and if f_1 and $-f_2$ are regular at x, then equality holds and f_1/f_2 is regular at x.

In the next theorem, we assume that p_1 and p_2 are regular.

THEOREM 5.1. *Consider the problem* (FI). *Let $u \in \Delta$. Assume that there exist (τ, λ) such that $\tau \geq 0$, $\tau \neq 0$, $\lambda \geq 0$,*

$$0 \in \sum_{i=1}^{r} \tau_i \partial \left(\frac{p_i}{q_i} \right)(u) + \sum_{j=1}^{m} \lambda_i \partial g_j(u)$$

and $\lambda_j g_j(u) = 0$, $j = 1, 2, \ldots, m$. Then u is a global weak minimum for (FI)$'$.

PROOF. The proof follows the lines of the proof of Theorem 4.1 of Jeyakumar and Mond [3] with appropriate changes in $(p_i/q_i)'$. Proposition 5.1 plays a crucial role in this proof.

For a V-invex nonsmooth multi-objective fractional programming problem (FI)$'$, the weak and strong duality properties hold with the following dual problem:

$$\begin{aligned} V\text{-maximise} \quad & \left(\frac{p_1(u)}{q_1(u)}, \dots, \frac{p_r(u)}{q_r(u)}\right) \\ \text{subject to} \quad & 0 \in \sum_{i=1}^{r} \tau_i \partial \left(\frac{p_i}{q_i}\right)(u) + \sum_{j=1}^{m} \lambda_j \partial g_j(u) \\ & \lambda_j g_j \geq 0, \quad 1, 2, \dots, m \\ & \lambda_j \geq 0, \quad \tau \geq 0, \quad \tau e = 1. \end{aligned}$$

6. Conclusion

The Kuhn-Tucker subgradient conditions are shown to be sufficient for a weak minimum of a multi-objective programming problem when generalised invexity (V-pseudo-invexity/V-quasi-invexity) is present. Weak and strong duality theorems have been established. We use the results of Section 4 to extend Egudo and Hanson [2] to the fractional case in Section 5. If $p = 1$, then our result extends the results on invexity used in Zhao [4] for the case of nonsmooth programming to pseudo-invexity and quasi-invexity.

Acknowledgement

The authors are indebted to Bruce Craven for his useful comments which led to the present improved version of the paper. The authors also wish to thank V. Jeyakumar and M. A. Hanson for sending their recent reprints and some of their unpublished works.

References

[1] F. H. Clarke, *Optimization and nonsmooth analysis* (Wiley-Interscience, New York, 1983).

[2] R. R. Egudo and M. A. Hanson, "On sufficiency of Kuhn-Tucker conditions in nonsmooth multi-objective programming", *FSU Technical Report M-888*, (1993).

[3] V. Jeyakumar and B. Mond, "On generalised convex mathematical programming", *J. Austral. Math. Soc. Ser. B* **34** (1992) 43–53.

[4] F. Zhao, "On sufficiency of the Kuhn-Tucker conditions in non-differentiable programming", *Bull. Austral. Math. Soc.* **46** (1992) 385–389.