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Theories of Polar Motion from Tisserand to Poincaré (1890 – 1910)

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Abstract. The discovery by Seth C. Chandler (1891) that the motion of the pole (the reality of which had been established by K.F. Küstner and by the simultaneous latitude observations at Honolulu and Berlin by German astronomers) resulted from two components *i.e.* a free circular motion with a period of 427 days and a forced elliptical motion with a period of 365.25 days, raised considerable interest in the scientific community of astronomers and geophysicists.

The celebrated *Mécanique Céleste* of Tisserand (1890) had been published just one year before at a time when doubts still persisted and arguments could be presented in favor of the fixed pole. Starting with Tisserand's arguments, we describe in this paper the impact of the successive contributions by A. Greenhill, S. Newcomb, Th. Sloudsky, S. Hough, G. Herglotz, A. Love, J. Larmor and H. Poincaré to the solution of the problems raised by the Chandler period.

The lines of reasoning taken by these eminent scientists were rigorously correct so that, after about one hundred years, contemporary researchers, who benefit from a far better knowledge of the inner structure of the Earth and are able to take advantage of modern computing power, do not contradict any of their conclusions and instead refine them with an accuracy which was not imaginable one century ago.

1. The rigid Earth rotation

F. Tisserand (1845–1896) and H. Poincaré (1854–1912) were both members of the Commission pour l'étude de la variabilité des latitudes as Poincaré succeeded Tisserand after his death. Both are eminent representatives of the famous French school of celestial mechanics having left monumental works and having provided major contributions to many problems in astronomy. Tisserand wrote the second volume of his celebrated *Mécanique Céleste*, issued in 1890, when the existence of the polar motion was only suspected but not proved.[1]

When he presented this volume at the 1890 Freiburg meeting of the Commission Permanente of the International Geodetic Association, he declared (page 77): "Le mouvement de rotation de la Terre a été étudié assez complétement, notamment ce qui concerne les petits déplacements du pôle á la surface. Quelques

[1] *"les portions conservées ainsi dans p et q ne pourrant pas être contrôlées par l'observation, qui donne á peine quelques présomptions de leur existence".* (Mécanique Céleste II, p. 395).

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uns des petits termes signalés pourront jouer un rôle dans la détermination de la nutation initiale. L'ouvrage se termine par un exposé des recherches récentes sur la variabilité possible des latitudes, en vertu des actions géologiques et météorologiques."

At the same meeting, when W. Förster (1832–1921) Director of the Berlin Observatory, pushed strongly for the organization of simultaneous latitude determinations at Honolulu and Berlin, Tisserand became sceptical and temporising, asking to delay the decision and to first use methods of observation other than the Horrebow - Talcott one in European observatories.

1.1. Astronomical Forced Nutations superposed on the Chandlerian free mode

It appears rather curious that in chapter XXVI entitled "Fixité des Pôles á la Surface de la Terre" of his *Mécanique Céleste*, Tisserand develops the very small terms of a nearly diurnal polar motion due to lunisolar torque expressed in centimeters of displacement at the Earth's surface as an argument that the polar motion can only be very small.

As a matter of fact these terms had been calculated and published in 1881, before Tisserand by Th. von Oppolzer (1841–1886) with respective amplitudes of $0''.009$, $0''.006$ and $0''.003$. They are nothing else than the radii of the circular cones rolling "á la Poinsot", without slipping, on the large space cones of precession and of fortnightly and semiannual nutations.

These terms can be obtained directly and more easily from the tidal potential development, corresponding to the tidal tesseral diurnal waves K_1, O_1 and P_1 (Melchior 1983). They have practically the same amplitudes as the corresponding tidal deviations of the vertical at the poles.

Due recognition was given to von Oppolzer by F. Ross (1912) and by the ILS Central Bureau who designated them as the "Oppolzer terms" and introduced them as corrections in the reduction of latitude observations. In a short list of six references, Tisserand mentions the 1886 von Oppolzer treatise but does not refer to him in the developments of his Chapter XXVI.

1.2. Rotation of a deformable Earth

The chapters written by Tisserand refer to a totally rigid Earth. The two last chapters (XXIX and XXX) of his treatise were written by R. Radau. They are more descriptive and relate the precursory research by W. Hopkins (1793–1866) about the role of a liquid core enclosed within a smooth ellipsoidal boundary inside a rigid crust to precession and nutation (1839, see §3.). Radau also reminds the reader of the objections of Lord Kelvin (1824–1907) against an internal structure of the Earth consisting of a massive liquid core inside a thin crust (1863), an idea deriving from some geological considerations related to the phenomena of volcanism.

The most interesting part of chapter XXIV is the development of simple calculations of possible changes of the axes of inertia and rotation due to displacements or additions of large masses at the Earth's surface. It refers to the original paper (1889) by G. V. Schiaparelli (1835–1910) who is probably the first to have used the wording "Geodynamics" to describe these kinds of phenomena.

Finally Radau derives the equations of the diurnal and semidiurnal nutations and, in the last chapter (XXX) develops the Liouville equations for the rotation of a body of variable form and the related researches of G. H. Darwin and Lord Kelvin. These developments have been largely used until now.

2. *Chandlerian Polar Motion and Earth's Elasticity*

The remarkable discovery of S. C. Chandler, in 1891, gave immediate rise to several important papers addressing the problem of the Earth's elasticity (Newcomb, Hough, Herglotz, Love, Larmor) and the problems raised by the presence of a liquid core (Sloudsky, Hough, Poincaré).

2.1. Newcomb interpretation

The *Astronomical Journal* issue 249 by which Chandler (1846–1913) announced his discovery is dated November 23, 1891. The reaction of Newcomb was quick as he gave his interpretation of the lengthening already in the December 23, 1891 issue 251 of the same journal. S. Newcomb (1835–1909) was thus the first to identify the Chandlerian circular component of the polar motion with the 304.4 sidereal day free Eulerian mode (1891, 1892) whose period is lengthened to 427 days because of the elastic deformation of the Earth induced by the centrifugal force tesseral disturbing potential:

$$W = -\frac{1}{2}\omega r^2 (p\cos\lambda + q\sin\lambda)\sin 2\theta. \tag{1}$$

(θ is the colatitude, λ the longitude, (p, q, r) the projections of the vector rotation ω on the axes fixed to the Earth's mantle).

He proposed a very simple and elegant geometric interpretation. Let (see Fig. 1) I_0 be the initial position of the pole of inertia, R_1 the instantaneous pole of rotation at time t. The centrifugal force potential (1) creates internal stresses which deform the Earth so that the principal axes of inertia are changed and the instantaneous pole of inertia moves to I_1. The Eulerian motion refers to I_1 while the astronomically observed motion is referred to I_0, the mean pole of inertia. It clearly appears that the Chandlerian angular speed χ around I_0 is smaller than the Eulerian angular speed α around I_1 which qualitatively explains the lengthening of the period.

2.2. Hough analysis

In his 1896 paper about the rotation of an elastic spheroid, S. S. Hough (1870–1923) re-examined Newcomb's interpretation of the solution of the problem in an analytical form.

The model is a homogeneous oceanless spheroid of revolution (moments of inertia A, A, C), composed of isotropic, incompressible, gravitating material whose figure conforms to that required for hydrostatic equilibrium, so that when the body is undisturbed one may suppose it free from strain in its interior.

Putting

$$\frac{\mu}{\rho} = n, \qquad \psi = V + \frac{1}{2}\omega^2(x^2 + y^2) - \frac{p}{\rho}, \tag{2}$$

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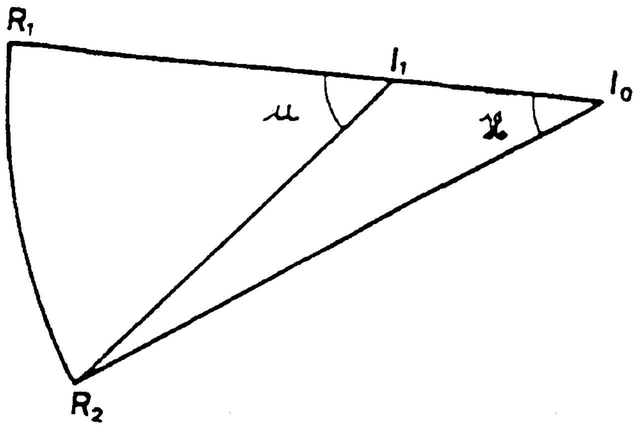

Figure 1. Newcomb explanation for the Chandler period.

where ρ, μ are the density and rigidity of the Earth's material, ω the angular velocity of rotation about the axis z, p the hydrostatic pressure at x, y, z and V the gravitational potential, Hough obtains the well known equation for ψ:

$$\left[\left[\nabla^2\left(n\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\right]^2 - 4\omega^2\frac{\partial^4}{\partial t^2 \partial z^2}\right]\psi = 0, \tag{3}$$

which reduces to the Poincaré equation when n = 0. If the motion relative to the moving axes consists of a simple harmonic vibration of period $2\pi/\lambda$, the Hough equation becomes

$$\left[\nabla^2(n\nabla^2 + \lambda^2)^2 - 4\omega^2\lambda^2\frac{\partial^2}{\partial z^2}\right]\psi = 0. \tag{4}$$

To solve the problem one has to take care of the boundary conditions which imply that the components of surface-tractions vanish at all points on the displaced surface. Their expressions are not too difficult to obtain but rather cumbersome (Hough's equations (16)). In the case of a spheroid of small flattening ε, one can neglect the square of ε and take as the equation of the free surface

$$r = a\{1 + \varepsilon T_2\} \qquad (T_2 \text{ is a spherical harmonic function of order 2}). \tag{5}$$

It is well known that a homogeneous spheroid, rotating in a sidereal day, has a geometrical flattening

$$\varepsilon = \frac{5}{4}\frac{\omega^2 a^3}{GM} = \frac{1}{232} \tag{6}$$

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so that

$$\frac{\lambda}{\omega} = \frac{C - A}{A} = \varepsilon, \tag{7}$$

and λ is a small quantity of the order ω^3.

The determination of the elastic distortions follows from a rather long but simple procedure adopted by Kelvin as well as by Love to obtain the flattening which would be induced in a sphere of radius a by a centrifugal force when distortion is resisted by elasticity alone:

$$\varepsilon' = \frac{5}{2}\frac{\omega^2 a^2}{19n} = \frac{1}{522} \qquad (\varepsilon'/\varepsilon = 0.444). \tag{8}$$

$(2n^2 + 4n + 3 = 19$ for $n = 2)$ if the rigidity is taken as the rigidity of steel:

$$\mu = 8.19 \times 10^{11} dyne/cm^2 = 8.19 \times 10^{10} Pa. \tag{9}$$

Taking the components of the angular momentum[2]

$$h_1 = \int\int\int[(\dot{w}y - \dot{v}z) - \omega xz]dm - \rho\int\int \xi\omega xz dS,$$

or

$$h_1 = \imath\lambda\int\int\int[(wy - vz)dm - \rho\omega\int\int \xi xz dS, \tag{10}$$

and

$$\dot{h}_1 = -\lambda^2\int\int\int[(wy - vz)dm - \rho\omega\imath\lambda\int\int \xi xz dS. \tag{11}$$

Introducing these developments into the equations of angular momentum:

$$\dot{h}_1 - h_2\omega = 0 \qquad \dot{h}_2 + h_1\omega = 0, \tag{12}$$

Hough has obtained:

$$\lambda = \frac{\omega\varepsilon}{1 + \varepsilon'/\varepsilon} = \frac{1}{335}\omega. \tag{13}$$

Thus, for an homogeneous elastic spheroid, the free period lengthens from 232 to 335 days. Hough, recognizing that the problem of the heterogeneous Earth is much more complicated, is, at his epoch (1896), only in a position to make speculations. He therefore proposes to decrease the flattening 1/232 to 1/304.4 the value of the constant of precession and, by a simple proportionality obtains:

[2](u,v,w) being the components of the elastic displacement of a particle, ξ the displacement along the normal to the surface.

$$\lambda = \frac{304.4}{232} \times 335 = 440 \text{ days}, \tag{14}$$

a period in excess of the Chandler period which let him conclude that the effective rigidity of the Earth is slightly greater than that of steel.

2.3. Love and Larmor Elastic Yielding Theory

To characterise the spherical elasticity properties, A.E.H. Love (1863–1940) introduced (1909) two dimensionless parameters describing the deformations of a spherical Earth body. These parameters, called "Love numbers" are defined as follows: h specifies the amount by which the surface of the Earth yields to tidal forces, k specifies the amount by which the potential of the Earth is altered through the rearrangement of the matter within it, due to the tidal yielding.

Lord Kelvin (1863) had previously related the same parameters to the mean density ρ and mean rigidity μ of an incompressible homogeneous elastic sphere of radius a and gravity g at its surface, obtaining the often reported relations

$$h = \frac{5}{2}\left(1 + \frac{19\mu}{2g\rho a}\right)^{-1}, \qquad k = \frac{3}{5}h, \tag{15}$$

which, for $h = 0.60$ and $g\rho a = 3.45 \times 10^{11}$ Pa would give $\mu = 1.15 \times 10^{11}$ Pa, is quite a bit higher than the rigidity of steel.[3]

For a spherically symmetrically stratified heterogeneous Earth whose rheological parameters ρ (density), λ (compressibility), μ (rigidity) depend only upon the radius vector r, the Love numbers are related to these parameters through a system of six linear differential equations which can also be transformed into a differential equation of the sixth order in h as given by G. Herglotz (1881–1953), (1905).

As the lengthening of the polar free motion period from 304.4 to 427 sidereal days is due to the perturbing tesseral potential W_2 (Eqn. 1.) one may expect a relation between the ratio of the Chandler to the Euler periods with the Love number k.

We follow here the short demonstration proposed (1909) by Sir Joseph Larmor (1857–1942). The perturbing potential can be written

$$W_2 = -\frac{1}{2}\omega^2 r^2 \sin^2\theta = -\frac{1}{3}\omega^2 r^2 (1 - P_2(\theta)). \tag{16}$$

$P_2(\theta) = \frac{3\cos^2\theta - 1}{2}$ is the second order zonal harmonic spherical function (θ: colatitude). The resulting deformation and rearrangement of the matter changes $(C - A)$ to $(C' - A')$ and gives rise to the additional potential

$$W_2' = k\frac{1}{3}\omega^2 r^2 P_2 = \frac{G}{r^3}\Big\{(C - A) - (C' - A')\Big\}P_2 \tag{17}$$

[3]The rigidity of a steel bar results from molecular cohesion while the rigidity μ of the planet Earth results from the enormous pressure $g\rho a$ prevailing in its interior: Kelvin called it "earth's tidal effective rigidity."

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as the gravitational potential of the Earth is

$$V_2 = G\left(\frac{M}{r} - \frac{C-A}{r^3}P_2 + \ldots\right) \tag{18}$$

with the radius vector

$$r = a(1+\varepsilon\sin^2\theta) = a\left(1+\frac{2}{3}\varepsilon\right)\left(1-\frac{2}{3}\varepsilon P_2\right). \tag{19}$$

The coefficient of P_2 in $V_2 - W_2$ at the surface $r = a$ is to be made equal to zero:

$$\frac{2}{3}ga\left(\varepsilon - \frac{\omega^2 a}{2g}\right) - \frac{G}{a^3}(C-A) = 0 \qquad \left(g = \frac{GM}{a^2}\right), \tag{20}$$

so that, combining (17), (19) and (20) one obtains

$$\frac{\tau_0}{\tau} = \frac{C-A}{C'-A'} = 1 - \frac{1}{3}k\omega^2 a^5 G^{-1}(C-A)^{-1} = 1 - \frac{k\omega^2 a/2g}{\varepsilon - \omega^2 a/2g} \tag{21}$$

or

$$1 - \frac{\tau_0}{\tau} = k\frac{\omega^2 a/2g}{\varepsilon - \omega^2 a/2g}, \tag{22}$$

which is the formula obtained by Love in his own paper. Larmor points out that "like Clairaut's formula for gravity, this relation is independent of any hypothesis as to the Earth's internal structure, except such as is involved in the definition and value of k." However this approach neglects the effect of rotation and uses a spherical model. With $\tau_0 = 304.4$ sidereal days, $\tau = 427$ or 432 s.d., $\omega^2 a/2g = 0.001729$ and $\varepsilon = 0.003353$, Equation (22) gives $k = 0.270$ or 0.277.

The value $k \sim 0.30$ is obtained with the most recent theoretical models for the main sectorial lunar semidiurnal wave M_2 or the main tesseral lunar diurnal wave O_1. Introducing this value in Eqn (22) would give a Chandler period τ of 449 sidereal days. This results from the fact that the liquid core of the Earth is decoupled from the mantle yielding.

As a matter of fact, Herglotz had already obtained Equation (22) in 1905, after more complicated developments in the form

$$\tau_0/\tau = (\varepsilon - \omega^2 a/2g)/[\varepsilon - q[1 + \Delta_4/(1 - 3\eta a/5)\Delta_1]],^4 \tag{23}$$

where

$$q = (5\omega^2 a/4g)(1 + 19\mu/2g\rho a)^{-1} = (\omega^2 a/2g)h.$$

Using $\tau_0 = 232$ s.d. as Hough did, Herglotz found $\tau = 343$ s.d. while $\tau_0 = 304.4$ s.d. would have given him 450 sidereal days.

[4] Δ_1 and Δ_4 being determinants of rather complicate constructions. Here $\Delta_1 = 0.0001292$, $\Delta_4 = 0.00007587$ (see Herglotz 1905).

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3. The 19th Century Earth model: an inviscid incompressible homogeneous fluid contained inside an ellipsoidal rigid shell

In the last century, the temperature gradient measured in deep mines (but no more than 2 000 m depth) was extrapolated by geologists which led us to believe that the Earth's constitution could be a thick fluid interior enclosed in a thin rigid shell. Volcanism was another argument in favour of this hypothesis.

The precursory papers by Hopkins (1839), Greenhill (1879–80) and Lord Kelvin (1885) took on great importance as soon as the existence of a polar motion was established and its characteristics described by Chandler. These fundamental papers obviously inspired the theoretical developments made by Th. Sloudsky (1841–1897), (1895), Hough (1895) and Poincaré (1910).

As early as 1839, W. Hopkins — as reported by Kelvin (W.Thomson, 1863) "to whom is due the grand idea of thus learning the physical condition of the interior from phenomena of rotary motion presented by the surface" — published two papers entitled "On the Phenomena of Precession and Nutation, assuming the Fluidity of the Interior of the Earth."

Considering the Earth's density as uniform throughout, Hopkins introduced the pressure on the spheroidal inner surface (of ellipticity α) of the rigid shell due to the centrifugal force resulting from the rotation of the liquid core. Taking $q = a/a(\text{core})$, with $C(\text{shell}) = C(1 - q^{-5})$ to the first order of the ellipticity, he found a free retrograde nutation of period $T(1 - q^{-5})\alpha$, that is practically C(shell)/Cα $\approx$ 358 sidereal days (Hopkins, 1839, pages 410–411). A nearly diurnal retrograde free wobble of very small amplitude (1/358 of the nutation amplitude) is associated with this free nutation as a result of the Poinsot representation.

The Hopkins conclusions were that: 1) the precession and the lunar nutation will be the same whatever the thickness of the shell; 2) that "in addition to the above motions of precession and nutation, the pole of the earth would have a small circular motion, depending on the internal fluidity."

Later Lord Kelvin taking again and developing the Hopkins argument strongly objected to the thin crust model on the basis of the precession and nutation observed amplitudes but this was refuted by Delaunay and Newcomb. Kelvin then abandoned this argument (Thomson, 1863, revised 1890) and based his view upon the small amplitude of the solid Earth tidal deformation, concluding that the Earth's global rigidity was that of steel (see §2.3).

Let us now consider the behaviour of this model when the solid shell is rotating about an axis inclined on its principal axis of inertia. If the core was spherical and its constituent fluid nonviscous, the Euler period would be shortened instead of lengthened.

In that case indeed, the core and mantle are decoupled and the core does not participate in the rotation and precession of the mantle. Let I be the moment of inertia of this spherical core and $A^* = A - I = B - IC^* = C - I$ then

$$\tau = \frac{2\pi}{\omega}\frac{A^*}{C^* - A^*} = \frac{2\pi}{\omega}\frac{A - I}{C - A} \tag{24}$$

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a period shorter than the 304.4 sidereal days of the completely solid Earth. It should be decreased by 10% to become about 270 sidereal days.[5]

Today we well know that a fluid contained inside an ellipsoidal shell in rotation supports one particular inertial oscillation, the spin-over mode which consists of a rotation about an axis slightly inclined on the rotation axis of the shell. It is now called the Poincaré mode to remind us of his famous 1910 paper. It should have been called "Hopkins-Hough-Sloudsky mode" (see §3.2).

However, a hundred years ago the question was not so evident until two major papers were quite simultaneously published in 1895 by S.S. Hough in Cambridge and by Th. Sloudsky in Moscow.

3.1. The ellipsoidal boundary condition

The incompressibility approximation (solenoidal flow: $div\ \bar{\mathrm{v}} = 0$) may be justified when the speed is much less than the speed of sound. Let the internal boundary of the ellipsoidal shell be

$$F \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{25}$$

which is the interface between the incompressible liquid core and the Earth's rigid mantle. Suppose that the shell is precessing with a slow rotation vector Ω of components $(\Omega_1, \Omega_2, \Omega_3)$ referred to the principal axes of the ellipsoid. The velocity of a point at the shell's internal boundary is $(\Omega \Lambda r)$ wth components

$$\Omega_2 z - \Omega_3 y \quad \Omega_3 x - \Omega_1 z \quad \Omega_1 y - \Omega_2 x.$$

To satisfy the boundary condition we take the fluid particles velocity (u, v, w) in the core as deriving from a potential ϕ. Then, the boundary condition is that, along the normal to the ellipsoidal boundary, one must have

$$(\Omega_2 z - \Omega_3 y)\frac{x}{a^2} + (\Omega_3 x - \Omega_1 z)\frac{y}{b^2} + (\Omega_1 y - \Omega_2 x)\frac{z}{c^2} = \frac{d\phi}{dx}\frac{x}{a^2} + \frac{d\phi}{dy}\frac{y}{b^2} + \frac{d\phi}{dz}\frac{z}{c^2} \tag{26}$$

which expresses a null flux across the boundary. We seek a solution of the form:

$$\phi = Ayz + Bzx + Cxy, \tag{27}$$

which satisfies the condition of incompressibility $div\ \bar{\mathrm{v}} = 0, \nabla^2\phi = 0$ (Lamb, 1879, art 110) and introduce it in the boundary condition (26). Identifying separately the terms in xy, in yz, in zx gives the velocity potential of the irrotational motion resulting from the shell's angular velocity $\bar{\Omega}$.

$$\phi = \frac{b^2 - c^2}{b^2 + c^2} yz\Omega_1 + \frac{c^2 - a^2}{c^2 + a^2} zx\Omega_2 + \frac{a^2 - b^2}{a^2 + b^2} xy\Omega_3 \tag{28}$$

from which we obtain the fluid particle velocities induced by the rotation of the ellipsoidal shell:

[5] $A = 8.01010^{37}\,kg\,m^2; I = 8.50810^{36}\,kg\,m^2; I/A \cong 0.10$

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$$\begin{aligned} u &= \frac{d\phi}{dx} = \frac{c^2 - a^2}{c^2 + a^2} z\Omega_2 + \frac{a^2 - b^2}{a^2 + b^2} y\Omega_3, \\ v &= \frac{d\phi}{dy} = \frac{a^2 - b^2}{a^2 + b^2} x\Omega_3 + \frac{b^2 - c^2}{b^2 + c^2} z\Omega_1, \\ w &= \frac{d\phi}{dz} = \frac{b^2 - c^2}{b^2 + c^2} y\Omega_1 + \frac{c^2 - a^2}{c^2 + a^2} x\Omega_2, \end{aligned} \tag{29}$$

$$curl \bar{\mathbf{v}} = 0, \qquad \nabla^2 \bar{\mathbf{v}} = 0$$

(Greenhill, 1879, page 239 ([6]); Kelvin, 1885, page 197; Hough, 1895, page 471; Sloudsky, 1895 page 303). The terms containing Ω_3 disappear when $a = b$.

Of course all terms are nullified for a spherical shell ($a = b = c$) because in the absence of viscosity there is no viscous boundary layer to transmit information about rotation perturbations from the shell to the liquid core.

3.2. Inertial rotation in the core: Poincaré artifice

A movement may exist in the fluid core, independent of the shell rotation. To describe it Poincaré artfully used a transformation of the ellipsoid (25) into the sphere

$$x'^2 + y'^2 + z'^2 = 1 \text{ by putting } x' = x/a \quad y' = y/b \quad z' = z/c \tag{30}$$

A rigid body rotation of this sphere (the simplest mode assumed by Poincaré that appears plausible from physical intuition) by a rotation vector $\varpi(\omega_1, \omega_2, \omega_3)$ slightly inclined upon the axis of rotation of the mantle results in velocities of the fluid particles:

$$\begin{aligned} u' &= \omega_2 z' - \omega_3 y', \\ v' &= \omega_3 x' - \omega_1 z', \\ w' &= \omega_1 y' - \omega_2 x'. \end{aligned} \tag{31}$$

The cavity being closed, the component u' cannot be a function of x', v' of y', w' of z', w'. The particles do not penetrate the core-mantle boundary and

[6] The kinetic energy

$$T = \frac{1}{2}\int\int\int \rho(u^2 + v^2 + w^2) dx dy dz$$

$$= \frac{1}{2}\frac{M}{5}\left\{ \frac{(b^2 - c^2)^2}{b^2 + c^2}\Omega_1^2 + \frac{(c^2 - a^2)^2}{c^2 + a^2}\Omega_2^2 + \frac{(a^2 - b^2)^2}{a^2 + b^2}\Omega_3^2 \right.$$

(the products of inertia being null) shows that the liquid core is kinetically equivalent to a mass M of principal moments of inertia:

$$\frac{M}{5}\frac{b^2 - c^2}{b^2 + c^2} \quad \frac{M}{5}\frac{c^2 - a^2}{c^2 + a^2}\,\frac{M}{5}\frac{a^2 - b^2}{a^2 + b^2} \quad \text{(Greenhill, 1879, page 239)}$$

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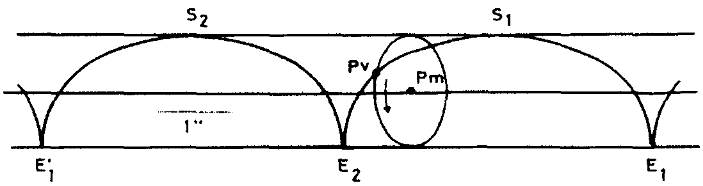

Figure 2. Annual displacement of the true celestial pole P_v as a result of the solar semiannual prograde nutation (semi axis 0″.55, 0″.50). The resulting curve is an epicycloid with amplitude 1″.10. E_1 = spring, S_1 = summer, E_2 = autumn, S_2=winter. The annual precession rate $\Delta\Psi$ being about 50″ and $\sin\theta\Delta\Psi = 20''$, one has $E_1E_2 = 10''$.

should not penetrate the inner core if it has the same flattening. Thus only a rotation is possible. Such a rotation is usually called the "spin-over" or "tilt-over" free mode (see Appendix 2). One may suggest that it could be excited by the periodic impulses exerted on the Earth's axis of rotation when a nutation path proceeds through a sharp regression as shown in Figure 2 (the very great difference of the periods of the nutations with respect to the precession makes each nutation component perform an epicycloid).

Returning to the ellipsoidal shape one has

$$\begin{aligned} u &= \frac{a}{c}\omega_2 z - \frac{a}{b}\omega_3 y = \frac{a}{c}\omega_2 z - \omega_3 y = (1+\varepsilon)\omega_2 z - \omega_3 y, \\ v &= \frac{b}{a}\omega_3 x - \frac{b}{c}\omega_1 z = \omega_3 x - \frac{a}{c}\omega_1 z = \omega_3 x - (1+\varepsilon)\omega_1 z, \\ w &= \frac{c}{b}\omega_1 y - \frac{c}{a}\omega_2 x = \frac{c}{a}(\omega_1 y - \omega_2 x) = (1-\varepsilon)(\omega_1 y - \omega_2 x), \end{aligned} \tag{32}$$

a velocity field which indeed does not penetrate the boundary (25) as $\sum u_i \frac{\partial F}{\partial x_i} = 0$.

When $a = b$, $\varepsilon = (a-c)/c$ (flattening)

$$curl\bar{v} \equiv \left(\frac{a}{c} + \frac{c}{a}\right)\omega_1, \quad \left(\frac{a}{c} + \frac{c}{a}\right)\omega_2, \quad 2\omega_3 \text{ (a flow of constant vorticity).}$$

If we add the irrotational velocity field induced by the ellipsoidal shell precession Ω (29) to this core motion with a small flattening of the interface such that

$$\alpha = \frac{a^2 - c^2}{a^2 + c^2}(ellipticity) \quad 1 + \alpha = \frac{2a^2}{a^2 + c^2} \quad 1 - \alpha = \frac{2c^2}{a^2 + c^2}$$

(numerical values corresponding to the Earth's liquid core are given in Appendix 1) we recover the explicit laminar solution obtained by A.G. Greenhill in 1880 and by Th. Sloudsky in 1895 (page 305), *i.e.* the toroidal velocity field with $a = b$:

$$
\begin{aligned}
u &= \left\{ \frac{2a^2}{a^2+c^2}\omega_2 - \frac{a^2-c^2}{a^2+c^2}\Omega_2 \right\} z - \omega_3 y, \\
v &= \omega_3 x - \left\{ \frac{2a^2}{a^2+c^2}\omega_1 - \frac{a^2-c^2}{a^2+c^2}\Omega_1 \right\} z, \\
w &= \left\{ \frac{2c^2}{a^2+c^2}\omega_1 - \frac{a^2-c^2}{a^2+c^2}\Omega_1 \right\} y - \left\{ \frac{2c^2}{a^2+c^2}\omega_2 - \frac{a^2-c^2}{a^2+c^2}\Omega_2 \right\} x,
\end{aligned}
\tag{33}
$$

$curl\mathbf{v} = 2\omega$ (Poincaré constant vorticity).

In the fluid core, the Navier-Stokes equation, expressed in a frame where the mantle diurnal rotation is stationary, is

$$\partial/\partial t\mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} + 2\Omega\Lambda\mathbf{v} = \nu\nabla^2\mathbf{v} + \nabla\psi, \tag{34}$$

where ψ is the reduced pressure which includes the centrifugal acceleration while ν is the dynamic viscosity, but we consider here a perfect nonviscous fluid core: $\nu = 0$.

Taking the curl:[7]

$$\partial/\partial t curl\mathbf{v} + curl[curl(\mathbf{v})\Lambda\mathbf{v}] + 2curl(\Omega\Lambda\mathbf{v}) = 0 \tag{35}$$

and with the components of $\mathbf{v}$ as given by eqn. (33), with $a = b$, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}curl\bar{\mathbf{v}} = \frac{1}{2}\left(\frac{a}{c}+\frac{c}{a}\right)\frac{\partial\omega_1}{\partial t}, \quad \frac{1}{2}\left(\frac{a}{c}+\frac{c}{a}\right)\frac{\partial\omega_2}{\partial t}, \quad \frac{\partial\omega_3}{\partial t} \\
&\frac{1}{2}curl\left[(curl\bar{\mathbf{v}})\Lambda\bar{\mathbf{v}}\right] = \frac{1}{2}\left(\frac{c}{a}-\frac{a}{c}\right)\omega_2\omega_3, \quad -\frac{1}{2}\left(\frac{c}{a}+\frac{a}{c}\right)\omega_1\omega_3, \quad 0 \\
&curl(\bar{\Omega}\Lambda\bar{\mathbf{v}}) = \omega_3\Omega_2 - \frac{a}{c}\omega_2\Omega_3, \quad \frac{a}{c}\omega_1\Omega_3 - \omega_3\Omega_1, \quad \frac{c}{a}\omega_2\omega_1 - \frac{c}{a}\omega_1\Omega_2
\end{aligned}
$$

Adding these three terms we recover the equations for the time variation of the rotation vector ω of the core which Greenhill in 1880 and Sloudsky (Eq. 45, page 304) also obtained in 1895:

$$
\begin{aligned}
\frac{d\omega_1}{dt} &= \left(\frac{a^2-c^2}{a^2+c^2}\right)\omega_2\Omega_3 + \omega_2\Omega_3 - \frac{2a^2}{a^2+c^2}\omega_3\Omega_2, \\
\frac{d\omega_2}{dt} &= -\left(\frac{a^2-c^2}{a^2+c^2}\right)\omega_3\Omega_1 - \omega_1\Omega_3 - \frac{2a^2}{a^2+c^2}\omega_3\Omega_1, \\
\frac{d\omega_3}{dt} &= -\frac{2c^2}{a^2+c^2}\left[\omega_1\Omega_2 - \omega_2\Omega_1\right].
\end{aligned}
\tag{36}
$$

[7] with $\mathbf{v}\nabla\mathbf{v}$=curl $\mathbf{v}\Lambda$ $\mathbf{v}$ - $\frac{1}{2}$ grad $\mathbf{v}^2$

For the steady case ($\frac{d}{dt}\bar{\omega} = 0$) one obtains the numerical values (Appendix 2):

$$\omega_1 = 1.657 \times 10^{-5}\omega_3 = 12.08 \times 10^{-10} rads^{-1}, (\text{spin} - \text{over})$$

$$arctg(\omega_1/\omega_3) = 3''.47, (\text{tilt} - \text{over})$$

and for the velocity at the core-mantle boundary:

$$a\omega_1 = 4.21 \times 10^{-3} ms^{-1}.$$

If the container is stationary ($\bar{\Omega} = 0$) the third equation gives ω_3 = constant and consequently the two other equations can be integrated:

$$\omega_1(t) = K \sin(\sigma t + \phi), \quad \omega_2(t) = K \cos(\sigma t + \phi).$$

The angular speed is

$$\sigma = \frac{a^2 - c^2}{a^2 + c^2}\omega = \frac{\omega}{392}(\text{Appendix 1})$$

which is called "free core nutation" with a period of 392 sidereal days. Finally Sloudsky reaches the same conclusion as Hough (§3.3) that there exists a nearly diurnal free wobble in addition to the Chandler free wobble but he does not evaluate the period of this principal motion concluding only that it may be twelve or fourteen months.

The motions described by equations (33) and (36) correspond to the inertial coupling between core and mantle. As the fluid was supposed to be perfect, *i.e.* nonviscous, there is no boundary layer along the mantle internal surface and no viscous coupling. The no slip nor stress-free boundary conditions *are not satisfied.*

The theory of boundary layers and of their role in the transmission of information from the solid envelope to the bulk of the liquid body has indeed been introduced by Prandtl in 1904, *i.e.* nine years after the papers by Sloudsky and by Hough were published. Also the Ekman boundary layer (and its famous spiral) applicable to the case of rotating bodies, was introduced in 1905.

Its results state that if the boundary was spherical ($a = c$) the mantle rotation (Ω) could not force any rotation of the fluid core which clearly appears in the equations (29) and (33) where all terms containing the components of Ω have $(a^2 - c^2)$ as factor.

We know presently that four different couplings exerted by the core on the mantle may act to tighten the core to the mantle rotation: inertial (ellipsoidal form of the boundary), topographical (roughness of the boundary), viscous (viscosity of the fluid) and electromagnetic (core dynamo). The couplings have been calculated and their strength evaluated in several recent papers.

3.3. The oscillations of a rotating ellipsoidal shell containing a homogeneous incompressible fluid according to S.S. Hough (1895)

With the same conventions as in §2.2, Hough uses Eqn (33) for the velocity components:

$$u = \frac{c^2 - a^2}{c^2 + a^2} z\Omega_2 - y\omega_3 + z\omega_2,$$

$$v = \frac{a^2 - c^2}{a^2 + c^2} z\Omega_1 - z\omega_1 + x\omega_3, \tag{37}$$

$$w = \frac{a^2 - c^2}{a^2 + c^2}(y\Omega_1 - x\Omega_2) + y\omega_1 - x\omega_2,$$

where $2a^2/(a^2 + c^2)$ was taken as unity.

The components of the total angular momentum $\bar{H}$ (shell + core) are of the form

$$h_1 = A_S(\Omega_1 + \omega_1) + \int\int\int \rho(wy - vz)dxdydz, \tag{38}$$

or

$$h_1 = A_S(\Omega_1 + \omega_1) + \frac{M}{5}\frac{(a^2 - c^2)^2}{a^2 + c^2}\Omega_1 + \frac{M}{5}(a^2 + c^2)\omega_1 \quad h_2, h_3, \ldots$$

M, ρ are the mass and the density of the fluid, A_S, A_S, C_S the principal moments of inertia of the shell. The equations of angular momentum with no external forces are:

$$\begin{aligned} &\{\dot{h}_1 - h_2 r + h_3 q = 0\} \quad p = \Omega_1 + \omega_1, \\ &\{\dot{h}_2 - h_3 p + h_1 r = 0\} \quad q = \Omega_2 + \omega_2, \\ &\{\dot{h}_3 - h_1 q + h_2 r = 0\} \quad r = \Omega_3 + \omega_3. \end{aligned} \tag{39}$$

The development is straightforward and, putting each variable proportional to $e^{-\imath\lambda t}$, Hough obtains the period biquadratic equation in the fourth and second powers of λ and of ω_3 with roots nearly equal in pairs. Supposing that the density of the rigid shell of mean radius r_1 is the same as the density of the liquid core, that the cavity is approximately spherical of mean radius r and treating its flattening ε as a very small quantity whose first power only is retained, one obtains, A_1, A_1, C_1 being the principal moments of inertia of the liquid core, the roots

$$\lambda^2 = \omega^2[(C_S - A_S)/A_S + \varepsilon q]^2 \text{ (the Chandler wobble),} \tag{40}$$

and

$$\lambda^2 = \omega^2(1 + 2E) \text{ or } \lambda = \pm(1 + E),$$

the free nearly diurnal wobble with

$$\mu = \frac{8\pi}{15}\rho r^5 \varepsilon = A_1 - C_1(Clairaut), \quad C_S = \frac{8\pi}{15}\rho(r_1^5 - r^5), \tag{41}$$

$$q = \frac{\mu}{\varepsilon C} = \frac{r^5}{r_1^5 - r^5}, \qquad E = \varepsilon(1 + q).$$

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The parameter q increases with the size of the core radius r. With the actual values $r_1 = 6371$ km, $r = 3485$ km, one has $q = 5.15\times10^{-2}$, $\varepsilon = 1.31\times10^{-4}$, $E = 0.00268$ and $\lambda = 15.081°$/hour, the frequency of the *nearly diurnal wobble* (period 23h 52m 16s) with which the Hopkins free nutation is associated (Poinsot) with a period of 392 days and an amplitude 392 times the wobble amplitude.

If we consider now the principal moments of inertia of the Earth as a whole:

$$\mathbf{C} = C_S + (2/5)Ma^2 = C_S[1 + q(1 + 2\varepsilon)],$$

$$\mathbf{A} = A_S + (1/5)M(a^2 + c^2) = A_S + qC_S(1 + \varepsilon), \tag{42}$$

we obtain the period of the free mode of a thin spheroidal shell containing a homogeneous fluid

$$\tau = \frac{2\pi}{\omega}\frac{C_S}{C_S - A_S} = \frac{2\pi}{\omega}\frac{\mathbf{C}}{\mathbf{C} - \mathbf{A}}\frac{1}{1+q}, \tag{43}$$

while for an Earth supposed solid throughout ($q = 0$) it is

$$\tau_0 = \frac{2\pi}{\omega}\frac{\mathbf{C}}{\mathbf{C} - \mathbf{A}} = 305.4 \text{ sidereal days, Euler period } + \text{ one day.}$$

As q is a positive parameter, the period of the free mode of a thin spheroidal shell containing a homogeneous fluid is *shorter* than the Euler period by a factor $(r/r_1)^5$.

The Hough formula (43) gives as shortenings of the Euler period related to different core radii:

Shell	804	1609	2900	Km
Core	5566	4762	3471	Km
shortening	−156	−71	−15	Days

Modern studies have given a quite larger shortening of 50 days with respect to the Euler period (Smith and Dahlen, 1981).

As a conclusion of a second paper, Hough (1896) referred to in the preceding section, stated that "the effects of the elastic deformations would more than counteract the influence of the reduced effective inertia due to internal fluidity and that with a given degree of rigidity the period of oscillation would be still further prolonged." This was definitely established by Jeffreys (1949) and fully confirmed by modern research.

3.4. The precession of deformable bodies according to Poincaré (1910)

We consider the spheroid as an ellipsoid of revolution ($a = b$) then, let A, A, C represent the principal moments of inertia of the whole Earth,
A_1, A_1, C_1 the principal moments of inertia of the fluid core,
$F, F, H = C_1$ the expressions $(b/c)\sum mz^2 + (c/b)\sum my^2, \ldots, \ldots$ in the core,
$\omega(p, q, r)$ the absolute rotation of the whole Earth, projections on moving axes
$\omega_1(p_1, q_1, r_1)$ the relative rotation of the fluid core, projections on axes taken as fixed, and
L, M, N the components of the moment of the external forces.

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Then the kinetic energy of the system is

$$2T = Ap^2 + Aq^2 + Cr^2 + A_1p_1^2 + A_1q_1^2 + C_1r_1^2 + 2Fpp_1 + 2Fqq_1 + 2Hrr_1. \quad (44)$$

The Lagrange equations are

$$d/dt(Ap + Fp_1) - r(Aq + Fq_1) + q(Cr + Hr_1) = L, \quad (45)$$

while the Helmholtz equations for the fluid core are

$$d/dt(A_1p_1 + Fp) + r_1(A_1q_1 + Fq) - q_1(C_1r_1 + Hr) = 0. \quad (46)$$

In the third Lagrange equation one takes $N = 0$ when there is no viscous effect:

$$d/dt(Cr + C_1r_1) + F(pq_1 - p_1q) = N = 0. \quad (47)$$

Subtracting the third Helmholtz equation

$$d/dt(C_1r + C_1r_1) + F(pq_1 - p_1q) = 0, \quad (48)$$

one obtains: $r = constant = \omega$, $r_1 = constant = \omega_1$ supposed to be very small. The four remaining equations may be written

$$\begin{aligned} A\dot{p} + F\dot{p}_1 + (C - A)\omega q - F\omega q_1 &= L = K\cos kt, \\ A\dot{q} + F\dot{q}_1 - (C - A)\omega p + F\omega p_1 &= M = -K\sin kt, \\ F\dot{p} + A_1\dot{p}_1 - C_1\omega q_1 &= 0, \\ F\dot{q} + A_1\dot{q}_1 + C1\omega p_1 &= 0, \end{aligned} \quad (49)$$

and, with $\omega = p + iq$, $\omega_1 = p_1 + iq_1$, $L + iM = Ke^{-i\omega_1 t}$, their combinations are

$$\begin{aligned} A\dot{\omega} - i(C - A)\omega\omega + F\dot{\omega}_1 + iF\omega\omega_1 &= Ke^{-i\omega_1 t}, \\ F\dot{\omega} + A_1\dot{\omega}_1 + iC_1\omega\omega_1 &= 0. \end{aligned} \quad (50)$$

The free oscillation frequencies λ ($\omega = me^{i\lambda t}, \omega_1 = ne^{i\lambda t}$) are given for K = 0 by the condition

$$Det(\lambda) = \begin{vmatrix} A\lambda - (C - A)\omega & F\lambda + F\omega \\ F\lambda & A_1\lambda + C_1\omega \end{vmatrix}. \quad (51)$$

Replacing A with $A_S + A_1$

$$Det(\lambda) = \begin{vmatrix} A_S\lambda - (C - A)\omega + A_1\lambda & F\lambda + F\omega \\ F\lambda & A_1(\lambda + \omega) + (C_1 - A_1)\omega \end{vmatrix} \quad (52)$$

or

$$\begin{aligned} &A_1(\lambda + \omega)\{A_S\lambda - (C - A)\omega\} + \varepsilon_1^2 A_1^2\lambda^2 + \\ &\{\varepsilon_1 A_S A_1 + \varepsilon_1 C_1 A_1\}\omega\lambda - \varepsilon_1 A_1(C - A)\omega^2 = 0. \end{aligned} \quad (53)$$

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if

$$\varepsilon = \frac{C_1 - A_1}{A_1}, \quad \varepsilon_1^2 = \frac{A_1^2 - F_1^2}{A_1^2}, \quad F^2 = 2A_1C_1 - C_1^2.$$

In the spherical case ($\varepsilon_1 = 0$) the condition is

$$(\lambda + \omega)\{A_S\lambda - (C - A)\omega\} = 0, \tag{54}$$

and we find again: a nearly diurnal free wobble $\lambda_1 = -\omega$ and an Eulerian free wobble with a period shorter than the Eulerian period for a rigid Earth $\lambda_2 = \omega(C - A)/A_S$, *i.e.* a period of about 270 sidereal days.

Considering now a forced nutation of frequency $\lambda = -\omega_i$ and putting $\omega - \omega_i = \Delta\omega_i$, the determinant is

$$Det(\omega_i) = \begin{vmatrix} A\Delta\omega_i - C\omega & F\Delta\omega_i \\ F\Delta\omega_i - F\omega & A_1\Delta\omega_i + (C_1 - A_1)\omega \end{vmatrix}. \tag{55}$$

The solution is

$$\bar{\omega} = \frac{(\Delta\omega_i + \varepsilon\omega)}{Det(\omega_i)}K, \tag{56}$$

and, for a totally solid Earth, without a fluid core ($A_S = 0$, F $= 0$)

$$\bar{\omega}_0 = \frac{\Delta\omega_i + \varepsilon_1\omega}{-\varepsilon_1 C\omega^2 - (C - A\varepsilon_1)\omega\Delta\omega_i + A\Delta\omega_i^2}K. \tag{57}$$

Thus

$$\frac{\bar{\omega}}{\bar{\omega}_0} = \left\{1 - \frac{A_1(\omega\Delta\omega_i - \Delta\omega_i^2)}{C\omega - A\Delta\omega_i)(\varepsilon_1\omega + \Delta\omega_i)}\right\}^{-1} \tag{58}$$

(details of this development are given in Melchior (1983).

Numerical values

$$A \sim C \sim 80.10 \times 10^{36} kg\ m^2, \quad A_1 \sim C_1 \sim 8.04 \times 10^{36} kg\ m^2, \quad A_1/A = 0.100,$$

$$\varepsilon_1 \cong 1/392.15 \text{ (hydrostatic model)}$$

give

$$\bar{\omega} = \bar{\omega}_0\left(1 + \frac{44.8}{392.15 + x}\right)^{-1}, x = \frac{\omega}{\Delta\omega_i}. \tag{59}$$

Thus a resonance takes place for

$$\begin{aligned} x = -\varepsilon_1 = -392.15 \quad &\text{or} \quad \Delta\omega_i = -\varepsilon_1\omega, \\ \omega_i = \omega - \Delta\omega_i = \omega(1 + \varepsilon_1) \quad &\text{with} \quad \omega = 15.041069°/hour. \end{aligned} \tag{60}$$

The period is

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$$T = \frac{2\pi}{\omega_i} = \frac{2\pi}{\omega}(1 - \varepsilon_1) = 24h \times (1 - 1/392.15)$$
$$= 23h56m20s. \ (frequency\ 15.074^\circ/hour).$$

We see that the diurnal free wobble period is determined by the flattening ε_1 of the fluid core.

In the same paper, another important conclusion of Poincaré concerns precession and nutation. Taking

$$p = \alpha \sin kt, \quad q = \alpha \cos kt,$$
$$p_1 = \alpha_1 \sin kt, \quad q_1 = \alpha_1 \cos kt, \tag{61}$$

the equations to be solved are

$$(A\alpha + F\alpha_1)(k + \omega) - \alpha C\omega = K,$$
$$(F\alpha + A\alpha_1)k + \alpha_1 C_1 \omega = 0. \tag{62}$$

Some elementary developments give

$$\frac{\alpha}{\alpha_0} = \frac{\varepsilon_1 n - 1}{\varepsilon_1 n - 1 + \eta}, \tag{63}$$

where $\eta = A_1/A \sim C_1/C$ characterizes the thickness of the mantle, ($\eta = 0.11$ for the real Earth.) $\alpha = \alpha_0$ when $\eta = 0$ (no fluid core) and n is the number of days of the forced period. Results show that if the product $\varepsilon_1 n$, with $\varepsilon_1 = 1/392.15$, is very large, which is the case for precession (25,800 years) the amplitude α is the same as the amplitude α_0 for the solid Earth. This is not true for those nutations whose period is not long with respect to 392 days, 392.15 being the inverse of the flattening, in particular the annual, semiannual and fortnightly nutations. This property was christened "gyrostatic rigidity" by Kelvin (1890, §14, page 442). When it is to be distinguished from the known natural rigidity of an elastic solid it will be called gyrostatic rigidity.

We can check this result very simply if we consider that the precession phenomenon is the result of the torque exerted by the tidal tesseral force associated with the sidereal tidal wave K_1, fixed in space, with a frequency $\lambda = -\omega$ with respect to the rotating Earth or $\Delta\omega_i = 0$. We obtain from eqn. (57) and (58)

$$\bar{\omega}_0 = \frac{K}{C\omega} e^{-i\omega t},$$

$$\bar{\omega}/\bar{\omega}_0 = 1,$$

the same frequency as for a totally solid Earth.

For the principal elliptical nutation of lunar node frequencies

$$\lambda = \omega \pm N, \Delta\omega_1 \mp N,$$

$$\frac{\varpi}{\varpi_0} = \left\{ 1 - \frac{A_1\left(\frac{N}{\omega} - \left(\frac{N}{\omega}\right)^2\right)}{\left(C - A\frac{N}{\omega}\right)\left(\varepsilon_1 + \frac{N}{\omega}\right)} \right\}^{-1},$$

$\frac{N}{\omega}$ being small ($\approx 1/6800$) compared to ε_1, results show that $\bar{\omega}$ is also close to the solid Earth response $\bar{\omega}_0$. This is no more true when $\Delta\omega_i/\omega$ is not small.

This result of Poincaré justifies the statement made by Lord Kelvin already in 1876 but without a mathematical proof published by him (1890, footnote at page 324).

4. Hundred years later ...

To conclude this memorial lecture, I feel "seasonable" enough to briefly describe the current state of our knowledge and understanding of the polar motion origin and characteristics. It seems that the 1910 Poincaré paper terminated a period of real interest about the theory of polar motion and the underlying geophysical phenomena. Fortunately the well established International Latitude Service continued its activity, apparently for practical applications in geodesy only, not without difficulties (closure of Cincinnati and Tschardjui stations, interruptions at Gaithersburg).

Only a few scientists continued to pay attention to the problem and, by chance, an outstanding genius, Sir Harold Jeffreys, maintained the attention of the scientific community on the unsolved problems of polar motion, nutations and Earth tides that he demonstrated to be key problems for internal geophysics. The International Geophysical Year (1957–1958) brought back the polar motion and Earth tides problem to the front of geodynamics, giving simultaneously a very strong impulse to all geophysics. As a result, hundreds of contributions to all aspects of Earth rotation have appeared during the last forty years.

4.1. The Chandler period

Thanks to the International Latitude Service activity, one hundred years of determination of the polar path have allowed us to perform many tentative determinations of the Chandler period. The conclusion upon which most of the authors agree now is that

$$\tau = 435.2 \pm 2.6 \text{ sidereal days (Wilson and Haubrich, 1976)},$$

with a quality factor Q about 100, that is a damping period of about 40 years. Earth models allow us to evaluate the contribution of each structural part (Smith and Dahlen, 1981):

- rigid earth eulerian period	304.4 sidereal days
- elasticity of the mantle (eqn. 22)	+ 143.0
- fluidity of the core	−50.5
- participation of the oceans (pole tide)	+ 29.8
- anelasticity of the mantle	+ 8.5
Altogether	435.2 sidereal days

The Chandler motion, being a free motion, is subject to systematic damping (anelasticity of the mantle, dissipation in the oceans, viscous coupling at the core-mantle interface). It can be maintained only if continuously excited by irregular impulses, and this is not yet fully explained despite many tentative evaluations of the probable role of seismic activity, snow, atmospheric pressure and ground water impulses.

4.2. The elliptic annual component

This component is obviously of atmospheric origin. Contrary to the opinion expressed in his time by Sloudsky, satisfactory explanations have been obtained by considering mainly the variations of air pressure (mass redistribution) and also the torques exerted by winds and currents (Wilson and Haubrich, 1976). The ocean water redistributions, acting as an inverted barometer are also taken into account. Nevertheless the amplitude of this annual component is not fully explained and new investigations are surely needed.

4.3. Free Core Nutation and nearly diurnal free wobble

The theoretical frequency of this suspected retrograde wobble is $-15.0737°$/hour (using the last century Hough equations (40) (41) we get $-15.081°$/hour) which, according to eqn. (60) corresponds to a core flattening $\varepsilon_1 = 1/460.9$ that is $a - c = 7.56$ km ($a = 3485$km). This is very close to the frequencies of diurnal barometric and thermic variations while its amplitude is certainly extremely small so that attempts to extract it from the ground based astronomical observations are not convincing.

However it is associated, in the Poinsot sense, to a retrograde free nutation in space of period 460 days having an amplitude 460 times the wobble amplitude which has been detected in VLBI measurements with an amplitude of the order of $0''.0002$ only.[8]

Moreover the existence of the corresponding resonance frequency had been observed already in tidal measurements with tiltmeters in 1960 (Melchior 1992) and demonstrated with much higher precision by accurate measurements with superconducting gravimeters of the tesseral diurnal Earth tides component waves with frequencies 13.943°/h (wave O_1), 14.959° /h (wave P_1), 15.041°/h (wave K_1) and, principally 15.082°/h (wave ψ_1). This last one unfortunately has a very small amplitude (Fig.3 — Melchior 1992). VLBI and tidal gravimeters agree very well about a diurnal resonance frequency of $-15.076°$/hour which, according to Eqn. (60) corresponds to a flattening $\varepsilon_1 = 1/431$ (thus $a - c$=8.09km) and consequently associated with a free core nutation period of 431 days.

Theory and observation can thus be reconciled if one increases the hydrostatic core flattening (9 km) by about 500 meters. Moreover these tidal tesseral diurnal waves are associated with the annual, semiannual and fortnightly nutations which are affected by the same resonance effect as anticipated by Lord Kelvin and demonstrated by Poincaré (§3.4, eqn 62). It made it compulsory to adopt new nutation tables taking that resonance into consideration (Melchior, 1971; Dehant and Defraigne, 1997).

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[8]Uncertainty in VLBI determinations is of the order of $0''.00015$.

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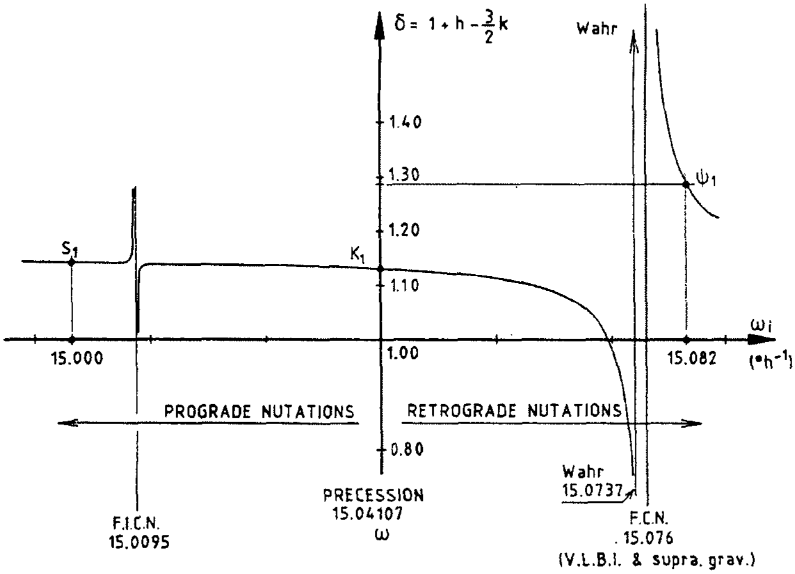

Figure 3. Free core nutation frequency.

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4.4. The future ...

The exceptional importance of the Chandler wobble results from the fact that, between the seismic high frequencies (up to one hour period for the free oscillations) and the very low frequencies of tectonic processes like the postglacial rebound there exist only a few periodic deformation processes: the Earth tides, the Chandler wobble and the astronomical nutations to allow investigations about the anelasticity in the Earth.

We must recognise that the existence of a solid inner core inside a stratified convecting liquid core makes highly improbable the Sloudsky - Hough - Poincaré description of the flow in the liquid core. Hydrodynamicists advocate indeed the formation, in the outer liquid core of a cylindric boundary layer tangent to the inner core and whose axis is the axis of rotation. The part of the liquid core external to this cylinder only would participate in the mantle rotation. Moreover any obstacle due to suspected topographical irregularities of the core mantle interface would generate Taylor columns (known since 1917) aligned with the rotation axis inside the liquid core (Melchior, §3.5, 1986). This corresponds to a geostrophic approximation justified by the extreme smallness of the Ekman and Rossby numbers of the liquid core.

Moreover the influence of the magnetic field and Lorentz force is not clear. The exact role of the inertial couplings (ellipticity and topography of the boundary) and the dissipative couplings (viscous and electromagnetic stresses in the hydromagnetic Ekman - Hartman boundary layer) remains a matter for more research. This will not be easy to model. Very fortunately, the extraordinary improvements due to the space techniques of measurements of geodetic and astrometric variable parameters shall surely give rise to new discoveries in the coming century.

We would finally conclude by stressing how the statement by Lord Kelvin about Hopkins "grand idea" was truly objective (§3). It is also remarkable that the precursors in the 19^{th} and beginning of the 20^{th} century have announced, only on theoretical grounds the main mechanical properties of the free motions of the axis of rotation of the Earth.

The lack of sufficient precise observational data prevent them going further. This may be now accomplished thanks to one century of persevering activity of the International Latitude Service and Polar Motion Service.

Acknowledgments. At the occasion of this commemoration, one has to pay tribute, not only to the successive directors in charge of the International Latitude Service: Th. Albrecht, B. Wanach, W. Mahnkopf, H. Kimura, L. Carnera, G. Cecchini, T. Hattori, Sh. Yumi but also, perhaps mainly, to all humble astronomers who devoted their time and energy to maintain the activity of the stations at Carloforte, Tschardjui and Kitab, Mizusawa, Ukiah, Cincinnati and Gaithersburg. Without their generous devotion, the theoreticians would have been deprived of the necessary experimental data they needed to correctly guide their analysis of the phenomenon and its causes.

5. Appendix 1

Earth's rotation angular velocity $\omega = 7.29211510^{-5} rads^{-1}$ Earth's lunisolar precession rate $\Omega = 7.73842310^{-12} rads^{-1}$

For the Earth's core one has

$a = 348510^3 m$, $a - c = 8.8910^3 m$

$e = (a-c)/a = 1/392.15 = 0.002550$ (hydrostatic flattening)

$\alpha = (a^2 - c^2)/(a^2 + c^2) = 0.0025533$ (ellipticity)

$e^2 = 0.000006502$

$e^3 = 0.000000016$,

$a/c + c/a = 2 + e^2 + e^3 = 2.00000652$

$c/a - a/c = -2 - e^2 - e^3 = -0.0051065$

$2a^2/(a^2 + c^2) = 1 + \alpha = 1.0025533$

$2c^2/(a^2 + c^2) = 1 - \alpha = 0.997447$

Obviously the oscillations of the mantle hardly communicate themselves to a nonviscous fluid core. If one increases the flattening by 500 meters, then

$a - c = 9.3910^3 m \quad e = 1/371 = 0.00269$

Remark: for a homogeneous ellipsoid of revolution

$$\frac{C-A}{A} = \frac{a^2 - c^2}{a^2 + c^2}.$$

6. Appendix 2

Considering the steady case ($\frac{d}{dt}\bar{\omega} = 0$) and taking a reference plane such that $\Omega_2 = 0$, the third equation (36) gives $\omega_2 = 0$ (because $\Omega_1 \neq 0$ necessarily) and the first equation is satisfied.

From the second equation (36) it then follows that

$$\omega_1/\omega_3 = \left(\frac{2a^2}{a^2 + c^2}\Omega_1\right) / \left(\frac{a^2 - c^2}{a^2 + c^2}\omega_3 - \Omega_3\right).$$

We take

$\omega_3 \cong 7.292 \times 10^{-5} rads^{-1}$ (the angle to Oz being very small) $\Omega = 1.064 \times 10^{-7}\omega_3$ $\Omega_1 = \Omega \sin 23.44° = 0.423 \times 10^{-7}\omega_3$ $\Omega_3 = \Omega \cos 23.44° = 0.976 \times 10^{-7}\omega_3$ obtaining the "spin-over" mode ω_1 or "tilt-over" arc tg (ω_1/ω_3). $\omega_1 = 1.657 \times 10^{-5}\omega_3 = 12.08 \times 10^{-10} rads^{-1}$ $arctg(\omega_1/\omega_3) = 3.47''$ (Rochester *et al.* 1975) $a\omega_1 = (3.48510^6 m)\omega_1 = 4.21 \times 10^{-3} ms^{-1}$

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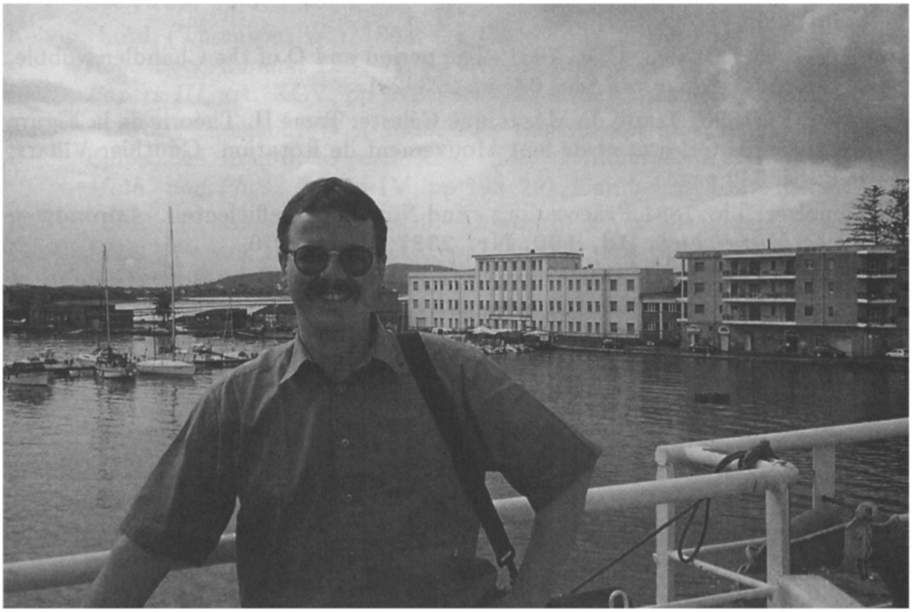

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