D. Popescu
Nagoya Math. J.
Vol. 118 (1990), 45-53

LETTER TO THE EDITOR

General Néron desingularization and approximation

DORIN POPESCU

This letter concerns our papers [4], [5] and its aim is to give a simplification to the proof of the General Néron Desingularization (see [5] (2.4) or here below) together with a small reparation; as T. Ogoma pointed out in [3], our Lemma (9.5) from [4] does not hold in the condition iii_2) (this is true because the "changing" from line 5 from down the page 123 [4] may not preserve iii_2)). However our results were not affected in characteristic zero (they use just iii_1) from [4] (9.5)). In [3] Ogoma gives a nice simplification of our proof. Though completely based on our papers his simplification contains two new ideas:

1) a procedure to pass from a system of elements which is regular in a localization to a "good enough" system (see [3] (4.3), (4.5) or here Lemma 6 and Corollary 7).

2) the so called "residual smoothing".

First idea is very important and should be part of all possible simplifications. The second idea is a difficult notion which hides a lot of details. Our simplification does not use such hard notions or hard results from characteristic $p > 0$ as the Nica-Popescu Theorem [2] (1.1) but certainly it is inspired by [3], [4] and [5]. Moreover we believe that our simplification preserves better the flavour of the old Néron desingularization (compare our Step 4 and [4] Section 6).

Let u: $A \to A'$ be a morphism of Noetherian rings, B a finite type A-algebra and f: $B \to A'$ an A-morphism. A *desingularization of* (B, f) *with respect to* u is a standard smooth A-algebra B' together with two A-morphisms g: $B \to B'$, h: $B' \to A'$ such that $f = hg$.

General Néron desingularization ([5] (2.4)). If u is regular then (B, f) has a desingularization with respect to u.

Received August 23, 1988.
Revised February 10, 1989.

The proof follows by Noetherian induction on $\sqrt{f(H_{B/A})A'}$ from the following Theorem (see e.g. [4] (5.2)), where $H_{B/A}$ is the ideal defining the nonsmooth locus of B over A.

THEOREM 1. *Suppose that $A_{u^{-1}q} \to A'_q$ is formally smooth for a minimal prime over-ideal q of $\mathfrak{a} := \sqrt{f(H_{B/A})A'}$ such that $u^{-1}q$ is a minimal prime over-ideal of $u^{-1}\mathfrak{a}$. Then there exist a finite type A-algebra B' and two A-morphisms $g\colon B \to B'$, $h\colon B' \to A'$ such that $hg = f$ and*

$$f(H_{B/A})A' \subset \sqrt{h(H_{B'/A})A'} \not\subset q\,.$$

Remark. i) The condition "$u^{-1}q$ is a minimal prime over-ideal of $u^{-1}\mathfrak{a}$" does not appear in [4], [5]. There we have another more complicated condition concerning the flatness of u. However for our Noetherian induction (see above) does not matter the order in which we choose for desingularization the minimal prime over-ideals of $\mathfrak{a}$ (if somebody insist to prove Theorem 1 without the above condition then Step 1 will be much more complicated; an idea is given at the end of Step 4 namely to pass from d to δd where $\delta \notin q$ belongs to all minimal prime over-ideals $\neq q$ of $\mathfrak{a}$).

ii) The Question [4] (1.3) seems to be older than we expect it (see e.g. [6]).

The proof of Theorem is based on [4] (9.1), (9.2) and some preliminaries which we present below.

LEMMA 2. *Let $q \subset A'$, be a prime ideal and $j\colon A' \to A'_q$ the canonical map. If (B, jf) has a desingularization with respect to ju then there exist a finite type B-algebra B' and a B-morphism $h\colon B' \to A'$ such that $h(H_{B'/A}) \not\subset q$.*

The Lemma is quite elementary. Given a desingularization (C, α, β) of (B, f) with respect to ju let us say $C \cong B[X]/(F)$, $\beta\colon C \to A'_q$, $X \to y/t$, $y, t \in A'$ then we may take $B' := B[Y, T]/(G)$, $h\colon B' \to A'$, $(Y, T) \to (y, t)$, where $G = T^s F(Y/T)$ for a certain high enough positive integer s.

LEMMA 3. *Let $q \subset A'$ be a minimal prime ideal and $j\colon A' \to A'_q$ the canonical map. If (B, jf) has a desingularization with respect to ju then there exists a finite type B-algebra B' and a B-morphism $h\colon B' \to A'$ such that*

$$f(H_{B/A}) \subset \sqrt{h(H_{B'/A})A'} \not\subset q\,.$$

Proof. By Lemma 2 there exist a finite type B-algebra $C \cong B[X]/(F)$, $X = (X_1, \cdots, X_r)$, $F = (F_1, \cdots, F_m)$ and a B-morphism $\alpha\colon C \to A'$ such that $\alpha(H_{C/A}) \not\subset q$. We may suppose that $q \supset \mathfrak{a} := \sqrt{f(H_{B/A})A'}$, otherwise $B' := B$, $h := f$ work. If $\mathfrak{a}$ is a nil ideal then $B' := C$, $h := \alpha$ work. Otherwise choose an element z in $\bigcap_{\substack{p \not\supset \mathfrak{a} \\ p \in \operatorname{Min} A'}} p$ which is not in q, $\operatorname{Min} A'$ being the set of minimal prime ideals of A'. Then $z\mathfrak{a}$ is a nil ideal. Let $y = (y_1, \cdots, y_n)$ be a system of elements from $\mathfrak{a}$ such that $\mathfrak{a} = \sqrt{yA'}$. We have $(zy_i)^s = 0$, $1 \leq i \leq n$ for a certain positive integer. Changing z, y by z^s, y^s we may suppose $zy_i = 0$, $1 \leq i \leq n$.

Let $B' := B[X, Y, Z, T]/(F - \sum_{i=1}^{n} Y_i T_i, ZY)$, $T_i = (T_{i1}, \cdots, T_{im})$, $T = (T_i)_i$, $ZY = (ZY_1, \cdots, ZY_n)$, $\cdots$ and $h\colon B' \to A'$ the B-morphism given by $X \to \alpha(X)_A$, $Y \to y$, $Z \to z$, $T \to 0$. Note that $B'_{Y_i} \cong B[X, Y, (T_j)_{j \neq i}, Y_i^{-1}]$ is smooth over B and so $\mathfrak{a} = \sqrt{yA'} \subset \sqrt{h(H_{B'/B})A'}$. Thus $\mathfrak{a} \subset \sqrt{h(H_{B'/A})A'}$ (see [4] (2.2)]. On the other hand $B'_Z \cong B[X, Z^{\pm 1}, T]/(F) = C[T, Z^{\pm 1}]$ is smooth over C. Thus $B'_{h^{-1}q}$ is smooth over A and so $h(H_{B'/A}) \not\subset q$.

LEMMA 4 (see e.g. [2] (3.7)). *Let $q \subset A'$ be a prime ideal, $r = \operatorname{ht} q - \operatorname{ht} u^{-1}$ and $x = (x_1, \cdots, x_r)$ a system of elements from q. Suppose that the map $A_{u^{-1}q} \to A'_q$ induced by u is flat, $R := A'_q/(u^{-1}q)A'_q$ is regular and x induces a regular system of parameters in R. Then the A-morphism v: $A[X] \to A'$, $X = (X_1, \cdots, X_r) \to x$ induces a flat map v_q: $A[X]_{v^{-1}q} \to A'_q$ and $(v^{-1}q)A'_q = qA'_q$.*

For the proof note that $A/u^{-1}q \otimes_A v_q$ is flat (see e.g. [1] (36.B)) and so v_q is also by [1] (20.G) applied to $A_{u^{-1}q} \to A[X]_{v^{-1}q} \to A'_q$.

LEMMA 5. *Let $q \subset A'$ be a prime ideal, $k \subset K$ the residue field extension of u_q: $A_{u^{-1}q} \to A'_q$, E/k a finite type field subextension of K/k and $y = (y_1, \cdots, y_s)$ a system of elements from A' inducing a p-basis $\bar{y}$ of E over k. Suppose that u_q is formally smooth. Then the A-morphism w: $A[Y] \to A'$, $Y = (Y_1, \cdots, Y_s) \to y$ induces a flat map w_q: $A[Y]_{w^{-1}q} \to A'_q$ and the ring $A'_q/(w^{-1}q)A'_q$ is regular of dimension r-rank $\Gamma_{E/k}$, where $\Gamma_{E/k}$ is the imperfection module of E over k (see e.g.* [1] (39.B)).

Proof. Applying [1] (20.G) to $A_{u^{-1}q} \to A[Y]_{w^{-1}q} \to A'_q$ we reduce to the case when A is a field. Now it is enough to apply [5] (7.1).

LEMMA 6 (Ogoma [3] (4.3)). *Let z, x be two elements in A and s, t two positive integers such that* $\operatorname{Ann}_{A_z} x^s = \operatorname{Ann}_{A_z} x^{s+1}$ *and* $\operatorname{Ann}_A z^t = \operatorname{Ann}_A z^{t+1}$.

Then $\mathrm{Ann}_A(z^t x)^s = \mathrm{Ann}_A(z^t x)^{s+1}$.

COROLLARY 7 (Ogoma [3]). *Let* $q \subset A'$ *be a prime ideal,* $x = (x_1, \cdots, x_r)$ *a system of elements from* A' *which induces a regular system of elements in* A'_q *and* s *a positive integer. Then there exists a system of elements* $z = (z_1, \cdots, z_r)$ *in* $A' \backslash q$ *such that*

$$((z_1^s x_1^s, \cdots, z_{i-1}^s x_{i-1}^s) \colon z_i x_i) = ((z_1^s x_1^s, \cdots, z_{i-1}^s x_{i-1}^s) \colon z_i^2 x_i^2)$$

for all $1 \leq i \leq r$, *where* $x_0 := 0$.

Proof. Applying induction on r we reduce to the case $r = 1$. Then $x = x_1$ induces a nonzero divisor in A'_q. Since $(\mathrm{Ann}_{A'} x)A'_q = \mathrm{Ann}_{A'_q} x = 0$ there exists an element $z \in A' \backslash q$ such that $z\, \mathrm{Ann}_{A'} x = 0$ and so x induces a nonzero divisor in A'_z. We have $\mathrm{Ann}_{A'_z} x = \mathrm{Ann}_{A'_z} x^2$ and by Noetherianity $\mathrm{Ann}_{A'} z^t = \mathrm{Ann}_{A'} z^{t+1}$ for a certain positive integer t. Changing z by z^t we get $\mathrm{Ann}_{A'} zx = \mathrm{Ann}_{A'}(zx)^2$ by Lemma 6.

LEMMA 8. *Suppose that* u *is a morphism of Artinian local rings such that the residue field extension* $k \subset K$ *induced by* u *has rank* $\Gamma_{K/k} < \infty$. *Then* A' *is a filtered inductive union of its local sub-*A*-algebras* $C \subset A'$ *essentially of finite type such that the inclusion* $C \hookrightarrow A'$ *is faithfully flat.*

The proof is given at the end.

Let $b = (b_1, \cdots, b_\mu)$ be a system of generators of $H_{B/A}$, $d \in A$ an element and s a positive integer. The B-algebra $B_1 := B[Z]/(d^s - \sum_{i=1}^{\mu} b_i Z_i)$, $Z = (Z_1, \cdots, Z_\mu)$ is called the *containerizer of* B *over* A *with respect to* d, b, s. Given a system of elements $d_1, \cdots, d_r$ *in* A we may speak by recurrence of the *containerizer of* B *over* A *with respect to* $d_1, \cdots, d_r, b, s$.

LEMMA 9 ([4] (2.4)). *Then the following conditions hold*:

i) $d \in H_{B_1/A}$,

ii) $H_{B/A} \subset H_{B_1/B}$ (*in particular* $H_{B/A} \subset H_{B_1/A}$).

iii) *if* $u(d)^s \in f(H_{B/A})A'$ *then* f *extends to an* A*-morphism* $f_1 \colon B_1 \to A'$ *by* $Z \to z$, *where* z *is chosen by* $u(d^s) = \sum_{i=1}^{\mu} f(b_i) z_i$.

Let $B \cong A[Y]/(F)$ be a presentation of B over A and S the symmetric A-algebra associated to $(F)/(F)^2$. We call S the *standardizer of* B *over* A (this notion and the "containerizer" appeared in [4] but Ogoma gave them names [3]). An element $x \in H_{B/A}$ is a *standard element for* the above *presentation of* B *over* A if there exists a system of polynomials $G = (G_1,$

https://doi.org/10.1017/S0027763000002981 Published online by Cambridge University Press

$\cdots, G_s)$ in the ideal (F) such that $x \in \sqrt{\Delta_G((G):(F))}$, where Δ_G is generated by all $s \times s$-minors of $(\partial G/\partial Y)$.

LEMMA 10 ([4] (3.4)). *The following conditions hold:*

i) $H_{B/A} \subset H_{S/B}$ *(in particular* $H_{B/A} \subset H_{S/A}$*),*

ii) *there exists a presentation of* S *over* A *for which all elements from* $H_{B/A}$ *are standard,*

iii) f *extends to an* A*-morphism* $\alpha: S \to A'$ *in a trivial way (by construction* $S = B[Z']/(F')$ *where* F' *is a homogeneous linear system of polynomials, then* α *is given by* $Z' \to 0$*).*

Proof of Theorem 1. We divide the proof in four steps.

Step 1. *Reduction to the case when* $\operatorname{ht}(u^{-1}q) = 0$.

Let $d_1, \cdots, d_t$ be a system of elements from $u^{-1}\mathfrak{a}$ which forms a system of parameters in $A_{u^{-1}q}$, $t = \operatorname{ht}(u^{-1}q)$ (by hypothesis $(u^{-1}\mathfrak{a})A'_{u^{-1}q} = (u^{-1}q)A_{u^{-1}q}$). Apply induction on t. The case $t = 0$ remains for the next steps. If $t > 0$ then by Lemmas 9, 10 there exist a finite type B-algebra $\hat{B}$ and an A-morphism $\beta: \hat{B} \to A'$ extending f such that

1) d_t is a standard element for $\hat{B}$ over A,

2) $H_{B/A} \subset H_{\hat{B}/A}$.

Changing (B, f) by $(\hat{B}, \beta)$ we may suppose that d_t is a standard element for B over A. Let n be the positive integer associated to d_t by [4] (9.2). Then it is enough to show our Lemma for $\tilde{A} := A/(d_t^n)$, $\tilde{A} \otimes_A A'$, $\tilde{A} \otimes_A B, \cdots$. But this follows by induction hypothesis.

Step 2. *Case when* $\operatorname{ht} q = 0$.

Then $A_{u^{-1}q} \to A'_q$ is a regular morphism of Artinian local rings. Thus A'_q is a filtered inductive limit of standard smooth $A_{u^{-1}q}$ algebras by [4] (3.3) and it is enough to apply Lemma 3.

Let $k \subset K$ be the residue field extension induced by $A_{u^{-1}q} \to A'_q$.

Step 3. *Case when* $k \subset K$ *is separable.*

The ring $R := A'_q/(u^{-1}q)A'_q$ is regular by formally smoothness and as $\mathfrak{a}A'_q = qA'_q$ we may choose in $\mathfrak{a}$ a system of elements $x = (x_1, \cdots, x_r)$, $r := \dim R$ which induces in R a regular system of parameters. By Lemma 4 the A-morphism $v: A[X] \to A'$, $X = (X_1, \cdots, X_r) \to x$ induces a flat map $v_q: A[X]_{v^{-1}q} \to A'_q$ and $(v^{-1}q)A[X]_{v^{-1}q} \supset (v^{-1}\mathfrak{a})A[X]_{v^{-1}q} \supset (u^{-1}\mathfrak{a}, X)A[X]_{v^{-1}q} = (u^{-1}q, X)A[X]_{v^{-1}q} = (v^{-1}q)A[X]_{v^{-1}q}$ since $(u^{-1}\mathfrak{a})A_{u^{-1}q} = (u^{-1}q)A_{u^{-1}q}$. Thus $v^{-1}q$ is a minimal prime over-ideal of $v^{-1}\mathfrak{a}$ and v_q is formally smooth because $k \otimes_{A_{u^{-1}q}} v_q$ is exactly the separable (i.e. formally smooth) extension $k \subset K$ (see [1] 43).

Clearly it is enough to show our Lemma for v: $A[X] \to A'$, $B[X]$, f': $B[X] \to A'$ being given by f and v because $A[X]$ is smooth over A and $H_{B[X]/A[X]} \supset H_{B/A}$. By Step 1 it is enough to treat the case when $v^{-1}q$ (and so q) is minimal. But this was done in Step 2.

Step 4. *General case*

Using Step 1 we suppose additionally that $u^{-1}q$ is minimal in A and so $(u^{-1}q)^{\lambda} A_{u^{-1}q} = 0$ for a certain positive integer λ. Choose a positive integer τ such that $q^{\tau} A'_q \subset f(H_{B/A})A'_q$ and let $b = (b_1, \cdots, b_{\mu})$ be a system of generators of $H_{B/A}$. Consider the containerizer B_1 of $B[X]$, $X = (X_1, \cdots, X_r)$ over $A[X]$ with respect to $X_1, \cdots, X_r, b, \tau$ and let B_2 be the standardizer of B_1 over $A[X]$. Then there exists a positive integer c' such that for every i, $X_i^{c'} \in \Delta_{F_i}((F_i): I_i)$ for some representation $B_2 = A[X, U]/I_i$ and some system F_i from I_i. Applying Lemma 8 to $A_{u^{-1}q} \to \tilde{A}' := A'_q/q^n A'_q$, $n := \sup\{\tau, \lambda + rc\}$, $c := 10c'$ we find an essentially of finite type, local sub-A-algebra $\tilde{D}$ of $\tilde{A}'$ containing the image of the composite map $B \xrightarrow{f} A' \to \tilde{A}'$ and such that $\tilde{D} \hookrightarrow \tilde{A}'$ is flat.

Let $k \subset L$ be the residue field extension given by $A_{u^{-1}q} \to \tilde{D}$ and $y = (y_1, \cdots, y_s)$ a system of elements from A' which induces a p-basis $\bar{y}$ of L over k. Clearly we may suppose that y belongs modulo $q^n A'_q$ to $\tilde{D}$. By Lemma 5 the A-morphism $\tilde{w}$: $A[Y] \to A'$, $Y = (Y_1, \cdots, Y_s) \to y$ induces a flat map $A[Y] \to A'_q$ such that $R' := A'_q/(\tilde{w}^{-1}q)A'_q$ is a regular local ring of dimension $t := r - \operatorname{rank} \Gamma_{L/k}$. Choose a system of elements $y' = (y'_1, \cdots, y'_t)$ in q which induces a regular system of parameters in R'. By Lemma 4 the $A[Y]$-morphism w: $A[Y, Y'] \to A'$, $Y' = (Y'_1, \cdots, Y'_t) \to y'$ induces a flat map w_q: $C_{w^{-1}q} \to A'_q$, $C := A[Y, Y']$ such that $(w^{-1}q)A'_q = qA'_q$. We may choose y' such that it belongs modulo $q^n A'_q$ to $\tilde{D}$. Then w_q induces a map η: $\tilde{C} \to \tilde{D}$, where $\tilde{C} := C/(w^{-1}q)^n C$.

Since $\tilde{C} \otimes w_q$ and $\tilde{D} \hookrightarrow A'$ and faithfully flat we get η flat and $\tilde{D}/(w^{-1}q)\tilde{D} \cong L$. The field extension $k(\bar{y}) \subset L$ is finite separable because $k \subset L$ is of finite type and $\bar{y}$ is a p-basis in L/k. Thus η is formally smooth by [1] 43 and so smooth because it is essentially of finite type.

Let $d = (d_1, \cdots, d_r)$ be a system of elements from C inducing a system of parameters in $C_{w^{-1}q}$. But $C_{w^{-1}}q$ is a Cohen Macaulay ring because it is a smooth algebra over an Artinian ring, $A_{u^{-1}q}$. Then d is regular in

$C_{w^{-1}q}$ and changing d_i, $1 \leq i \leq r$ by some of their multiples we may suppose by Corollary 7 that

$$((d_1^c, \cdots, d_{i-1}^c) : d_i) = ((d_1^c, \cdots, d_{i-1}^c) : d_i^2)$$

for all $1 \leq i \leq r$, where $d_0 := 0$.

The linear equation

$$(*) \qquad w(d_i^\tau) = \sum_{j=1}^{\mu} f(b_j) Z_{ji}$$

has sure a solution $z_i = (z_{1i}, \cdots, z_{\mu i})$ in A'_q because $q^\tau A'_q \subset f(H_{B/A})A'_q$ and we claim that we may choose z_i such that it belongs modulo q^n to $\tilde{D}$. Indeed, $(*)$ has a solution $\tilde{z}_i$ in $\tilde{D}$ because $\tilde{D} \hookrightarrow \tilde{A}'$ is faithfully flat, let us say $\tilde{z}_i$ is induced by a system of elements $\hat{z}_i$ from A'_q. Changing Z_{ji} by $\hat{Z}_{ji} + \hat{z}_{ji}$ it remains to show that

$$\rho_i := w(d_i^\tau) - \sum_j f(b_j)\hat{z}_{ji} = \sum_j f(b_j)\hat{Z}_{ji}$$

has a solution in A'_q. But this is trivial because $\rho_i \in q^n A'_q \subset q^\tau A'_q \subset f(H_{B/A})A'_q$.

Let $\delta = (\delta_1, \cdots, \delta_r)$ be a system of elements in $A' \backslash q$ such that $\delta_i z_i \in A'^\mu$ for all i, $1 \leq i \leq r$. By flatness $w(d)$ induces a regular system in A'_q and so changing δ_i, $1 \leq i \leq r$ by some of their multiples we may suppose by Corollary 7 that

$$((\delta_1^c w(d_1^c), \cdots, \delta_{i-1}^c w(d_{i-1}^c)) : \delta_i w(d_i)) = ((\delta_1^c w(d_1^c), \cdots, \delta_{i-1}^c w(d_{i-1}^c)) : \delta_i^2 w(d_i^2))$$

for all $1 \leq i \leq r$.

Consider the A-morphism ε: $A[X] \to C[T]$, $T = (T_1, \cdots, T_r)$ given by $X_i \to d'_i := T_i d_i$ and the C-morphism w': $C[T] \to A'$, $T \to \delta$. The correspondence $Z_i \to \delta_i z_i$, $1 \leq i \leq r$ defines a $C[T]$-morphism $C[T] \otimes_{A[X]} B_1 \to A'$ (see Lemma 9) which extends trivially ($Z' \to 0$, see Lemma 10) to a $C[T]$-morphism β_2: $B'_2 \to A'$, where $B'_2 := C[T] \otimes_{A[X]} B_2$. Note that $\tilde{C} \otimes_{C[T]} \beta_2$ factorizes through $\tilde{D}[T]$ because z belongs modulo $q^n A'_q$ to $\tilde{D}$.

Since $\tilde{D}[T]$ is smooth over $\tilde{C}[T]$ our Theorem holds for $\tilde{B}_2 := \tilde{C} \otimes_{C[T]} B'_2$, $\tilde{C} \otimes \beta_2$ with respect to $\tilde{C} \otimes w'$ (see Lemma 2). Note that

$$q^n A'_q = (u^{-1}q, d')^n A'_q \subset (d')^{rc} A'_q \subset (d_1'^c, \cdots, d_r'^c) A'_q$$

and so applying Lemma 3 and by recurrence [4] (9.1) for d'_i and $e = 2$ we get a finite type B'_2-algebra B' and a B'_2-morphism h: $B' \to A'$ such that

$$h(H_{B/A}B_2') \subset h(H_{B_2'/C[T]}) \subset \sqrt{h(H_{B'/C[T]})A_q'} \not\subset q\,.$$

As $C[T]$ is smooth over A we are ready.

Proof of Lemma 8. Given a field L consider the Cohen ring R_L of residue field L, i.e. L if $p := \text{char } L = 0$ or a complete DVR of residue field L which is an unramified extension of $Z_{(p)}$. By Cohen Structure Theorem we have $A \cong R_k[X]/\mathfrak{a}$, $A' \cong R_K[Y]/\mathfrak{a}$, $X = (X_1, \cdots, X_r)$, $Y = (Y_1, \cdots, Y_s)$. Let $\mathscr{L}$ be the set of all subfields L, $k \subset L \subset K$ such that $k \subset L$ is of finite type and a' is defined over R_L, i.e. a' is an extension of $a_L' := a' \cap R_L[Y]$ to $R_K[Y]$. Then A' is a filtered inductive union of $D_L := R_L[Y]/a_L'$, $L \in \mathscr{L}$ and by base change $D_L \hookrightarrow D_K = A'$ is flat. Since A is a finite type R_k-algebra it is enough to show that there exists $L \in \mathscr{L}$ such that D_L contains $u(\bar{R}_k)$, where $\bar{R}_k := R_k/(p^r)$ and $(p^r) = a \cap R_k$.

If $k \subset K$ is separable then we may suppose that u extends the map $\bar{R}_k \to \bar{R}_K$ and our wish is trivially fulfilled. Otherwise, by [2] (2.14) there exist two subfields $E \subset F \subset k$ such that

1) $E \subset F$ is of finite type,
2) $E \subset K$ is separable,
3) $F \subset k$ is etale.

As above we may suppose that $u|_{\bar{R}_E}$ extends the map $\bar{R}_E \to \bar{R}_K$ because of 2) and so every D_L, $L \in \mathscr{L}$ contains $u(\bar{R}_E)$. Since $\bar{R}_F$ is essentially of finite type over $\bar{R}_E$ we can choose $L' \in \mathscr{L}$ such that $D_{L'}$ contains $u(\bar{R}_F)$.

We claim that $u(\bar{R}_k) \subset \mathrm{D}_{L'}$. Indeed as $\bar{R}_F \hookrightarrow \bar{R}_k$ is smooth there exists a ring morphism $v\colon \bar{R}_k \to \mathrm{D}_{L'}$, extending $u|_{\bar{R}_F}$ which lifts the composite map $\bar{R}_k \to k \hookrightarrow L'$. Then $u|_{\bar{R}_k}$ and the composite map v', $\bar{R}_k \xrightarrow{v} D_{L'} \hookrightarrow A'$ lift both the map $\bar{R}_k \to k \to K$. Thus $u = v'$ (in particular $D_{L'} \supset u(\bar{R}_k)$) because $\bar{R}_F \subset \bar{R}_k$ is also etale.

References

[1] Matsumura, H., Commutative Algebra, Benjamin, New-York, 1980.
[2] Nica, V. and Popescu, D., A structure Theorem on formally smooth morphisms in positive characteristic, J. Algebra, **100** (1986), 436–455.
[3] Ogoma, T., General Néron Desingularization based on idea of Popescu, Preprint 1988.
[4] Popescu, D., General Néron desingularization, Nagoya Math. J., **100** (1985), 97–126.
[5] ——, General Néron desingularization and approximation, Nagoya Math. J., **104** (1986), 85–115.

[6] Raynaud, M., Anneaux henséliens et approximations, in Colloque d'Algébre de Rennes, Publ. Sem. Math. Univ. Rennes, 1972.

Department of Mathematics
INCREST
Bdul Păcii 220, 79622
Bucharest, Romania