MATRIX CHARACTERIZATIONS OF TOPOLOGICAL PROPERTIES

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1. Introduction. In [S], H. Sharp characterizes each topology on a finite set $S = \{s_1, s_2, \ldots s_n\}$ with a $n \times n$ zero-one matrix $T = (t_{ij})$ where $t_{ij} = 1$ if and only if $s_j \epsilon \{\overline{s_i}\}$. In this paper we seek matrix characterizations of certain topological properties of finite spaces. Such characterizations will provide purely mechanical ways of determining if a space has a certain topological property.

We give matrix characterizations of the T_Y separation axiom of Youngs [Y]; the R_0 and R_1 separation axioms of Davis [D], the strong T_0 separation axiom of Robinson and Wu [RW], and the six separation axioms introduced by Aull and Thron [AT]. Also, we include matrix characterizations of regular, completely regular, normal, completely normal, 0-dimensional, and extremally disconnected spaces.

2. Preliminaries. If $(S, \mathcal{T})$ is a finite topological space and if T is the matrix corresponding to $\mathcal{T}$, we will denote the space $(S, \mathcal{T})$ by (S, T). For each $s_i \epsilon S$, we let $F_i = \{\overline{s_i}\}$ and B_i be the minimal open set containing s_i. For each $A \subset S$, let $B_A = \cup B_i$ $(s_i \epsilon A)$; clearly, B_A is the minimal open set containing A. A useful fact proven in [S] is that $B_i \subset B_j$ if and only if $F_j \subset F_i$. The identity matrix is denoted by **1**.

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For each $s_i \in S$ we associate the $1 \times n$ row vector $\varepsilon_i = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{ni})$ where δ_{ij} is the Kronecker delta. For each $A \subset S$ we associate the vector $A_v = \Sigma\, \epsilon_i\ (s_i \in A)$. In particular, the ith row of T, T_i, is $(\{\bar{s}_i\})_v$, and the ith column of T, T^i, is $(B_i)_v$.

THEOREM 2.1. Let A be a subset of a finite topological space (S, T), then $(\bar{A})_v = A_v \cdot T$ where matrix multiplication is with respect to Boolean algebra.

Proof. Since $\bar{A} = \bigcup \{\bar{s}_i\} (s_i \in A)$, we need only show that $(\{\bar{s}_i\})_v = \{s_i\}_v \cdot T$; that is, $T_i = \varepsilon_i \cdot T$ since $\{s_i\}_v = \varepsilon_i$ and $(\{\bar{s}_i\})_v = T_i$. Let V_j denote the jth entry of $\varepsilon_i \cdot T$.

Thus, $V_j = \sum_{k=1}^{n} \delta_{ki} t_{kj} = t_{ij}$ and $T_i = \varepsilon_i \cdot T$.

THEOREM 2.2. Let A be a subset of a finite topological space (S, T), then $T \cdot (A_v)' = ((B_A)_v)'$ where matrix multiplication is with respect to Boolean algebra.

Proof. The proof is similar to the proof of 2.1.

COROLLARY 2.3. Let A be a subset of a finite topological space (S, T). A is open if and only if $T \cdot (A_v)' = (A_v)'$ where matrix multiplication is with respect to Boolean algebra.

Let $\mathcal{T}' = \{U \subset S \mid S - U \in \mathcal{T}\}$; it is straightforward to prove that $\mathcal{T}'$ is a topology. Sharp [S] has proved that the matrix associated with $\mathcal{T}'$ is the transpose of T, T'. The conclusion of 2.2 may be restated as $A_v \cdot T' = (B_A)_v$.

3. Separation Axioms. Let $(X, \mathcal{T})$ be a topological space (not necessarily finite). $(X, \mathcal{T})$ is T_Y if and only if for x, y in X, $x \neq y$, $\{\bar{x}\} \cap \{\bar{y}\}$ is degenerate ($\{\bar{x}\} \cap \{\bar{y}\}$ is either empty or a singleton). The next six separation axioms were introduced in [AT]. $(X, \mathcal{T})$ is T_D if and only if for each x in X, $\{x\}'$ is a closed set. $(X, \mathcal{T})$ is T_{UD} if and only if for each $x \in X$, $\{x\}'$ is the union of disjoint closed sets.

(X, τ) is T_{DD} if and only if (X, τ) is T_D and for every x, y in X, $x \neq y$, $\{x\}' \cap \{y\}' = \phi$. (X, τ) is T_F if and only if for each x in X and disjoint finite set F, either $x \notin \bar{F}$ or $F \cap \overline{\{x\}} = \phi$. (X, τ) is T_{FF} if and only if given any two disjoint finite sets F_1 and F_2 in X, either $F_1 \cap \bar{F}_2 = \phi$ or $\bar{F}_1 \cap F_2 = \phi$. (X, τ) is said to be T_{YS} if and only if for all $x, y \in X$, $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is either ϕ, $\{x\}$, or $\{y\}$. To this list we add three additional separation axioms. For each x in X, let $\{\hat{x}\}$ denote the intersection of all open sets in (X, τ) containing x. (X, τ) is T_γ if and only if x, y in X, $x \neq y$, $\{\hat{x}\} \cap \{\hat{y}\}$ is either ϕ, $\{x\}$, or $\{y\}$. (X, τ) is T_α if and only if $\{\hat{x}\} - \{x\}$ is empty for all but at most one x in X. (X, τ) is T_β if and only if $\{x\}'$ is empty for all but at most one x in X. (X, τ) is $T_{\alpha\beta}$ if and only if it is both T_α and T_β. The strong T_D (denoted by T_{SD}) and the strong T_0 (denoted by T_{S0}) were introduced in [WR]. (X, τ) is T_{S0} if and only if for each $x \in X$, $\{x\}' = \bigcup \overline{\{y\}}$ $(y \in \{x\}')$, $\phi = \bigcap \overline{\{y\}}$ $(y \in \{x\}')$, and $\overline{\{y\}}$ is compact for some $y \in \{x\}'$. (X, τ) is T_{SD} if and only if for each $x \in X$, $\{x\}'$ is the union of a finite family of closed sets, such that the intersection of the non-empty members of this family is empty. (X, τ) is R_0 if and only if every open set contains the closure of each of its points. (X, τ) is R_1 if and only if for every pair of points x, y in X, $\overline{\{x\}} \neq \overline{\{y\}}$ implies that $\overline{\{x\}}$ and $\overline{\{y\}}$ have disjoint neighborhoods. Aull and Thron [AT] proved that a space is T_{FF} if and only if it is either T_α or T_β.

In finite topological spaces, some of the above axioms become equivalent. In [AT], T_0 was shown to be equivalent to T_D and T_{UD}. From the definitions we observe that T_{S0} and T_{SD} are equivalent. In theorem 3.6 we show that R_0 is equivalent to R_1. In [AT], a space satisfying T_{DD} was proven to satisfy T_{YS}, and since T_0 is equivalent to T_D in finite spaces, it follows immediately that T_{DD} and T_{YS}

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are equivalent. Figure 1 shows the order relationship between the separation axioms for finite spaces.

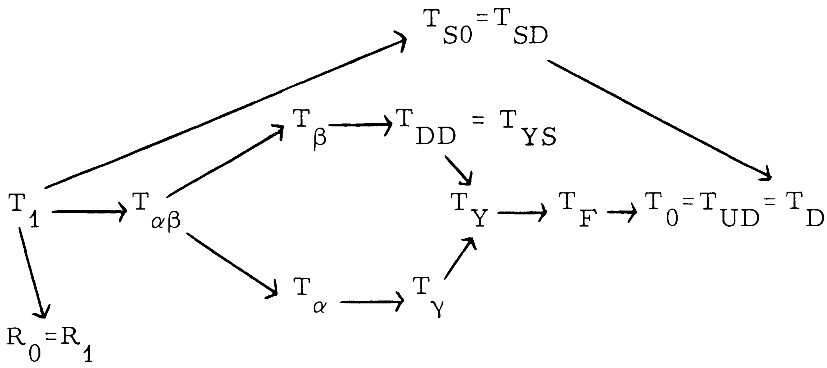

FIGURE 1

It is clear that (S, T) is T_1 if and only if T is the identity matrix. Sharp [S] proved that (S, T) is T_0 if and only if T is anti-symmetric.

THEOREM 3.1. In a finite topological space (S, T), the following are equivalent:

(a) (S, T) is T_{S0},

(b) (S, T) is T_0 and the derived set of any singleton is never a singleton, and

(c) T is anti-symmetric and no ith row of T-1 is a unit vector.

<u>Proof</u>. Clearly, (b) is equivalent to (c) and (a) implies (b). We need only show that (b) implies (a). Suppose that (S, T) is T_0 and the derived set of any singleton is never a singleton. Let $N = \{a \in S \mid \{a\}' = \{a_1, \ldots, a_m\}$ where $m \geq 2$ and $\bigcap_{j=1}^{m} \{\bar{a}_j\} \neq \phi\}$. We will show $N = \phi$ by an induction proof on the cardinality of the derived set of any element of N. First, assume that there is an element in N, say a, such that the cardinality of $\{a\}' = 2$; so $\{a\}' = \{a_1, a_2\}$ and $\bigcap_{i=1}^{2} \{\bar{a}_i\} \neq \phi$.

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Without loss of generality we may assume $a_1 \in \overline{\{a_2\}}$. Since (S, T) is T_0, $\{a\}'$ is closed; so, $\overline{\{a_2\}} = \{a_1, a_2\}$ implying that $\{a_2\}' = \{a_1\}$ which is a contradiction to the fact that no derived set of any singleton of S is a singleton. We conclude that no element of N has a two element derived set. Now suppose that there is no $a \in N$ such that $2 \leq$ cardinality of $\{a\}' < m$. If $a \in N$ with the cardinality of $\{a\}' = m$, we would have $\{a\}' = \bigcup_{j=1}^{m} \overline{\{a_j\}}$ and $\bigcap_{j=1}^{m} \overline{\{a_j\}} \neq \phi$. Since $\bigcap_{j=1}^{m} \overline{\{a_j\}} \neq \phi$, there is an a_k in $\overline{\{a_i\}}$ for each i. Let $j \in \{1, 2, \ldots, m\} - \{k\}$. $a_k \in \{a_j\}'$, so $\{a_j\}' \neq \phi$. Since no derived set of any singleton is a singleton, we know that the cardinality of $\{a_j\}' \geq 2$. Since $\{a\}'$ is closed, then $\{a_j\}' \subset \{a\}' - \{a_j\}$. Therefore, $2 \leq$ cardinality of $\{a_j\}' \leq m-1$. Also, $a_k \in \bigcap_{b \in \{a_j\}'} \overline{\{b\}}$. Thus, $a_k \in N$ which is a contradiction to the induction hypothesis. Hence, $N = \phi$ and (S, T) is T_{S0}.

THEOREM 3.2. Let (S, T) be a finite topological space. The following are equivalent:

(a) (S, T) is T_F,

(b) for each s_i in S, either $F_i = \{s_i\}$ or $B_i = \{s_i\}$, and

(c) for each i, either the ith row or the ith column of T-1 is zero.

Proof. (b) is clearly equivalent to (c). It remains to show that (a) is equivalent to (b). Suppose (S, T) is T_F, and let $s_i \in S$. Since $s_i \notin S-\{s_i\}$, we have that either $\overline{\{s_i\}} \cap (S-\{s_i\}) = \phi$ or $\{s_i\} \cap \overline{S-\{s_i\}} = \phi$. Thus, $\{s_i\} = F_i$ or $\{s_i\} = B_i$. Clearly (b) implies (a).

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DEFINITION. Let $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ be vectors with n real-valued components. The intersection of v and w is $w \wedge v = (\min(v_1, w_1), \min(v_2, w_2), \ldots, \min(v_n, w_n))$.

THEOREM 3.3. Let (S, T) be a finite topological space. The following are equivalent:

(a) (S, T) is T_Y,

(b) the intersection of T_i and T_j has at most one non-zero entry for all $i \neq j$, and

(c) the matrix $T \cdot T'$, with respect to ordinary multiplication, is zero or one everywhere except possibly on the diagonal.

Proof. Since the intersection of T_i and T_j is the intersection of $(\{\overline{s_i}\})_v$ and $(\{\overline{s_j}\})_v$ which is $(\{\overline{s_i}\} \cap \{\overline{s_j}\})_v$, then (a) is equivalent to (b). Clearly, (b) is equivalent to (c).

THEOREM 3.4. Let (S, T) be a finite topological space.

(a) (S, T) is T_{DD} if and only if T is anti-symmetric and $(T-1)(T-1)'$ is a diagonal matrix.

(b) (S, T) is T_Y if and only if T is anti-symmetric and $(T-1)'(T-1)$ is a diagonal matrix.

Proof of (a). Suppose T is anti-symmetric and $(T-1)(T-1)' = [v_{ij}] = T^*$ is a diagonal matrix; that is, $v_{ij} = \sum_{k=1}^{n} t_{ik} \cdot t_{jk} = 0$ for $i \neq j$; therefore, for each k, $t_{ik} \cdot t_{jk} = 0$, implying that $t_{ik} = 0$ or $t_{jk} = 0$. So, for each k, $s_k \notin \{s_i\}'$ or $s_k \notin \{s_j\}'$, thus giving $\{s_i\}' \cap \{s_j\}' = \phi$. Since T is anti-symmetric $s_i \notin \{s_j\}'$ or $s_j \notin \{s_i\}'$. Thus, (S, T) is T_{DD}. Conversely, suppose (S, T) is T_{DD}. For $i \neq j$, $\{s_i\}' \cap \{s_j\}' = \phi$. For each k, $s_k \notin \{s_i\}'$ or $s_k \notin \{s_j\}'$ implying that $t_{ik} = 0$ or $t_{jk} = 0$. So, $t_{ik} \cdot t_{jk} = 0$ and $v_{ij} = 0$ for $i \neq j$. Thus, $T^* = (T-1)(T-1)'$

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is a diagonal matrix, and T is anti-symmetric since T_{DD} implies T_0.

Proof of (b). (b) follows from an argument similar to that of (a).

THEOREM 3.5. Let (S, T) be a finite topological space.

(a) (S, T) is T_β if and only if (T-1) has at most one non-zero row.

(b) (S, T) is T_α if and only if (T-1)' has at most one non-zero row.

(c) (S, T) is $T_{\alpha\beta}$ if and only if both (T-1) and (T-1)' have at most one non-zero row.

(d) (S, T) is T_{FF} if and only if (T-1) or (T-1)' has at most one non-zero row.

Proof. To prove (a), note that since the ith row of T - 1 is $(\{s_i\}')_v$, then T-1 has at most one non-zero row if and only if $\{s_i\}' \neq \phi$ for at most one s_i in S which is equivalent to (S, T) being T_β. (b) and (c) follow similarly. (d) follows since (S, T) is T_{FF} if and only if it is T_α or T_β.

We now show that R_0 is equivalent to R_1 in finite spaces and give a matrix characterization. Also, we prove that R_0 is equivalent to 0-dimensional, regular, and completely regular.

THEOREM 3.6. Let (S, T) be a finite topological space. The following are equivalent;

(a) T is a symmetric matrix,

(b) (S, T) is 0-dimensional,

(c) (S, T) is completely regular,

(d) (S, T) is regular,

(e) (S, T) is R_1, and

(f) (S, T) is R_0.

Proof. It is well known that (b) implies (c), (c) implies (d), and (e) implies (f).

(f) implies (e): Let (S, T) be R_0. Let $s_i \in S$, then $s_i \in \{\bar{s_i}\} = F_i \subset B_i$. If $s_j \in B_i - F_i$, then $F_j \subset B_i - F_i$, so $B_i - F_i$ is an open set containing s_j and properly contained in B_i which is a contradiction. So, $F_i = B_i$. If $F_i \cap F_j \neq \phi$, then $s_k \in F_i \cap F_j$. By a similar argument $B_k = F_j$ and $B_k = F_i$, so, $F_i = F_j$. Thus, if $F_i \neq F_j$, then $F_i \cap F_j = \phi$. Since $B_i = F_i$, $B_j = F_j$ and $B_i \cap B_j = \phi$, we have proven that (S, T) is R_1.

(f) implies (a): Suppose (S, T) is R_0 and $s_i \in S$. By the argument presented in the proof that (f) implies (e), we have proven that $F_i = B_i$. Hence, $B_i = S-(S-F_i) \in \mathcal{T}'$ and $\mathcal{T} \subset \mathcal{T}'$. If $A \in \mathcal{T}'$, then $S-A \in \mathcal{T}$. $S-A = \bigcup_{i \in \Lambda} B_i$ for some subset Λ of $\{1, 2, 3, \ldots, n\}$. Therefore,

$A = \bigcap_{i \in \Lambda} S-B_i = \bigcap_{i \in \Lambda} S-F_i$. which is in $\mathcal{T}$. So, $\mathcal{T}' \subset \mathcal{T}$ which proves that $\mathcal{T} = \mathcal{T}'$ and, hence, T is symmetric.

(a) implies (b): If T is symmetric, then $B_i \in \mathcal{T}'$ for all $s_i \in S$. So, $S-B_i \in \mathcal{T}$ and B_i is closed. Since B_i is smallest open set containing s_i, (S, T) is zero dimensional.

(d) implies (f): Suppose (S. T) is regular and $U \in \mathcal{T}$ with $s_i \in U$. There is a V in $\mathcal{T}$ such that $s_i \in V \subset \bar{V} \subset U$, so $\{\bar{s_i}\} \subset U$ and (S, T) is R_0.

THEOREM 3.7. Let (S, T) be a finite topological space. The following are equivalent:

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(a) (S, T) is normal,

(b) for each F_i, B_{F_i} is closed, and

(c) $((F_i)_v \cdot T') \cdot T = (F_i)_v \cdot T'$.

Proof. By 2.1 and the comment following 2.3, (b) is equivalent to (c). Clearly, (b) implies (a). To prove (a) implies (b), suppose (S, T) is normal. Let $s_i \in S$; there is an open set V such that $F_i \subset V \subset \bar{V} \subset B_{F_i}$. Since B_{F_i} is the smallest open set containing F_i, then $V = \bar{V} = B_{F_i}$ and B_{F_i} is closed.

THEOREM 3.8. Let (S, T) be a finite topological space. Let $T' \cdot T = (t^*_{ij})$ where multiplication is with respect to Boolean algebra. The following are equivalent:

(a) (S, T) is completely normal,

(b) $s_i \notin F_j$, $s_j \notin F_i$ imply $B_i \cap B_j = \phi$, and

(c) $t^*_{ij} = 1$ implies $t_{ij} = 1$ or $t_{ji} = 1$.

Proof.

(a) implies (b): Suppose (S, T) is completely normal, and suppose $s_i \notin F_j$ and $s_j \notin F_i$. So, $\{s_i\} \cap \{\bar{s}_j\} = \phi$ and $\{\bar{s}_i\} \cap \{s_j\} = \phi$. By complete normality, $B_i \cap B_j = \phi$ since B_i and B_j are the smallest open sets containing s_i and s_j, respectively.

(b) implies (c): Suppose (b) is true and $t^*_{ij} = 1$; there is a k such that $t_{ki} = 1$ and $t_{kj} = 1$. So, $B_i \cap B_j \neq \phi$; hence, either $s_i \in F_j$ or $s_j \in F_i$ implying $t_{ji} = 1$ or $t_{ij} = 1$, respectively.

(c) implies (a): Suppose (c) is true and $C, D \in S$ such that $C \cap \bar{D} = \bar{C} \cap D = \phi$. Let $s_i \in C$ and $s_j \in D$; $s_i \notin F_i$ and $s_j \notin F_i$. So, $t_{ij} = t_{ji} = 0$. By (c), $t^*_{ij} = 0$. Hence, $t_{kj} = t_{ki} = 0$

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for all k implying $B_i \cap B_j = \phi$. Since this is true for any $s_i \in C$ and any $s_j \in D$, then $B_C \cap B_D = \phi$. This proves that (S, T) is completely normal.

4. Connectedness. In [S], Sharp gives a matrix characterization of connectedness. Clearly, in finite spaces, totally disconnected is equivalent to T_1. Theorem 3.6 gives a matrix characterization of 0-dimensional. We conclude the article by giving a matrix characterization of extremally disconnected.

THEOREM 4.1. Let (S, T) be a finite topological space. (S, T) is extremally disconnected if and only if for each B_i in S, $T \cdot ((\overline{B_i})_v)' = ((\overline{B_i})_v)'$.

Proof. Let (S, T) be extremally disconnected; that is the closure of each open set is also open. By 2.3, we have that $T \cdot ((\overline{B_i})_v)' = ((\overline{B_i})_v)'$. Conversely, suppose that for each B_i in S, $T \cdot ((\overline{B_i})_v)' = ((\overline{B_i})_v)'$. By 2.3, $\bar{B}_i$ is open. Let U be an open subset of S, $U = \bigcup_{s_i \in U} B_i \cdot \bar{U} = \bigcup_{s_i \in U} \bar{B}_i$.

Since each B_i is open, $\bar{U}$ is open; thus, (S, T) is extremally disconnected.

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