RIGHT INVARIANT INTEGRALS ON LOCALLY COMPACT SEMIGROUPS

J. H. MICHAEL

(received 11 August 1963)

1. Introduction

An integral on a locally compact Hausdorff semigroup S is a non-trivial, positive, linear functional μ on the space $\mathscr{F}$ of continuous real-valued functions on S with compact supports. If S has the property:

(A) for each pair of compact sets C, D of S, the set

$$CD^{-1} = \{x; x \in S \text{ and } xy \in C \text{ for some } y \in D\} = \bigcup_{y \in D} Cy^{-1}$$

is compact; then, whenever $f \in \mathscr{F}$ and $a \in S$, the function f_a, defined by

$$f_a(x) = f(xa)$$

is also in $\mathscr{F}$. An integral μ on a locally compact semigroup S with the property (A) is said to be right invariant if

$$\mu(f) = \mu(f_a)$$

for all $f \in \mathscr{F}$ and all $a \in S$.

It is shown (5.1) that if S is a separable, metric, locally compact semigroup with the properties: (A),

(B) for each pair of compact sets C, D of S, the set

$$D^{-1}C = \{x; x \in S \text{ and } yx \in C \text{ for some } y \in D\} = \bigcup_{y \in D} y^{-1}C$$

is compact;

(C) S has a left ideal K that is contained in every other left ideal; then S admits a right invariant integral.

One of the results of W. G. Rosen [6] is that a necessary and sufficient condition for a compact, Hausdorff semigroup to admit a right invariant integral is the existence of a unique minimal left ideal. Since compact metric semigroups automatically satisfy conditions (A) and (B), the above theorem (except for the restriction to separable metric semigroups) is a generalisation

https://doi.org/10.1017/S1446788700024034 Published online by Cambridge University Press

of the sufficiency part of Rosen's result to locally compact semigroups. However, the proof is not a generalisation, because I have to assume that the semigroup is not compact. It is not known whether the condition (A), (B) and (C) are necessary.

It will frequently be necessary to assume that the semigroups under consideration have one or both of the properties (A) and (B). Whenever this is so, it will always be stated.

2. Some preliminary theorems

In this section some of the properties of the inverse image Ax^{-1} are investigated. Some results, that are needed later, are obtained. It is assumed throughout 2, that S is a locally compact Hausdorff semigroup with the property (A).

THEOREM 2.1. *If F is a closed subset of S and C is compact, then FC is closed.*

PROOF. Let y be a point of accumulation of FC. Let D be a compact neighbourhood of y. Since

$$(FC) \cap D \subset \{F \cap (DC^{-1})\}C,$$

y is a point of accumulation of the compact set $\{F \cap (DC^{-1})\}C$, so that $y \in FC$.

LEMMA 2.2. *If F is a closed subset of S, C is a compact subset and b is an element of S such that Fb does not intersect C, then there exists an open neighbourhood U of b such that FU does not intersect C.*

PROOF. (i) Assume first of all that F is compact. For each point ξ of F, $\xi b \notin C$, hence we can let $P(\xi)$, $Q(\xi)$ be open neighbourhoods of ξ, b respectively, such that

$$P(\xi)Q(\xi) \subset S \sim C.$$

Since F is compact, there exists a finite number $\xi_1, \cdots, \xi_p$ of ξ's so that

$$F \subset \bigcup_{i=1}^{p} P(\xi_i).$$

Put

$$U = \bigcap_{i=1}^{p} Q(\xi_i).$$

Then $FU \subset S \sim C$.

(ii) Now let F be an arbitrary closed set. Let D be a compact neighbourhood of b and put $E = CD^{-1}$. By (i), there exists an open neighbour-

https://doi.org/10.1017/S1446788700024034 Published online by Cambridge University Press

hood U of b such that $U \subset D$ and $(E \cap F)U$ does not intersect C. But $(F \sim E)U$ does not intersect C, hence FU does not intersect C.

THEOREM 2.3. *If C is a compact subset of S, $a \in S$ and U is an open set containing Ca^{-1}, then there exist open sets V, W containing C, a respectively, and such that $VW^{-1} \subset U$.*

PROOF. By 2.2, there exists a compact neighbourhood D of a such that $(S \sim U)D$ does not intersect C. By 2.1, $(S \sim U)D$ is closed, hence there is an open set V containing C and not intersecting $(S \sim U)D$. Put $W = \text{Int}\,(D)$.

Let ξ be an arbitrary element of VW^{-1}. There is a $\zeta \in W$ with $\xi\zeta \in V$. If $\xi \notin U$, then $\xi \in S \sim U$, hence $\xi\zeta \in (S \sim U)D$, so that $\xi\zeta \notin V$; a contradiction. Thus $VW^{-1} \subset U$.

THEOREM 2.4. *Let S also have the property* (B) *and suppose that S is not compact. Then, for every compact subset C of S, there exist elements x, y of S such that Cx^{-1}, Cy^{-1} are disjoint.*

PROOF. Suppose that C is a compact subset of S with Cx^{-1} intersecting Cy^{-1} for all $x, y \in S$. Let $a \in S$ and put $D = Ca^{-1}$. Then D is compact and Cx^{-1} intersects D for all $x \in S$. Hence $x \in D^{-1}C$ for all $x \in S$, so that $S \subset D^{-1}C$, from which it follows that S is compact; a contradiction.

3. The construction of a special metric

In this section a separable, metrizable, locally compact semigroup S with the property (B) is considered. It is shown (3.3), that there exists a metric d for S such that

$$d(xz, yz) \leqq d(x, y) \tag{1}$$

for all $x, y, z \in S$. In the special case where S is a group the existence of the inverse z^{-1} implies equality in (1), a well-known result (cf. [1, 3 and 5]).

THEOREM 3.1. *Let X be a locally compact metric space. The one point compactification of X is metrizable if and only if X is separable.* (This answers Advanced Problem 5058, American Mathematical Monthly 69 (Dec. 1962), 1012.)

PROOF. Since every compact metric space is separable, the "only if" part of the theorem is trivial. Assume X is separable and denote its one point compactification by X'. There exists an increasing sequence $\{C_r\}$ of compact subsets of X such that

$$X = \liminf_{r \to \infty} (C_r).$$

https://doi.org/10.1017/S1446788700024034 Published online by Cambridge University Press

Hence there exists a countable system of neighbourhoods of ∞ in X'. Then X' has a countable base for its topology. Since X' is T_1 and regular, then by [4, Theorem 17, p. 125] X' is metrizable.

THEOREM 3.2. *If S has a metric ρ for its topology such that: for every $\varepsilon > 0$, there exists a compact subset C of S with $\rho(x, y) < \varepsilon$ for all $x, y \in S \sim C$, then there exists a metric d for S such that*

$$d(xz, yz) \leqq d(x, y)$$

for all $x, y, z \in S$.

PROOF. It can be assumed that ρ is bounded because otherwise it could be replaced by $\rho_1(x, y) = \rho(x, y)/\{1+\rho(x, y)\}$. Define

$$d_1(x, y) = \sup_{w \in S} \rho(xw, yw).$$

Then

$$d_1(x, y) = d_1(y, x)$$
$$d_1(x, y) \leqq d_1(x, u)+d_1(u, y)$$

and

$$\begin{aligned} d_1(xz, yz) &= \sup_{w \in S} \rho(xzw, yzw) \\ &= \sup_{zw \in zS} \rho(xzw, yzw) \leqq d_1(x, y). \end{aligned}$$

Now define

$$d(x, y) = \max \{\rho(x, y), d_1(x, y)\}.$$

Then d satisfies the triangle inequality, $d(x, y) = d(y, x)$ and

$$\begin{aligned} d(xz, yz) &= \max \{\rho(xz, yz), d_1(xz, yz)\} \\ &\leqq d_1(x, y) \leqq d(x, y). \end{aligned}$$

Obviously, $d(x, y) \geqq \rho(x, y)$. Thus d is a metric. It remains to prove that d generates the topology of S.

Take $\varepsilon > 0$. Let C be a compact subset of S such that

$$\rho(x, y) < \varepsilon \tag{1}$$

for all $x, y \in S \sim C$. Let x' be an arbitrary point of S. Let D be a compact neighbourhood of x'. Then $D^{-1}C$ is a compact subset of S, hence $D \times (D^{-1}C)$ is compact. Therefore, the function

$$f(x, w) = xw$$

is uniformly continuous on $D \times (D^{-1}C)$ (with respect to ρ), hence there exists an $\eta > 0$ and such that

$$\rho(xw, x'w) < \varepsilon \tag{2}$$

https://doi.org/10.1017/S1446788700024034 Published online by Cambridge University Press

for all $x \in D$ with $\rho(x, x') < \eta$ and all $w \in D^{-1}C$. Let $\delta > 0$ be smaller than η, ε and such that the set

$$U = \{x; x \in S \text{ and } \rho(x, x') < \delta\}$$

is contained in D. Let x be an arbitrary point of U and w be an arbitrary point of S. If neither of xw, $x'w$ is in C, then by (1), $\rho(xw, x'w) < \varepsilon$. If one or both of xw, $x'w$ is in C, then $w \in D^{-1}C$, so that by (2), $\rho(xw, x'w) < \varepsilon$. Thus $d_1(x, x') < \varepsilon$ for all $x \in S$ with $\rho(x, x') < \delta$ and since $\delta < \varepsilon$, $d(x, x') < \varepsilon$ for all $x \in S$ with $\rho(x, x') < \delta$. Thus d generates the topology of S.

THEOREM 3.3. *There exists a metric d for S such that*

$$d(xz, yz) \leqq d(x, y)$$

for all $x, y, z \in S$.

PROOF. Let X be the one point compactification of S (X is a topological space, not a semigroup). By 3.1, there exists a metric ρ, generating the topology of X. For each $\varepsilon > 0$, there exists a compact subset C of S such that $\rho(x, \infty) < \frac{1}{2}\varepsilon$ for all $x \in S \sim C$. Then $\rho(x, y) < \varepsilon$ for all $x, y \in S \sim C$.

Thus the restriction of ρ to $S \times S$ is a metric for S satisfying the hypotheses of 3.2, hence there is a metric d for S such that $d(xz, yz) \leqq d(x, y)$ for all $x, y, z \in S$.

4. The existence of a right invariant integral on a left simple semigroup

Throughout 4, S will be a separable metrizable locally compact semigroup with the properties (A) and (B). It is assumed that S is not compact and that $Sx = S$ for all $x \in S$. It is shown (4.13) that there exists a right invariant integral for S. In 5, the condition $Sx = S$ will be replaced by the minimal ideal condition, described in the introduction. The assumption that S is not compact is justified because of Rosen's proof of the existence of a right invariant integral for a compact semigroup satisfying a minimal ideal condition.

Denote by $\mathscr{C}$ the collection of all non-empty compact subsets of S. Let $\mathfrak{S}$ denote the collection of all finite sequences of members of $\mathscr{C}$. (For each sequence $\mathscr{S} = S_1, \cdots, S_s$ of $\mathfrak{S}$, repetitions are allowed among the S_i's. Sequences with no terms are not admitted; i.e. it is required that $s \geqq 1$.) If $\mathscr{R}, \mathscr{S} \in \mathfrak{S}$ and $\mathscr{R} = R_1, \cdots, R_r$, $\mathscr{S} = S_1, \cdots, S_s$, then $\mathscr{R}\mathscr{S}$ denotes the sequence $R_1, \cdots, R_r, S_1, \cdots, S_s$.

A relation $\vdash$ is defined on $\mathfrak{S}$ as follows. We let

$$R_1, \cdots, R_r \vdash S_1, \ldots, S_s,$$

if there exist elements $x_1, \cdots, x_r, y_1, \cdots, y_s \in S$ such that

https://doi.org/10.1017/S1446788700024034 Published online by Cambridge University Press

$$\bigcup_{i=1}^{r} R_i x_i^{-1} \supset \bigcup_{j=1}^{s} S_j y_j^{-1}$$

and $S_1 y^{-1}, \cdots, S_s y_s^{-1}$ are mutually disjoint. Clearly, this relation does not depend on the order in which the R_i's are written or the S_j's are written.

THEOREM 4.1. *If* $\mathscr{P} \vdash \mathscr{Q}$ *and* $\mathscr{R} \vdash \mathscr{S}$, *then* $\mathscr{P}\mathscr{Q} \quad \mathscr{Q}\mathscr{S}$.

PROOF. Let $\mathscr{P} = P_1, \cdots, P_p$, $\mathscr{Q} = Q_1, \cdots, Q_q$, $\mathscr{R} = R_1, \cdots, R_r$ and $\mathscr{S} = S_1, \cdots, S_s$. Let $u_i, v_j, x_k, y_l \in S$ be such that

$$\bigcup_{i=1}^{p} P_i u_i^{-1} \supset \bigcup_{j=1}^{q} Q_j v_j^{-1},$$

with the terms on the right mutually disjoint and

$$\bigcup_{k=1}^{r} R_k x_k^{-1} \supset \bigcup_{l=1}^{s} S_l y_l^{-1}$$

with the terms on the right mutually disjoint. Let C be a compact set containing $Q_j v_j^{-1}$, $S_l y_l^{-1}$ for all j, l. By 2.4, there exist elements $a, b \in S$ such that Ca^{-1} does not intersect Cb^{-1}. Then

$$Q_j(av_j)^{-1} = (Q_j v_j^{-1})a^{-1} \subset Ca^{-1},$$
$$S_l(by_l)^{-1} = (S_l y_l^{-1})b^{-1} \subset Cb^{-1},$$

hence $Q_j(av_j)^{-1}$ and $S_l(by_l)^{-1}$ are disjoint for all j, l. Since

$$\bigcup_{i=1}^{p} P_i(au_i)^{-1} = \bigcup_{i=1}^{p} (P_i u_i^{-1})a^{-1} \supset \bigcup_{j=1}^{q} (Q_j v_j^{-1})a^{-1}$$
$$= \bigcup_{j=1}^{q} Q_j(av_j)^{-1}$$

and similarly

$$\bigcup_{k=1}^{r} R_k(bx_k)^{-1} \supset \bigcup_{k=1}^{s} S_l(by_l)^{-1},$$

if follows immediately that $\mathscr{P}\mathscr{R} \vdash \mathscr{Q}\mathscr{S}$.

THEOREM 4.2. $\mathscr{R} \vdash \mathscr{R}$.

PROOF. (i) When $\mathscr{R}$ has a single term C, then since $Cx^{-1} \supset Cx^{-1}$ for any $x \in S$, the theorem is trivial in this case.

(ii) When $\mathscr{R} = R_1, \cdots, R_r$, with $r > 1$, then by (i), $R_i \vdash R_i$ for all i, hence by 4.1 $R_1, \cdots, R_r \vdash R_1, \cdots, R_r$.

We now define a relation $>$ on $\mathfrak{S}$ as follows: we put

$$\mathscr{R} > \mathscr{S},$$

if there exists a $\mathscr{P} \in \mathfrak{S}$ such that

$$\mathscr{R}\mathscr{P} \vdash \mathscr{S}\mathscr{P}.$$

Clearly, if $\mathscr{R}_1$, $\mathscr{S}_1$ are rearrangements of $\mathscr{R}$ and $\mathscr{S}$, then $\mathscr{R} > \mathscr{S}$ if, and only if, $\mathscr{R}_1 > \mathscr{S}_1$.

By 4.2, $\mathscr{R} > \mathscr{R}$.

THEOREM 4.3. *If $\mathscr{R} \vdash \mathscr{S}$, then $\mathscr{R} > \mathscr{S}$.*

PROOF. Let $\mathscr{R} \vdash \mathscr{S}$ and let $\mathscr{P} \in \mathfrak{S}$.

By 4.1 and 4.2, $\mathscr{R}\mathscr{P} \vdash \mathscr{S}\mathscr{P}$ so that $\mathscr{R} > \mathscr{S}$.

THEOREM 4.4. *The relation $>$ is transitive.*

PROOF. Let $\mathscr{R} > \mathscr{S}$ and $\mathscr{S} > \mathscr{T}$. Take $\mathscr{P}$, $\mathscr{Q}$ such that $\mathscr{R}\mathscr{P} \vdash \mathscr{S}\mathscr{P}$ and $\mathscr{S}\mathscr{Q} \vdash \mathscr{T}\mathscr{Q}$. By 4.1, $\mathscr{R}\mathscr{P}\mathscr{S}\mathscr{Q} \vdash \mathscr{S}\mathscr{P}\mathscr{T}\mathscr{Q}$, hence $\mathscr{R}\mathscr{P}\mathscr{S}\mathscr{Q} \vdash \mathscr{T}\mathscr{P}\mathscr{S}\mathscr{Q}$, so that $\mathscr{R} > \mathscr{T}$.

THEOREM 4.5. *If $\mathscr{P} > \mathscr{Q}$ and $\mathscr{R} > \mathscr{S}$, then $\mathscr{P}\mathscr{R} > \mathscr{Q}\mathscr{S}$.*

PROOF. Let $\mathscr{A}$, $\mathscr{B}$ be such that $\mathscr{P}\mathscr{A} \vdash \mathscr{Q}\mathscr{A}$ and $\mathscr{R}\mathscr{B} \vdash \mathscr{S}\mathscr{B}$. By 4.1, $\mathscr{P}\mathscr{A}\mathscr{R}\mathscr{B} \vdash \mathscr{Q}\mathscr{A}\mathscr{S}\mathscr{B}$, hence $\mathscr{P}\mathscr{R}\mathscr{A}\mathscr{B} \vdash \mathscr{Q}\mathscr{S}\mathscr{A}\mathscr{B}$, so that $\mathscr{P}\mathscr{R} > \mathscr{Q}\mathscr{S}$.

THEOREM 4.6. *If $\mathscr{P}\mathscr{R} > \mathscr{Q}\mathscr{R}$, then $\mathscr{P} > \mathscr{Q}$.*

PROOF. Let S be such that $\mathscr{P}\mathscr{R}\mathscr{S} \vdash \mathscr{Q}\mathscr{R}\mathscr{S}$.

Then $\mathscr{P} > \mathscr{Q}$.

THEOREM 4.7. *If $\mathscr{P}\mathscr{R} > \mathscr{Q}\mathscr{S}$ and $\mathscr{S} > \mathscr{R}$, then $\mathscr{P} > \mathscr{Q}$.*

PROOF. By 4.5, $\mathscr{P}\mathscr{R}\mathscr{S} > \mathscr{Q}\mathscr{S}\mathscr{R}$, hence $\mathscr{P}\mathscr{R}\mathscr{S} > \mathscr{Q}\mathscr{R}\mathscr{S}$, so that by 4.6, $\mathscr{P} > \mathscr{Q}$.

LEMMA 4.8. *There do not exist compact sets $C_1, \cdots, C_p$ and elements $x_1, \cdots, x_p$ of S such that $C_1x_1^{-1}, \cdots, C_px_p^{-1}$ are mutually disjoint and $\bigcup_{i=1}^p C_ix_i^{-1}$ is properly contained in $\bigcup_{i=1}^p C_i$.*

PROOF. Suppose such sets C_i do exist. Let d be a metric for S such that $d(xz, yz) \leqq d(x, y)$ for all $x, y, z \in S$. For each $\delta > 0$ and each compact subset D of S, let $\mu_\delta(D)$ denote the minimum number of sets with diameter less than δ that are required to cover D.

Let $\eta > 0$ be such that $C_ix_i^{-1}$, $C_jx_j^{-1}$ have a distance $> \eta$ for all $i \neq j$ and there is a point of $\bigcup_{i=1}^p C_i$ with distance greater than η from every $C_ix_i^{-1}$. Let

$$q = \mu_\eta\left(\bigcup_{i=1}^p C_i\right)$$

and let $A_1, \cdots, A_q$ be sets with diameters less than η covering $\bigcup_{i=1}^p C_i$. There is at least one A_j not intersecting $\bigcup_{i=1}^p C_ix_i^{-1}$. Also, no A_j intersects more than one $C_ix_i^{-1}$. Let $\mathscr{B}_i$ be the collection consisting of all the A_j's intersecting $C_ix_i^{-1}$. Put

https://doi.org/10.1017/S1446788700024034 Published online by Cambridge University Press

$$\mathscr{B}'_i = \{Ax_i;\ A \in \mathscr{B}_i\}.$$

Then, the collection

$$\mathscr{B} = \bigcup_{i=1}^{p} \mathscr{B}'_i$$

covers $\bigcup_{i=1}^{p} C_i$ and $\mathscr{B}$ contains fewer than q sets. Since $d(xz, yz) \leqq d(x, y)$ for all $x, y, z \in S$, the diameter of each set in $\mathscr{B}$ is less than η. Thus $\mu_\eta(\bigcup_{i=1}^{p} C_i) < q$; a contradiction.

THEOREM 4.9. *If $C \in \mathscr{C}$, we never have $\mathscr{R} > \mathscr{R}C$.*

PROOF. Suppose $\mathscr{R} > \mathscr{R}C$ and let $\mathscr{P}$ be such that $\mathscr{R}\mathscr{P} \vdash \mathscr{R}\mathscr{P}C$. Let $\mathscr{R}\mathscr{P} = R_1, \cdots, R_r$ and let $x_1, \cdots, x_r, y_1, \cdots, y_r, y \in S$ be such that

$$\bigcup_{i=1}^{r} R_i x_i^{-1} \supset Cy^{-1} \cup \bigcup_{i=1}^{r} R_i y_i^{-1}$$

and the sets on the right are mutually disjoint. For each i, let $u_i \in S$ be such that $u_i x_i = y_i$. Put $S_i = R_i x_i^{-1}$. Then each S_i is compact and $S_i u_i^{-1} = (R_i x_i^{-1}) u_i^{-1} = R_i y_i^{-1}$. Thus, the sets $S_i u_i^{-1}$ are mutually disjoint and their union is properly contained in $\bigcup_{i=1}^{r} S_i$. This contradicts 4.8.

THEOREM 4.10. *Let $\mathscr{D}$ be a countable, infinite collection of non-empty compact subsets of S.*

There exists a non-negative, non-trivial real-valued function λ on $\mathscr{D}$ such that for all sequences $\mathscr{R} = R_1, \cdots, R_r$, $\mathscr{S} = S_1, \cdots, S_s$ in $\mathscr{D}$ with $\mathscr{R} > \mathscr{S}$, it is always true that

$$\sum_{i=1}^{r} \lambda(R_i) \geqq \sum_{j=1}^{s} \lambda(S_j). \tag{1}$$

PROOF. Let

$$\mathscr{D} = \{C_1, C_2, \cdots\}$$

and put

$$\mathscr{D}_n = \{C_1, \cdots, C_n\}.$$

Define $\lambda(C_1) = 1$.

In order to define $\lambda(C_{n+1})$ we suppose that λ has already been defined on $\mathscr{D}_n$ in such a way that (1) is satisfied, whenever $\mathscr{R} > \mathscr{S}$ and $R_i, R_j \in \mathscr{D}_n$ for all i, j. Put

$$\alpha = \sup q^{-1}\left[\sum_{j=1}^{t} \lambda(F_j) - \sum_{i=1}^{p} \lambda(E_i)\right],$$

the supremum being taken over all

$$E_1, \cdots, E_p, C_{n+1}, \cdots, C_{n+1} > F_1, \cdots, F_t,$$

where $p \geqq 0$, $q \geqq 1$, $t \geqq 1$, C_{n+1} is repeated q times and $E_i, F_j \in \mathscr{D}_n$.

Let

$$\beta = \inf q^{-1}\left[\sum_{i=1}^{p}\lambda(E_i) - \sum_{j=1}^{t}\lambda(F_j)\right],$$

the infimum being taken over all

$$E_1, \cdots, E_p > F_1, \cdots, F_t, \; C_{n+1}, \cdots, C_{n+1},$$

where $p \geqq 1$, $t \geqq 0$, $q \geqq 1$, C_{n+1} is repeated q times and E_i, $F_j \in \mathscr{D}_n$. If there do not exist any such sets E_i, F_j, put $\beta = \alpha$. Now $\beta \geqq \alpha$, because if

$$E_1, \cdots, E_p > F_1, \cdots, F_t, \; C_{n+1}, \cdots, C_{n+1}$$

gives an approximation for β, while

$$E'_1, \cdots, E'_{p'}, \; C_{n+1}, \cdots, C_{n+1} > F'_1, \cdots, F'_{t'}$$

gives an approximation for α (C_{n+1} being repeated q' times), then, denoting the sequence of E_i's by $\mathscr{E}$, F_j's by $\mathscr{F}$, etc. and a sequence of k C'_{n+1}'s by $\mathscr{C}_k$, one has by **4.5**

$$\mathscr{E} \cdots \mathscr{E} > \mathscr{F} \cdots \mathscr{F} C_{qq'},$$

where $\mathscr{E}$ is repeated q' times and $\mathscr{F}$ q' times. Similarly

$$\mathscr{E}' \cdots \mathscr{E}' \mathscr{C}_{qq'} > \mathscr{F}' \cdots \mathscr{F}',$$

where $\mathscr{E}'$ is repeated q times and $\mathscr{F}'$ q times. Hence by **4.5** and **4.6**

$$\mathscr{E} \cdots \mathscr{E}\, \mathscr{E}' \cdots \mathscr{E}' > \mathscr{F} \cdots \mathscr{F}\, \mathscr{F}' \cdots \mathscr{F}',$$

therefore

$$q' \sum_{i=1}^{p}\lambda(E_i) + q\sum_{i=1}^{p'}\lambda(E'_i) \geqq q' \sum_{j=1}^{t}\lambda(F_j) + q\sum_{j=1}^{t'}\lambda(F'_j),$$

so that $\beta \geqq \alpha$.

Now define $\lambda(C_{n+1}) = \alpha$. It remains to prove that (1) is satisfied, whenever $\mathscr{R} > \mathscr{S}$ and R_i, $R_j \in D_{n+1}$ for all i, j. If $\mathscr{R}$ and $\mathscr{S}$ consist entirely of C_{n+1}'s, then by **4.9**, $r \geqq s$ and (1) is trivially satisfied. If $\mathscr{R}$ consists entirely of C_{n+1}'s but $\mathscr{S}$ does not, then by **4.9**, $\mathscr{R}$ has more C_{n+1}'s than $\mathscr{S}$; by a repeated application of **4.6**, one can remove all the C_{n+1}'s from $\mathscr{S}$ and an equal number from $\mathscr{R}$; (1) now follows from $\lambda(C_{n+1}) = \alpha$. If $\mathscr{S}$ consists entirely of C_{n+1}'s but $\mathscr{R}$ does not, one can assume that the number of C_{n+1}'s in $\mathscr{S}$ is greater than the number in $\mathscr{R}$, because otherwise (1) is trivially true; all the C_{n+1}'s can now be removed from $\mathscr{R}$ and an equal number from $\mathscr{S}$; (1) now follows from $\lambda(C_{n+1}) \leqq \beta$. If neither $\mathscr{R}$ nor $\mathscr{S}$ consists entirely of C_{n+1}'s, one can remove an equal number of C_{n+1}'s

https://doi.org/10.1017/S1446788700024034 Published online by Cambridge University Press

from each until one of them has no C_{n+1}'s; (1) then follows from $\alpha = \lambda(C_{n+1}) \leqq \beta$.

This completes the proof.

LEMMA 4.11. *If A is a countable subset of S and $\mathscr{D}$ is a countable collection of compact subsets of S, then there exists a countable collection $\mathscr{F}$ of compact sets such that*

(i) $\mathscr{D} \subset \mathscr{F}$,

(ii) *for every $C \in \mathscr{F}$ and every $a \in A$, $Ca^{-1} \in \mathscr{F}$, and*

(iii) *for all C, $D \in \mathscr{F}$, $C \cup D \in \mathscr{F}$ and $C \cap D \in \mathscr{F}$.*

PROOF. Define a string to be a finite sequence $P_1, \cdots, P_n$ of symbols such that each P_i is either a member of $\mathscr{D}$, a symbol a^{-1}, where $a \in A$, or one of the symbols $\cup$, $\cap$, (,). A string $\mathscr{P}$ will be called a word if there exists a finite sequence

$$\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_p = \mathscr{P}$$

of strings such that (i) $\mathscr{P}_1$ consists of a member of $\mathscr{D}$ and (ii) for each $i > 1$, $\mathscr{P}_i$ is either a member of $\mathscr{D}$ or has one of the forms

$$\mathscr{R}a^{-1}, \ (\mathscr{R}) \cup (\mathscr{S}), \ (\mathscr{R}) \cap (\mathscr{S}),$$

where $a \in A$ and $\mathscr{R}, \mathscr{S}$ are strings that precede $\mathscr{P}_i$ in the sequence $\mathscr{P}_1, \ldots, \mathscr{P}_p$.

Let $\mathscr{F}$ consist of all (compact) subsets of S that correspond to words. Since the number of words is countable, $\mathscr{F}$ is countable. $\mathscr{F}$ evidently has the required properties.

THEOREM 4.12. *If A is a countable dense subset of S, then there exists a non-trivial regular Borel measure μ on S such that*

$$\mu(Ba^{-1}) = \mu(B)$$

for every Borel set B of S and every $a \in A$ (μ is a Borel measure in the sense of Halmos [2, p. 223]. $\mu(C)$ is therefore finite for every compact set C.)

PROOF. Let $\mathscr{E}$ be a countable collection of compact subsets of S whose interiors form a base for the topology of S. Since S is not compact $\mathscr{E}$ is infinite. By 4.11, there exists a countable collection $\mathscr{F}$ of compact subsets of S such that (i) $\mathscr{E} \subset \mathscr{F}$; (ii) for every $C \in \mathscr{F}$ and $a \in A$, $Ca^{-1} \in \mathscr{F}$; and (iii) for all $C, D \in \mathscr{F}$, $C \cup D \in \mathscr{F}$ and $C \cap D \in \mathscr{F}$. Put $\mathscr{F}_1 = \mathscr{F} \sim \{\phi\}$.

By 4.10, there exists a non-negative non-trivial real-valued function λ on $\mathscr{F}_1$ such that for all sequences

$$R_1, \cdots, R_r > S_1, \cdots, S_s$$

with R_i, $R_j \in \mathscr{F}_1$ it is always true that

$$\sum_{i=1}^{r} \lambda(R_i) \geqq \sum_{j=1}^{s} \lambda(S_j). \tag{1}$$

Extend λ to $\mathscr{F}$ by putting $\lambda(\phi) = 0$. By (1)

$$\lambda(Ca^{-1}) = \lambda(C) \tag{2}$$

for all $C \in \mathscr{F}$ and all $a \in A$.

For each subset A of S, define

$$\mu(A) = \inf \sum_r \lambda(D_r), \tag{3}$$

where the infimum is taken over all (finite or infinite) sequences $D_1, D_2, \cdots$ of members of $\mathscr{F}$ such that $A \subset \bigcup_r \operatorname{Int}(D_r)$. Then $0 \leqq \mu(C) < \infty$ for every compact set C and $\mu(C) \geqq \lambda(C)$ for all $C \in \mathscr{F}$, hence u is not trivial. If $A_1, A_2, \cdots$ is a sequence of subsets of S, then clearly

$$\mu(\bigcup_r A_r) \leqq \sum_r \mu(A_r).$$

If $A \subset B$, then evidently $\mu(A) \leqq \mu(B)$. If A, B have a positive distance and $\mu(A \cup B) < \infty$, then take $\varepsilon > 0$ and members $D_1, D_2, \cdots$ of $\mathscr{F}$ such that $A \cup B \subset \bigcup_r \operatorname{Int}(D_r)$ and $\sum_r \lambda(D_r) < \mu(A \cup B) + \varepsilon$; for each r, let E_r, F_r be disjoint members of $\mathscr{F}$ such that $A \cap D_r \subset \operatorname{Int}(E_r)$ and $B \cap D_r \subset \operatorname{Int}(F_r)$; then $D_r \cap E_r$, $D_r \cap F_r \in \mathscr{F}$ and $A \subset \bigcup_r \operatorname{Int}(D_r \cap E_r)$, $B \subset \bigcup_r \operatorname{Int}(D_r \cap F_r)$, so that since $D_r > D_r \cap E_r$, $D_r \cap F_r$ and $\lambda(D_r) \geqq \lambda(D_r \cap E_r) + \lambda(D_r \cap F_r)$, $\mu(A \cup B) + \varepsilon > \sum_r \lambda(D_r \cap E_r) + \sum_r \lambda(D_r \cap F_r) \geqq \mu(A) + \mu(B)$.

Since $\mu(A \cup B)$ is trivially $\geqq \mu(A) + \mu(B)$ when $\mu(A \cup B) = \infty$, we have shown that, for all A, B with a positive distance, $\mu(A \cup B) = \mu(A) + \mu(B)$. Thus we have shown that μ is a Carathéodory outer measure [7, Chapter II]. Hence its restriction to the Borel sets is a Borel measure. It follows immediately from (3), that μ is regular.

It remains only to prove that

$$\mu(Ba^{-1}) = \mu(B) \tag{4}$$

for every Borel set B and every $a \in A$.

To prove (4), suppose first of all that B is compact. Observe that for each compact set C

$$u(C) = \inf \lambda(D),$$

where the infimum is taken over all $D \in F$ such that $C \subset \operatorname{Int}(D)$. Take $\varepsilon > 0$ and let $D \in F$ be such that $B \subset \operatorname{Int}(D)$ and $\lambda(D) < \mu(B) + \varepsilon$. Then $Ba^{-1} \subset \{\operatorname{Int}(D)\}a^{-1} \subset \operatorname{Int}(Da^{-1})$, hence $\mu(Ba^{-1}) \leqq \lambda(Da^{-1}) = \lambda(D) < \mu(B) + \varepsilon$, so that

$$(5) \qquad \mu(Ba^{-1}) \leqq \mu(B).$$

To prove the inequality opposite to (5), take $\varepsilon > 0$ and let $E \in \mathscr{F}$ be such that $Ba^{-1} \subset \operatorname{Int}(E)$ and $\lambda(E) < \mu(Ba^{-1})+\varepsilon$. Put $V = \operatorname{Int}(E)$. The set $(S \sim V)a$ does not intersect B and, by 2.1, is closed. Let $G \in \mathscr{F}$ be such that $B \subset \operatorname{Int}(G)$ and G does not intersect $(S \sim V)a$. Then $Ga^{-1} \subset \operatorname{Int}(E)$, hence $\mu(Ba^{-1})+\varepsilon > \lambda(Ga^{-1}) = \lambda(G) \geqq \mu(B)$. Thus

$$(6) \qquad \mu(Ba^{-1}) \geqq \mu(B)$$

(4) now follows from (5) and (6), when B is compact. When B is not compact, then, since μ is regular, $\mu(B)$ can be approximated by $\mu(C)$, where C is a compact set contained in B. Then $Ca^{-1} \subset Ba^{-1}$, so that (since $\mu(Ca^{-1}) = \mu(C)$)

$$\mu(Ba^{-1}) \geqq \mu(B).$$

But we can also approximate $\mu(Ba^{-1})$ by $\mu(C)$, where C is a compact set contained in Ba^{-1}. Then $\mu(C) \leqq \mu\{(Ca)a^{-1}\}$ and, since $Ca \subset B$ and $\mu\{(Ca)a^{-1}\} = \mu(Ca)$, it follows that

$$\mu(Ba^{-1}) \leqq \mu(B).$$

Thus (4) is true for an arbitrary Borel set.

THEOREM 4.13. *There exists a right invariant integral ν on S.*

PROOF. Let A be a countable dense subset of S. By 4.12, there exists a non-trivial regular Borel measure μ on S such that

$$\mu(Ba^{-1}) = \mu(B)$$

for every Borel set B and every $a \in A$.

Let $\mathscr{F}$ denote the linear space of all real-valued continuous functions on S with compact supports. For each $f \in \mathscr{F}$, define

$$\nu(f) = \int_S f(x)d\mu.$$

Then ν is a positive, non-trivial Radon measure and for all $a \in A$,

$$\nu(f_a) = \nu(f),$$

where $f_a(x) = f(xa)$.

It remains to prove that

$$(1) \qquad \nu(f_\xi) = \nu(f)$$

for all $\xi \in S$ and $f \in \mathscr{F}$.

To prove (1), let ρ be a metric for S such that $\rho(zx, zy) \leqq \rho(x, y)$ for all $x, y, z \in S$. (The existence of such a metric follows from 3.3). Take $\varepsilon > 0$. Let $E = \operatorname{spt} f$ and C be a compact set with E contained in its interior.

By 2.3, there exists a $\delta_1 > 0$ and such that

(2) $$Ex^{-1} \subset C\xi^{-1}$$

for all $x \in S$ with $\rho(x, \xi) < \delta_1$. Let g be a continuous non-negative function with compact support and such that $g(x) = 1$ for all $x \in C\xi^{-1}$. Since f is uniformly continuous, there exists a δ such that $0 < \delta < \delta_1$ and

$$\nu(g)|f(x)-f(x')| < \varepsilon$$

for all $x, x' \in S$ with $\rho(x, x') < \delta$. Let $a \in A$ be such that $\rho(a, \xi) < \delta$. Then, since $\rho(xa, x\xi) \leqq \rho(a, \xi) < \delta$ for all $x \in S$, we have

$$\eta\nu(g) \leqq \varepsilon,$$

where

$$\eta = \sup_{x \in S} |f(xa) - f(x\xi)|.$$

But it follows from (2) that the supports of f_ξ and f_a are both contained in $C\xi^{-1}$, hence

$$\eta g \geqq |f_a - f_\xi|$$

so that

$$\begin{aligned} \varepsilon \geqq \eta\nu(g) = \nu(\eta g) \geqq \nu(|f_a - f_\xi|) \geqq |\nu(f_a - f_\xi)| \\ = |\nu(f_a) - \nu(f_\xi)| = |\nu(f) - \nu(f_\xi)|. \end{aligned}$$

Thus $\nu(f_\xi) = \nu(f)$.

5. The main theorem

In this section we prove our main result concerning the existence of a right invariant integral.

THEOREM 5.1. *Let S be a separable, metrizable, locally compact semigroup, with the properties* (A) *and* (B) *and possessing a left ideal K that is contained in every other left ideal.*

Then S admits a non-trivial right-invariant integral.

PROOF. If S is compact, then K is a unique, mimimal left ideal; hence, by Rosen's theorem, S admits a non-trivial right-invariant integral. We can assume therefore, that S is not compact.

K is certainly a semigroup and it has the properties

(1) $$Kx \supset K$$

for all $x \in S$ and

(2) $$Kx = K$$

for all $x \in K$. Since $Sx = K$ when $x \in K$, it follows from 2.1 that K is a closed subset of S. Then K (regarded as a metric semigroup) is locally compact and separable and it has the properties (A) and (B). Consequently, by 4.13, there is a right-invariant integral ν for K. Let $\mathscr{F}$ denote the linear space of all real-valued continuous functions on S with compact supports. For each $f \in \mathscr{F}$, let f^* denote the restriction of f to K and define

$$\mu(f) = \nu(f^*).$$

Then μ is a positive Radon measure. Since, to every function g on K with compact support, there corresponds an $f \in \mathscr{F}$ with $f^* = g$, it follows that μ is non-trivial. It remains to prove that

$$\mu(f_\xi) = \mu(f) \tag{3}$$

for all $\xi \in S$ and $f \in \mathscr{F}$.

When $\xi \in K$, it is sufficient for the proof of (3) to show that

$$(f_\xi)^* = (f^*)_\xi, \tag{4}$$

because **4.13** shows that $\nu\{(f^*)_\xi\} = \nu(f^*)$.

Now, for all $x \in K$

$$(f_\xi)^*(x) = f_\xi(x) = f(x\xi) = f^*(x\xi) = (f^*)_\xi(x),$$

so that (4) is true. When $\xi \notin K$, let η be an element of K; then, since $K\xi \supset K$, there exists an element ζ of K with $\zeta\xi = \eta$; hence

$$\mu(f) = \mu(f_\eta) = \mu\{(f_\xi)_\zeta\} = \mu(f_\xi).$$

so that (1) is also true when $\xi \notin K$.

Thus μ is a right-invariant integral.

References

[1] Birkhoff, G., A note on topological groups, Compositio Math. 3 (1936), 427—430.
[2] Halmos, P. R., Measure theory. Van Nostrand (1950).
[3] Kakutani, S., Über die Metrization der topologischen Gruppen, Proc. Imp. Acad. Jap. 12 (1936), 82—84.
[4] Kelley, J. L., General topology. Van Nostrand (1955).
[5] Montgomery, D. and Zippin, L., Topological transformation groups. Interscience tracts in pure and applied mathematics, Tract 1 (1955).
[6] Rosen, W. G., On invariant means over compact semigroups, Proc. Amer. Math. Soc. 7 (1956), 1076—1082.
[7] Saks, S., Theory of the integral. Hafner (1937).

University of Adelaide
South Australia.