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DIFFERENTIAL ALGEBRAIC FUNCTION FIELDS DEPENDING RATIONALLY ON ARBITRARY CONSTANTS

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§1. Introduction

The general solution of an algebraic differential equation depends on the initial conditions, though it is in general too difficult to make explicit the shape of the relationship. Painlevé studied in [8] algebraic differential equations of second order with the general solutions depending rationally on the initial conditions and the solvability of such equations. Giving the precise definition of the notion "rational dependence on the initial conditions", Umemura [10] revived and generalized rigorously the discussion of Painlevé in the language of modern algebraic geometry. The theorem of Umemura is as follows; Let K be a differential field extension of complex number field $\boldsymbol{C}$ generated by a finite number of meromorphic functions on some domain in $\boldsymbol{C}$. Let y be the general solution of a given algebraic differential equation over K. Suppose that y depends rationally on the initial conditions. Then it is contained in the terminal K_m of a finite chain of differential field extensions: $K = K_0 \subset K_1 \subset \cdots \subset K_m$ such that each K_i is strongly normal over K_{i-1}.

In [5] the author defined the following: Let K be an ordinary differential field of characteristic zero. A differential field extension L of K is said to *depend rationally on arbitrary constants* if there exists a differential field extension M of K such that L and M are free over K and $LM = MC_{LM}$, where C_{LM} denotes the field of constants of LM. Two notions "the rational dependence on the initial conditions" and "the rational dependence on arbitrary constants" are equivalent. The later originates directly in the author's [4] (see also Matsuda [3]). The objective of this paper is to prove the following:

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THEOREM. *Let K be an algebraically closed ordinary differential field of characteristic zero. Let R be a differential field extension of K generated by a single element which is differentially algebraic over K. Then the following are equivalent*:

(i) $C_R = C_K$ *and R depends rationally on arbitrary constants*;

(ii) *there exists a strongly normal extension of K which contains R.*

This tells us that the length of the chain needed in the conclusion of Umemura's theorem is at most 2. It will be worthy to remark a well-known fact that if K contains nonconstants any differential field extension of K that is a finitely generated field extension of K contains an element y with $R = K\langle y\rangle$ (for instance see Ritt [9]).

The proof of Theorem will be divided into 4 steps. We utilize some basic facts in Kolchin [2] without warning and discuss entirely from differential-algebraic viewpoint.

§ 2. The proof of Theorem

The deduction of (ii) from (i) is a straightforward result from Lemma 1 of [7] (see below). So it is sufficient to prove (i) under the condition (ii). Let U be a universal differential field extension of K. For any intermediate differential field L between K and U we denote by C_L the field of constants of L. Here we recall the definition of strongly normal extensions: A finitely generated differential field extension N of K is called a strongly normal extension of K if $C_N = C_K$ and there exists a differential subfield M of U containing K such that M is differentially isomorphic to N over K, M and N are linearly disjoint over K and $MN = MC_{MN}$ (cf. Proposition 1 of [1]). We extract Lemma 1 of [7] for reader's convenience:

LEMMA. *Let L and M be two intermediate differential fields between K and U. Suppose M is a finitely generated differential field extension of L which is contained in LC_U. Then $M = LC_M$.*

Now let us return to the proof of our theorem. By assumption of Theorem there is an element y of R with $R = K\langle y\rangle$ and $\operatorname{tr.deg} R/K$ is finite. Assume (i). Then there exists a finitely generated differential field extension E of K such that E and R are linearly disjoint over K and $ER = EC_{ER}$. We have generators $e = (e_i)_{1\leqq i\leqq p}$ of E over K: $E = K\langle e\rangle$.

Step 1. There exists a finitely generated differential field extension E_1 of K such that E_1 and R are linearly disjoint over K, $C_{E_1} = C_K$ and $E_1R = E_1C_{E_1R}$.

Proof. Since y is in $ER = EC_{ER} = K\langle e\rangle C_{ER}$, we have the representation:

$$y = \sum a_j x_j / \sum b_j x_j\,, \qquad 1 \leqq j \leqq q$$

where the a's and b's are elements of C_{ER}, the x's are elements of $K\{e\}$ which are linearly independent over C_{ER} and b_i does not vanish for some i. Since a_j and b_j lie in ER, we have the representations:

$$a_j = \sum a_{jh} y_h / \sum b_{jh} y_h\,, \qquad 1 \leqq h \leqq r$$
$$b_j = \sum c_{jh} y_h / \sum d_{jh} y_h\,, \qquad 1 \leqq h \leqq r$$

where the y's are elements of R which are linearly independent over K and a_{jh}, b_{jh}, c_{jh}, d_{jh} are elements of $K\{e\}$ such that for any j there are j_1 and j_2 with $b_{jj_1}d_{jj_2} \neq 0$ and $c_{ih} \neq 0$ for some h. By w we denote the wronskian of (x_j). Then the linear independence of (x_j) over constants implies $w \neq 0$. Note that w is an element of $K\{e\}$. By Kolchin [2] there exists a differential homomorphism ϕ of $K\{e\}$ to U over K such that $\phi(b_{jj_1}d_{jj_2}) \neq 0$ for any j, $\phi(wc_{ih}) \neq 0$ and $C_{K\langle\phi e\rangle} = C_K$. It is readily seen from the universality of U that there exists a differential field extension E_1 of K such that E_1 is differentially isomorphic to $K\langle\phi e\rangle$ and E_1 and R are linearly disjoint over K. Then ϕ can be extended to a differential homomorphism of $R\{e\}$ to E_1R over R. We denote this by the same symbol ϕ. Since $\phi w \neq 0$ we see (ϕx_j) are linearly independent over constants. Noting also ϕb_i is defined and nonzero because (y_h) are linearly independent over E_1 and $\phi c_{ih} \neq 0$ we have

$$\phi \sum b_j x_j = \sum \phi b_j \phi x_j \neq 0\,.$$

And

$$y = \phi y = \sum \phi a_j \phi x_j / \sum \phi b_j \phi x_j$$

is contained in $E_1C_{E_1R}$ since ϕa_j and ϕb_j belong to C_{E_1R}. Thus we conclude R is contained in $E_1C_{E_1R}$ and so $E_1R = E_1C_{E_1R}$.

Step 2. There exists a finitely generated differential field extension E_2 of K such that E_2 and R are linearly disjoint over K, $C_{E_2} = C_K$, $E_2R = E_2C_{E_2R}$ and E_2 contains an element z which is a generic differential specialization of y over K.

Proof. Take a generic differential specialization z of y over K with the property that $K\langle z\rangle$ and E_1R are linearly disjoint over K. This is possible according to the universality of U. Let L denote the algebraic closure of $K\langle z\rangle$ in U. Then LE_1 and LR are linearly disjoint over L. In fact L and E_1R are linearly disjoint over K and so E_1R and E_1L are linearly disjoint over E_1. Hence R and LE_1 are linearly disjoint over K. This implies our assertion. Clearly $LE_1R = LE_1C_{E_1R}$. Similarly to Step 1 we can find a finitely generated differential field extension F of L such that LR and F are linearly disjoint over L, $C_F = C_L = C_K$ and $FR = FC_{FR}$. Take a differential subfield F_1 of F which is a finitely generated differential field extension of K and satisfies $R = F_1C_{FR}$. Then F_1 and R are linearly disjoint over K, $C_{F_1} = C_K$ and $F_1R = F_1C_{F_1R}$ by virtue of Lemma. As E_2 we may take $F_1\langle z\rangle$. The verification of required properties is easy.

By the finite generatedness of E_2, C_{E_2R} is finitely generated over C_K in the ordinary sense. Hence we have elements $u = (u_j)_{1\leqq j\leqq s}$ with $C_{E_2R} = C_K(u)$ and the representation:

$$y = f_1(u)/f(u)\,,$$

where f_1 and f are in $E_2[u]$, $f \neq 0$. We may write

$$u_j = g_j(y)/g(y)\,,$$

where g_j and g are in $E_2\{y\}$, $g \neq 0$. By substitution we get

$$g(f_1(u)/f(u)) = h(u)/f(u)^d\,,$$

where h is in $E_2[u]$ and d is a natural number. Let X be the set of all c in C_K^p that are specializations of u over C_K and V be the set of all c in X with $f(c)h(c) \neq 0$. Then X is an irreducible affine variety, V is open and dense in X. In fact noting K is algebraically closed we see $C_{E_2} = C_K$ is also algebraically closed. Hence

$$E_2[u] = E_2 \otimes_{C_K} C_K[u]\,.$$

From this we have a representation $fh = \sum a_jp_j$, where the a_j are in E_2, linearly independent over C_K, the p_j are in $C_K[u]$. Using this representation we find that c is an element of V is equivalent to that $fh(c) \neq 0$ and therefore that $p_j(c) \neq 0$ for some j. This shows V is open in X. Denote by E_3 the differential field extension of K generated with z and all $y(c)$, c being in V. Then E_3 is a differential subfield of E_2 with the finite transcendence degree over K since the generators are all differentially algebraic over K and E_2 is a finitely generated differential field extension of K. The element $y(c)$ of E_3 is a differential specialization of y over K associated with the specialization c of u over C_K. Note that any element c of V is characterized by $f(c) \neq 0$, $g(y(c)) \neq 0$.

Step 3. $E_3R = E_3C_{E_3R}$.

Proof. The field $E_2(u)$, the quotient field of $E_2 \otimes_{C_K} C_K[u]$, is the function field of the irreducible variety X' determined from X by base extension $C_K \to E_2$. Let us show that V is dense in X'. Let t be a regular function on X'. Then we can express as $t = \sum a_i t_i$, a_i in E_2, t_i in $C_K[u]$, the a_i being linearly independent over C_K. For any c in V we have $t(c) = \sum a_i t(c)$. If $t(c) = 0$ then $t_i(c) = 0$ for all i. Each t_i would be identically 0 on X and hence 0 on X'. Thus $t = 0$ on X'. This shows V is dense in X'. Let F be the algebraic closure of E_3 in E_2. We regard E_2 as a field with operators in the sense of [1], where as operators we take $\mathrm{Der}\,(E_2/E_3)$. Then F is the field of constants of E_2. Since E_2 and $E_3(u)$ are linearly disjoint over E_3, we may set $Du_i = 0$ for each i and each derivative D in $\mathrm{Der}\,(E_2/E_3)$, and $E_2(u)$ becomes a field extension of $E_3(u)$ in the sense of [1]. For any D in $\mathrm{Der}\,(E_2/E_3)$ we have $Dy(c) = 0$ and

$$f(c)Df_1(c) - f_1(c)Df(c) = 0$$

holds for each c in V. From this and the fact V is dense in X' it follows

$$fDf_1 - f_1Df = 0$$

holds in $E_2(u)$. Thus $Dy = 0$ and hence y is in $F(u)$ since $F(u)$ is the field of constants of the field $E_2(u)$ with operators $\mathrm{Der}\,(E_2/E_3)$ according to Lemma 1 of [1]. Thus FR is included in FC_{E_2R} and by Lemma it follows $FR = FC_{FR}$. We assume F is normal over E_3 by enlarging F if necessary. We have $C_F = C_K$ recalling $C_{E_2} = C_K$. By defining $\tau u_i = u_i$ for each i and any τ of $G(F/E_3)$, the Galois group of F over E_3, we find $F(u)$ is normal over $E_3(u)$ and the Galois group is identified with $G(F/E_3)$. Now from $\tau y(c) = y(c) \in E_3$ it follows

$$f(c)\tau f_1(c) - f_1(c)\tau f(c) = 0$$

holds for any c in V, hence

$$f\tau f_1 - f_1\tau f = 0$$

holds in $F(u)$ because V is dense in X'. This shows $\tau y = y$ and y lies in $E_3(u)$.

By Step 3 we may assume from the first that E has the finite transcendence degree over K, $C_E = C_K$ and E has an element z which is a generic differential specialization of y over K. Take an E_4 among such E's with the least transcendence degree over K. Define M from E_4 in the same manner as we defined E_3 from E_2. Then the degree of E_4 over M is finite. We use the same notations such as u, f_1, f, g_j, g. Let F be a normal algebraic extension of M which contains E_4. As before we suppose $F(u)$ is normal over $E_4(u)$ and identify the Galois group $G(F(u)/E_4(u))$ with $G(F/F_4)$. Then

$$y = f_1/f = f_1^*/f^*\,,$$

where $f_1^* = f_1 \prod_{\tau\neq 1} \tau f$, $f^* = \prod_\tau \tau f \in F[u]$, $\tau \in G(F/E_4)$. Since f^* is left invariant under any τ it is contained in $M(u)$ and therefore in $M[u]$. This is derived from the fact F and $M(u)$ are linearly disjoint over M. By $MR = MC_{MR} = M(u)$ it follows $f_1^* = yf^*$ belongs to $M[u]$. Similarly we obtain the representation:

$$u_j = g_j/g = g_j^*/g^*\,,$$

where $g_j^* = g_j \prod_{\tau\neq 1} \tau g$, $g^* = \prod_\tau \tau g \in M\{y\}$, $\tau \in G(F/E_4)$. And from the fact F and $M(u) = M\{y\}$ are linearly disjoint over M it follows that g^* and g_j^* are contained in $M\{y\}$. By the definitions of f^* and g^* every element c of V is characterized by $f^*(c) \neq 0$, $g^*(y(c)) \neq 0$. Take a differential field extension N of K such that N and M are differentially isomorphic and linearly disjoint over K and N contains R.

Step 4. N is a strongly normal extension of K.

Proof. There are a finite number of elements $(c_h)_{1\leqq h\leqq m}$ of V for which $M = K\langle z, z_1, \cdots, z_m\rangle$, $z_h = y(c_h)$. We can write $N = K\langle y, y_1, \cdots, y_m\rangle$, where each y_h is a generic differential specialization of z_h over K. Since N and M are linearly disjoint over K we see z_h is a differential specialization of y_h over M and y_h is a differential specialization of y over M. From $g^*(z_h) \neq 0$ we get $g^*(y_h) \neq 0$. Define $v_h = (v_{hj})$ by

$$v_{hj} = g_j^*(y_h)/g^*(y_h)\,.$$

Then (z_h, c_h) is a differential specialization of (y_h, v_h) over M and (y_h, v_h) is a differential specialization of (y, u) over M. All v_h are elements of $C_{M\langle y_h\rangle}$. The equality $yf^* = f_1^*$ implies $y_h f^*(v_h) = f_1^*(v_h)$ and so y_h is contained in $M(v_h)$. This derives $MN = M(u, v_1, \cdots, v_m)$ and completes the proof of Theorem.

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