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ON THE ANALYTICITY OF WLUD$^\infty$ FUNCTIONS OF ONE VARIABLE AND WLUD$^\infty$ FUNCTIONS OF SEVERAL VARIABLES IN A COMPLETE NON-ARCHIMEDEAN VALUED FIELD

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Abstract Let $\mathcal{N}$ be a non-Archimedean-ordered field extension of the real numbers that is real closed and Cauchy complete in the topology induced by the order, and whose Hahn group is Archimedean. In this paper, we first review the properties of weakly locally uniformly differentiable (WLUD) functions, k times weakly locally uniformly differentiable (WLUD^k) functions and WLUD$^\infty$ functions at a point or on an open subset of $\mathcal{N}$. Then, we show under what conditions a WLUD$^\infty$ function at a point $x_0 \in \mathcal{N}$ is analytic in an interval around x_0, that is, it has a convergent Taylor series at any point in that interval. We generalize the concepts of WLUD^k and WLUD$^\infty$ to functions from $\mathcal{N}^n$ to $\mathcal{N}$, for any $n \in \mathbb{N}$. Then, we formulate conditions under which a WLUD$^\infty$ function at a point $\boldsymbol{x_0} \in \mathcal{N}^n$ is analytic at that point.

Keywords: Taylor series expansion; analytic functions; non-Archimedean analysis; non-Archimedean-valued fields

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1. Introduction

Let $\mathcal{N}$ be a non-Archimedean-ordered field extension of $\mathbb{R}$ that is real closed and complete in the order topology and whose Hahn group $S_{\mathcal{N}}$ is Archimedean, i.e. (isomorphic to) a subgroup of $\mathbb{R}$. Recall that $S_{\mathcal{N}}$ is the set of equivalence classes under the relation $\sim$ defined on $\mathcal{N}^* := \mathcal{N}\backslash\{0\}$ as follows: For $x, y \in \mathcal{N}^*$, we say that x is of the same order as y and write $x \sim y$ if there exist $n, m \in \mathbb{N}$ such that $n|x| > |y|$ and $m|y| > |x|$, where $|\cdot|$ denotes the ordinary absolute value on $\mathcal{N}$: $|x| = \max\{x, -x\}$. $S_{\mathcal{N}}$ is naturally endowed with an addition via $[x] + [y] = [x \cdot y]$ and an order via $[x] < [y]$ if $|y| \ll |x|$ (which means $n|y| < |x|$ for all $n \in \mathbb{N}$), both of which are readily checked to be well defined. It follows that $(S_{\mathcal{N}}, +, <)$ is an ordered group, often referred to as the Hahn group or skeleton group, whose neutral element is $[1]$, the class of 1.

The theorem of Hahn [3] provides a complete classification of non-Archimedean-ordered field extensions of $\mathbb{R}$ in terms of their skeleton groups. In fact, invoking the axiom of choice, it is shown that the elements of our field $\mathcal{N}$ can be written as (generalized) formal power series (also called Hahn series) over its skeleton group $S_{\mathcal{N}}$ with real coefficients, and the set of appearing exponents forms a well-ordered subset of $S_{\mathcal{N}}$. That is, for all $x \in \mathcal{N}$, we have that $x = \sum_{q \in S_{\mathcal{N}}} a_q d^q$; with $a_q \in \mathbb{R}$ for all q, d a positive infinitely small element of $\mathcal{N}$, and the support of x, given by $\operatorname{supp}(x) := \{q \in S_{\mathcal{N}} : a_q \neq 0\}$, forming a well-ordered subset of $S_{\mathcal{N}}$.

We define for $x \neq 0$ in $\mathcal{N}$, $\lambda(x) = \min(\operatorname{supp}(x))$, which exists since $\operatorname{supp}(x)$ is well ordered. Moreover, we set $\lambda(0) = \infty$. Given a non-zero $x = \sum_{q \in \operatorname{supp}(x)} a_q d^q$, then $x > 0$ if and only if $a_{\lambda(x)} > 0$.

The smallest such field $\mathcal{N}$ is the Levi-Civita field $\mathcal{R}$, first introduced in [5, 6]. In this case, $S_{\mathcal{R}} = \mathbb{Q}$, and for any element $x \in \mathcal{R}$, $\operatorname{supp}(x)$ is a left-finite subset of $\mathbb{Q}$, i.e. below any rational bound r there are only finitely many exponents in the Hahn representation of x. The Levi-Civita field $\mathcal{R}$ is of particular interest because of its practical usefulness. Since the supports of the elements of $\mathcal{R}$ are left-finite, it is possible to represent these numbers on a computer. Having infinitely small numbers allows for many computational applications; one such application is the computation of derivatives of real functions representable on a computer [13, 15], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved. For a review of the Levi-Civita field $\mathcal{R}$, see [11] and references therein.

In the wider context of valuation theory, it is interesting to note that the topology induced by the order on $\mathcal{N}$ is the same as the valuation topology τ_v introduced via the non-Archimedean (ultrametric) valuation $|\cdot|_v : \mathcal{N} \to \mathbb{R}$, given by

$$|x|_v = \begin{cases} \exp\left(-\lambda(x)\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

It follows, therefore, that the field $\mathcal{N}$ is just a special case of the class of fields discussed in [9]. For a general overview of the algebraic properties of formal power series fields, we refer to the comprehensive overview by Ribenboim [8], and for an overview of the related valuation theory the book by Krull [4]. A thorough and complete treatment of ordered structures can also be found in [7]. A more comprehensive survey of all non-Archimedean fields can be found in [1].

2. Weak local uniform differentiability and review of recent results

Because of the total disconnectedness of the field $\mathcal{N}$ in the order topology, the standard theorems of real calculus like the intermediate value theorem, the inverse function theorem, the mean value theorem, the implicit function theorem and Taylor's theorem require stronger smoothness criteria of the functions involved in order for the theorems to hold. In this section, we will present one such criterion: the so-called weak local uniform differentiability, we will review recent work based on that smoothness criterion and then present new results.

In [2], we focus our attention on $\mathcal{N}$-valued functions of one variable. We study the properties of weakly locally uniformly differentiable (WLUD) functions at a point $x_0 \in \mathcal{N}$ or on an open subset A of $\mathcal{N}$. In particular, we show that WLUD functions are C^1, they include all polynomial functions, and they are closed under addition, multiplication and composition. Then, we generalize the definition of weak local uniform differentiability to any order. In particular, we study the properties of WLUD^2 functions at a point $x_0 \in \mathcal{N}$ or on an open subset A of $\mathcal{N}$; and we show that WLUD^2 functions are C^2, they include all polynomial functions, and they are closed under addition, multiplication and composition. Finally, we formulate and prove an inverse function theorem as well as a local intermediate value theorem and a local mean value theorem for these functions.

Here, we only recall the main definitions and results (without proofs) in [2] and refer the reader to that paper for the details.

Definition 1. Let $A \subseteq \mathcal{N}$ be open, let $f : A \to \mathcal{N}$, and let $x_0 \in A$ be given. We say that f is WLUD at x_0 if f is differentiable in a neighbourhood Ω of x_0 in A and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $(x_0 - \delta, x_0 + \delta) \subset \Omega$, and for every $x, y \in (x_0 - \delta, x_0 + \delta)$ we have that $|f(y) - f(x) - f'(x)(y - x)| \leq \epsilon |y - x|$. Moreover, we say that f is WLUD on A if f is WLUD at every point in A.

We extend the WLUD concept to higher orders of differentiability and we define WLUD^k as follows.

Definition 2. Let $A \subseteq \mathcal{N}$ be open, let $f : A \to \mathcal{N}$, let $x_0 \in A$, and let $k \in \mathbb{N}$ be given. We say that f is WLUD^k at x_0 if f is k times differentiable in a neighbourhood Ω of x_0 in A and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $(x_0 - \delta, x_0 + \delta) \subset \Omega$, and for every $x, y \in (x_0 - \delta, x_0 + \delta)$ we have that

$$\left| f(y) - \sum_{j=0}^{k} \frac{f^{(j)}(x)}{j!}(y - x)^j \right| \leq \epsilon \, |y - x|^k .$$

Moreover, we say that f is WLUD^k on A if f is WLUD^k at every point in A. Finally, we say that f is WLUD$^\infty$ at x_0 (respectively, on A) if f is WLUD^k at x_0 (respectively, on A) for every $k \in \mathbb{N}$.

Theorem 1 (Inverse Function Theorem). *Let $A \subseteq \mathcal{N}$ be open, let $f : A \to \mathcal{N}$ be WLUD on A, and let $x_0 \in A$ be such that $f'(x_0) \neq 0$. Then, there exists a neighbourhood Ω of x_0 in A such that*

(1) *$f|_\Omega$ is one-to-one;*

(2) *$f(\Omega)$ is open and*

(3) *f^{-1} exists and is WLUD on $f(\Omega)$ with $(f^{-1})' = 1/(f' \circ f^{-1})$.*

Theorem 2 (Local Intermediate Value Theorem). *Let $A \subseteq \mathcal{N}$ be open, let $f : A \to \mathcal{N}$ be WLUD on A, and let $x_0 \in A$ be such that $f'(x_0) \neq 0$. Then, there exists a neighbourhood Ω of x_0 in A such that for any $a < b$ in $f(\Omega)$ and for any $c \in (a, b)$, there is an $x \in (\min\{f^{(-1)}(a), f^{(-1)}(b)\}, \max\{f^{(-1)}(a), f^{(-1)}(b)\})$ such that $f(x) = c$.*

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Theorem 3 (Local Mean Value Theorem). *Let $A \subseteq \mathcal{N}$ be open, let $f : A \to \mathcal{N}$ be WLUD^2 on A, and let $x_0 \in A$ be such that $f''(x_0) \neq 0$. Then, there exists a neighbourhood Ω of x_0 in A such that f has the mean value property on Ω. That is, for every $a, b \in \Omega$ with $a < b$, there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In [12], we formulate and prove a Taylor theorem with remainder for WLUD^k functions from $\mathcal{N}$ to $\mathcal{N}$. Then, we extend the concept of WLUD to functions from $\mathcal{N}^n$ to $\mathcal{N}^m$ with $m, n \in \mathbb{N}$ and study the properties of those functions as we did for functions from $\mathcal{N}$ to $\mathcal{N}$. Then, we formulate and prove the inverse function theorem for WLUD functions from $\mathcal{N}^n$ to $\mathcal{N}^n$ and the implicit function theorem for WLUD functions from $\mathcal{N}^n$ to $\mathcal{N}^m$ with $m < n$ in $\mathbb{N}$.

As in the real case, the proof of Taylor's theorem with remainder uses the mean value theorem. However, in the non-Archimedean setting, stronger conditions on the function are needed than in the real case for the formulation of the theorem.

Theorem 4 (Taylor's Theorem with Remainder). *Let $A \subseteq \mathcal{N}$ be open, let $k \in \mathbb{N}$ be given, and let $f : A \to \mathcal{N}$ be WLUD^{k+2} on A. Assume further that $f^{(j)}$ is WLUD^2 on A for $0 \leq j \leq k$. Then, for every $x \in A$, there exists a neighbourhood U of x in A such that, for any $y \in U$, there exists $c \in [\min\{y, x\}, \max\{y, x\}]$ such that*

$$f(y) = \sum_{j=0}^{k} \frac{f^{(j)}(x)}{j!}(y - x)^j + \frac{f^{(k+1)}(c)}{(k + 1)!}(y - x)^{k+1}. \tag{1}$$

Before we define weak local uniform differentiability for functions from $\mathcal{N}^n$ to $\mathcal{N}^m$ and then state the inverse function theorem and the implicit function theorem, we introduce the following notation.

Notation 1. Let $A \subset \mathcal{N}^n$ be open, let $\boldsymbol{x_0} \in A$ be given, and let $\boldsymbol{f} : A \to \mathcal{N}^m$ be such that all the first-order partial derivatives of $\boldsymbol{f}$ at $\boldsymbol{x_0}$ exist. Then, $\boldsymbol{Df}(\boldsymbol{x_0})$ denotes the linear map from $\mathcal{N}^n$ to $\mathcal{N}^m$ defined by the $m \times n$ Jacobian matrix of $\boldsymbol{f}$ at $\boldsymbol{x_0}$:

$$\begin{pmatrix} \boldsymbol{f}_1^1(\boldsymbol{x_0}) & \boldsymbol{f}_2^1(\boldsymbol{x_0}) & \dots & \boldsymbol{f}_n^1(\boldsymbol{x_0}) \\ \boldsymbol{f}_1^2(\boldsymbol{x_0}) & \boldsymbol{f}_2^2(\boldsymbol{x_0}) & \dots & \boldsymbol{f}_n^2(\boldsymbol{x_0}) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{f}_1^m(\boldsymbol{x_0}) & \boldsymbol{f}_2^m(\boldsymbol{x_0}) & \dots & \boldsymbol{f}_n^m(\boldsymbol{x_0}) \end{pmatrix}$$

with $\boldsymbol{f}_j^i(\boldsymbol{x_0}) = \frac{\partial f_i}{\partial x_j}(\boldsymbol{x_0})$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Moreover, if $m = n$ then the determinant of the $n \times n$ matrix $\boldsymbol{Df}(\boldsymbol{x_0})$ is denoted by $J\boldsymbol{f}(\boldsymbol{x_0})$.

Definition 3 (WLUD). Let $A \subset \mathcal{N}^n$ be open, let $\boldsymbol{f} : A \to \mathcal{N}^m$, and let $\boldsymbol{x_0} \in A$ be given. Then, we say that $\boldsymbol{f}$ is WLUD at $\boldsymbol{x_0}$ if $\boldsymbol{f}$ is differentiable in a neighbourhood Ω of $\boldsymbol{x_0}$ in A and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $B_\delta(\boldsymbol{x_0}) :=$

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$\{\boldsymbol{t} \in \mathcal{N} : |\boldsymbol{t} - \boldsymbol{x_0}| < \delta\} \subset \Omega$, and for all $\boldsymbol{x}$, $\boldsymbol{y} \in B_\delta(\boldsymbol{x_0})$ we have that

$$|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{Df}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})| \le \epsilon|\boldsymbol{y} - \boldsymbol{x}|.$$

Moreover, we say that $\boldsymbol{f}$ is WLUD on A if $\boldsymbol{f}$ is WLUD at every point in A.

We show in [12] that if $\boldsymbol{f}$ is WLUD at $\boldsymbol{x_0}$ (respectively on A) then $\boldsymbol{f}$ is C^1 at $\boldsymbol{x_0}$ (respectively, on A). Thus, the class of WLUD functions at a point $\boldsymbol{x_0}$ (respectively, on an open set A) is a subset of the class of C^1 functions at $\boldsymbol{x_0}$ (respectively, on A). However, this is still large enough to include all polynomial functions. We also show in [12] that if $\boldsymbol{f}$, $\boldsymbol{g}$ are WLUD at $\boldsymbol{x_0}$ (respectively, on A) and if $\alpha \in \mathcal{N}$ then $\boldsymbol{f} + \alpha\boldsymbol{g}$ and $\boldsymbol{f} \cdot \boldsymbol{g}$ are WLUD at $\boldsymbol{x_0}$ (respectively, on A). Moreover, we show that if $\boldsymbol{f} : A \to \mathcal{N}^m$ is WLUD at $\boldsymbol{x_0} \in A$ (respectively, on A) and if $\boldsymbol{g} : C \to \mathcal{N}^p$ is WLUD at $\boldsymbol{f}(\boldsymbol{x_0}) \in C$ (respectively, on C), where A is an open subset of $\mathcal{N}^n$, C an open subset of $\mathcal{N}^m$ and $\boldsymbol{f}(A) \subseteq C$, then $\boldsymbol{g} \circ \boldsymbol{f}$ is WLUD at $\boldsymbol{x_0}$ (respectively, on A).

Theorem 5 (Inverse Function Theorem). *Let $A \subset \mathcal{N}^n$ be open, let $\boldsymbol{f} : A \to \mathcal{N}^n$ be WLUD on A and let $\boldsymbol{t_0} \in A$ be such that $J\boldsymbol{f}(\boldsymbol{t_0}) \neq 0$. Then, there is a neighbourhood Ω of $\boldsymbol{t_0}$ such that:*

(1) *$\boldsymbol{f}|_\Omega$ is one-to-one;*

(2) *$\boldsymbol{f}(\Omega)$ is open and*

(3) *the inverse $\boldsymbol{g}$ of $\boldsymbol{f}|_\Omega$ is WLUD on $\boldsymbol{f}(\Omega)$; and $\boldsymbol{Dg}(\boldsymbol{x}) = [\boldsymbol{Df}(\boldsymbol{t})]^{-1}$ for $\boldsymbol{t} \in \Omega$ and $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{t})$.*

As in the real case, the inverse function theorem is used to prove the implicit function theorem. But before we state the implicit function theorem, we introduce the following notation.

Notation 2. Let $A \subseteq \mathcal{N}^n$ be open and let $\boldsymbol{\Phi} : A \to \mathcal{N}^m$ be WLUD on A. For $\boldsymbol{t} = (t_1, \ldots, t_{n-m}, t_{n-m+1}, \ldots, t_n) \in A$, let

$$\hat{\boldsymbol{t}} = (t_1, \ldots, t_{n-m}) \text{ and } \tilde{J}\boldsymbol{\Phi}(\boldsymbol{t}) = \det\left(\frac{\partial(\Phi_1, \ldots, \Phi_m)}{\partial(t_{n-m+1}, \ldots, t_n)}\right).$$

Theorem 6 (Implicit Function Theorem). *Let $\boldsymbol{\Phi} : A \to \mathcal{N}^m$ be WLUD on A, where $A \subseteq \mathcal{N}^n$ is open and $1 \le m < n$. Let $\boldsymbol{t_0} \in A$ be such that $\boldsymbol{\Phi}(\boldsymbol{t_0}) = \boldsymbol{0}$ and $\tilde{J}\boldsymbol{\Phi}(\boldsymbol{t_0}) \neq 0$. Then, there exist a neighbourhood U of $\boldsymbol{t_0}$, a neighbourhood R of $\hat{\boldsymbol{t_0}}$ and $\boldsymbol{\phi} : R \to \mathcal{N}^m$ that is WLUD on R such that*

$$\tilde{J}\boldsymbol{\Phi}(\boldsymbol{t}) \neq 0 \text{ for all } \boldsymbol{t} \in U,$$

and

$$\{\boldsymbol{t} \in U : \boldsymbol{\Phi}(\boldsymbol{t}) = \boldsymbol{0}\} = \{(\hat{\boldsymbol{t}}, \boldsymbol{\phi}(\hat{\boldsymbol{t}})) : \hat{\boldsymbol{t}} \in R\}.$$

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3. New results

This paper is a continuation of the work done in [2, 12]. In the following section, we will generalize in Definition 5 and Definition 6 the concepts of WLUD^k and WLUD$^\infty$ to functions from $\mathcal{N}^n$ to $\mathcal{N}$; and we will formulate (in Theorem 8 and Theorem 9 and their proofs) conditions under which a WLUD$^\infty$ $\mathcal{N}$-valued function at a point $x_0 \in \mathcal{N}$ or a WLUD$^\infty$ $\mathcal{N}$-valued function at a point $\boldsymbol{x_0} \in \mathcal{N}^n$ will be analytic at that point.

Theorem 7. *Let $A \subseteq \mathcal{N}$ be open, let $x_0 \in A$, and let $f : A \to \mathcal{N}$ be WLUD$^\infty$ at x_0. For each $k \in \mathbb{N}$, let $\delta_k > 0$ in $\mathcal{N}$ correspond to $\epsilon = 1$ in Definition 2. Assume that*

$$\limsup_{j\to\infty}\left(\frac{-\lambda\left(f^{(j)}(x_0)\right)}{j}\right) < \infty \text{ and } \limsup_{k\to\infty}\lambda\left(\delta_k\right) < \infty.$$

Then, there exists a neighbourhood U of x_0 in A such that, for any x, $y \in U$, we have that

$$f(y) = \sum_{j=0}^{\infty}\frac{f^{(j)}(x)}{j!}(y-x)^j.$$

That is, the Taylor series $\sum_{j=0}^{\infty}\frac{f^{(j)}(x)}{j!}(y-x)^j$ converges in $\mathcal{N}$ to $f(y)$; and hence f is analytic in U.

Proof. First, we note that

$$\begin{aligned}
\limsup_{j\to\infty}\left(\frac{-\lambda(f^{(j)}(x_0))}{j}\right) < \infty &\iff \limsup_{j\to\infty}\exp\left(\frac{-\lambda(f^{(j)}(x_0))}{j}\right) < \infty\\
&\iff \limsup_{j\to\infty}\sqrt[j]{\exp(-\lambda(f^{(j)}(x_0)))} < \infty\\
&\iff \frac{1}{\limsup_{j\to\infty}\sqrt[j]{\exp(-\lambda(f^{(j)}(x_0)))}} > 0\\
&\iff \frac{1}{\limsup_{j\to\infty}\sqrt[j]{|f^{(j)}(x_0)|_v}} > 0\\
&\iff \frac{1}{\limsup_{j\to\infty}\sqrt[j]{|f^{(j)}(x_0)/j!|_v}} > 0.
\end{aligned}$$

Thus, the Taylor series $\sum_{j=0}^{\infty}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$ of f at x_0 has a positive radius of convergence [9]

$$R := \frac{1}{\limsup_{j\to\infty}\sqrt[j]{|f^{(j)}(x_0)/j!|_v}} = \frac{1}{\limsup_{j\to\infty}\sqrt[j]{|f^{(j)}(x_0)|_v}} > 0.$$

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Let

$$\lambda_0 = \limsup_{j\to\infty}\left(\frac{-\lambda\left(f^{(j)}(x_0)\right)}{j}\right).$$

Then, $\lambda_0 \in \mathbb{R}$ and $\lambda_0 < \infty$. Then, for all $x \in \mathcal{N}$ satisfying $\lambda(x - x_0) > \lambda_0$, we have that

$$\begin{aligned}
|x - x_0|_v &= \begin{cases} \exp(-\lambda(x-x_0)) & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0 \end{cases} \\
&< \exp(-\lambda_0) = \frac{1}{\exp(\lambda_0)} \\
&= \frac{1}{\exp(\limsup_{j\to\infty}(-\lambda(f^{(j)}(x_0))/j))} \\
&= \frac{1}{\limsup_{j\to\infty}\exp(-\lambda(f^{(j)}(x_0))/j)} \\
&= \frac{1}{\limsup_{j\to\infty}\sqrt[j]{|f^{(j)}(x_0)|_v}} = R.
\end{aligned}$$

Thus, for all $x \in \mathcal{N}$ satisfying $\lambda(x - x_0) > \lambda_0$, we have that $|x - x_0|_v < R$ and, by [9, p. 59], $\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j$ converges in $(\mathcal{N}, \tau_v)$; that is, it converges with respect to both the ultrametric absolute value $|\cdot|_v$ and the ordinary absolute value $|\cdot|$.

For all $k \in \mathbb{N}$, we have that $(x_0 - \delta_k, x_0 + \delta_k) \subset A$, f is k times differentiable on $(x_0 - \delta_k, x_0 + \delta_k)$, and

$$\left| f(x) - \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \right| \leq |x - x_0|^k \text{ for all } x \in (x_0 - \delta_k, x_0 + \delta_k).$$

Since $\limsup_{k\to\infty} \lambda(\delta_k) < \infty$, there exists $t > 0$ in $\mathbb{Q}$ such that $\limsup_{k\to\infty} \lambda(\delta_k) < t < \infty$. Thus, there exists $N \in \mathbb{N}$ such that

$$\lambda(\delta_k) < t \text{ for all } k > N. \tag{2}$$

Let $\delta > 0$ in $\mathcal{N}$ be such that $\lambda(\delta) > \max\{\lambda_0, t, 0\}$; this is possible since $\max\{\lambda_0, t, 0\} < \infty$. It follows from (2) that $\lambda(\delta) > \lambda(\delta_k)$ and hence $0 < \delta \ll \delta_k$ for all $k > N$. Thus, $(x_0 - \delta, x_0 + \delta) \subset A$, f is infinitely often differentiable on $(x_0 - \delta, x_0 + \delta)$, and

$$\left| f(x) - \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j \right| \leq |x - x_0|^k \,\forall\, x \in (x_0 - \delta, x_0 + \delta) \text{ and } \forall\, k > N. \tag{3}$$

Moreover, for all $x \in (x_0 - \delta, x_0 + \delta)$, we have that $\lambda(x - x_0) \geq \lambda(\delta) > \lambda_0$ and hence $\sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j$ converges in $\mathcal{N}$. Let $U = (x_0 - \delta, x_0 + \delta)$.

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First, we show that

$$f(x)=\sum_{j=0}^{\infty}\frac{f^{(j)}\left(x_0\right)}{j!}\left(x-x_0\right)^j \text{ for all } x\in U.$$

Let $x\in U$ be given. Taking the limit in (3) as $k\to\infty$, we get:

$$0\le\lim_{k\to\infty}\left|f(x)-\sum_{j=0}^{k}\frac{f^{(j)}\left(x_0\right)}{j!}\left(x-x_0\right)^j\right|\le\lim_{k\to\infty}\left|x-x_0\right|^k,$$

from which we obtain

$$0\le\left|f(x)-\lim_{k\to\infty}\sum_{j=0}^{k}\frac{f^{(j)}\left(x_0\right)}{j!}\left(x-x_0\right)^j\right|\le\lim_{k\to\infty}\left|x-x_0\right|^k.$$

Since $\lambda(x-x_0)\ge\lambda(\delta)>0$, we obtain that $\lim_{k\to\infty}\left|x-x_0\right|^k=0$. It follows that

$$0\le\left|f(x)-\sum_{j=0}^{\infty}\frac{f^{(j)}\left(x_0\right)}{j!}\left(x-x_0\right)^j\right|\le 0$$

from which we infer that $f(x)=\sum_{j=0}^{\infty}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$ or, equivalently,

$$f(x)=\sum_{l=0}^{\infty}\frac{f^{(l)}(x_0)}{l!}(x-x_0)^l. \tag{4}$$

Since the convergence of the Taylor series above is in the order (valuation) topology, we will show that the derivatives of f at x to any order are obtained by differentiating the power series in Equation (4) term by term. That is, for all $j\in\mathbb{N}$,

$$f^{(j)}(x)=\sum_{l=j}^{\infty}l(l-1)\dots(l-j+1)\frac{f^{(l)}(x_0)}{l!}(x-x_0)^{l-j}. \tag{5}$$

First, note that since $\lambda(l(l-1)\dots(l-j+1))=0$, it follows that $\sum_{l=j}^{\infty}l(l-1)\dots(l-j+1)\frac{f^{(l)}(x_0)}{l!}(x-x_0)^{l-j}$ converges in $\mathcal{N}$ for all $j\in\mathbb{N}$. Using induction on j, it suffices

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to show that

$$f'(x) = \sum_{l=1}^{\infty} l \frac{f^{(l)}(x_0)}{l!} (x - x_0)^{l-1} = \sum_{l=1}^{\infty} \frac{f^{(l)}(x_0)}{(l-1)!} (x - x_0)^{l-1}.$$

Let $h \in \mathcal{N}$ be such that $x + h \in U$. We will show that

$$\lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\} = \sum_{l=1}^{\infty} \frac{f^{(l)}(x_0)}{(l-1)!} (x - x_0)^{l-1}.$$

Thus,

$$\begin{aligned}
\lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\} &= \lim_{h \to 0} \left\{ \sum_{l=0}^{\infty} \frac{f^{(l)}(x_0)}{l!} \frac{(x+h-x_0)^l - (x-x_0)^l}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \sum_{l=1}^{\infty} \frac{f^{(l)}(x_0)}{l!} \frac{(x+h-x_0)^l - (x-x_0)^l}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \sum_{l=1}^{\infty} \frac{f^{(l)}(x_0)}{l!} [(x+h-x_0)^{l-1} \right. \\
&\qquad \left. + (x+h-x_0)^{l-2}(x-x_0) + \cdots + (x-x_0)^{l-1}] \right\} \\
&= \sum_{l=1}^{\infty} \frac{f^{(l)}(x_0)}{l!} [l(x-x_0)^{l-1}] \\
&= \sum_{l=1}^{\infty} \frac{f^{(l)}(x_0)}{(l-1)!} (x-x_0)^{l-1}.
\end{aligned}$$

Now, let $y \in U$ be given. Then

$$\begin{aligned}
f(y) &= \sum_{l=0}^{\infty} \frac{f^{(l)}(x_0)}{l!} (y - x_0)^l \\
&= \sum_{l=0}^{\infty} \frac{f^{(l)}(x_0)}{l!} [(y-x) + (x-x_0)]^l \\
&= \sum_{l=0}^{\infty} \sum_{j=0}^{l} \frac{f^{(l)}(x_0)}{l!} \binom{l}{j} (y-x)^j (x-x_0)^{l-j} \\
&= \sum_{l=0}^{\infty} \sum_{j=0}^{l} \frac{l(l-1)\ldots(l-j+1)}{j!} \frac{f^{(l)}(x_0)}{l!} (x-x_0)^{l-j} (y-x)^j.
\end{aligned}$$

Since convergence in the order topology (valuation topology) entails absolute convergence, we can interchange the order of the summations in the last equality [10, 14]. We get:

$$\begin{aligned} f(y) &= \sum_{j=0}^{\infty} \frac{1}{j!} \left[\sum_{l=j}^{\infty} l(l-1)\ldots(l-j+1) \frac{f^{(l)}(x_0)}{l!} (x-x_0)^{l-j} \right] (y-x)^j \\ &= \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} (y-x)^j \end{aligned}$$

where we made use of Equation (5) in the last step. □

Replacing m by 1 in Definition 3, then the $m \times n$ matrix $\boldsymbol{Df}(\boldsymbol{x})$ is replaced by the gradient of $\boldsymbol{f}$ at $\boldsymbol{x}$: $\boldsymbol{\nabla} f(\boldsymbol{x})$, and we readily obtain the definition of a WLUD $\mathcal{N}$-valued function at a point $\boldsymbol{x_0}$ or on an open subset A of $\mathcal{N}^n$.

Definition 4. Let $A \subset \mathcal{N}^n$ be open, let $f : A \to \mathcal{N}$, and let $\boldsymbol{x_0} \in A$ be given. Then, we say that f is WLUD at $\boldsymbol{x_0}$ if f is differentiable in a neighbourhood Ω of $\boldsymbol{x_0}$ in A and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $B_\delta(\boldsymbol{x_0}) \subset \Omega$, and for all $\boldsymbol{x}, \boldsymbol{y} \in B_\delta(\boldsymbol{x_0})$ we have that

$$|f(\boldsymbol{y}) - f(\boldsymbol{x}) - \boldsymbol{\nabla} f(\boldsymbol{x}) \cdot (\boldsymbol{y} - \boldsymbol{x})| \leq \epsilon |\boldsymbol{y} - \boldsymbol{x}|.$$

Moreover, we say that f is WLUD on A if f is WLUD at every point in A.

Using Definition 2 and Definition 4, the natural way to define k times weak local uniform differentiability (WLUD^k) at a point $\boldsymbol{x_0}$ or on an open subset A of $\mathcal{N}^n$ is as follows.

Definition 5. Let $A \subset \mathcal{N}^n$ be open, let $f : A \to \mathcal{N}$, and let $\boldsymbol{x_0} \in A$ be given. Then, we say that f is WLUD^k at $\boldsymbol{x_0}$ if f is k-times differentiable in a neighbourhood Ω of $\boldsymbol{x_0}$ in A and if for every $\epsilon > 0$ in $\mathcal{N}$ there exists $\delta > 0$ in $\mathcal{N}$ such that $B_\delta(\boldsymbol{x_0}) \subset \Omega$, and for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in B_\delta(\boldsymbol{x_0})$ we have that

$$\left| f(\boldsymbol{\eta}) - f(\boldsymbol{\xi}) - \sum_{j=1}^{k} \frac{1}{j!} \left[(\boldsymbol{\eta} - \boldsymbol{\xi}) \cdot \nabla \right]^j f(\boldsymbol{\xi}) \right| \leq \epsilon |\boldsymbol{\eta} - \boldsymbol{\xi}|^k,$$

where

$$\begin{aligned} [(\boldsymbol{\eta} - \boldsymbol{\xi}) \cdot \nabla]^j f(\boldsymbol{\xi}) &= [(\eta_1 - \xi_1) \frac{\partial}{\partial x_1} + \cdots + (\eta_n - \xi_n) \frac{\partial}{\partial x_n}]^j f(\boldsymbol{x}) \Big|_{\boldsymbol{x}=\boldsymbol{\xi}} \\ &= \sum_{l_1, \ldots, l_j = 1}^{n} \left(\frac{\partial^j f(\boldsymbol{x})}{\partial_{x_{l_1}} \cdots \partial_{x_{l_j}}} \right|_{\boldsymbol{x}=\boldsymbol{\xi}} \prod_{m=1}^{j} (\eta_{l_m} - \xi_{l_m})). \end{aligned}$$

Moreover, we say that f is WLUD^k on A if f is WLUD^k at every point in A.

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Definition 6. Let $A \subset \mathcal{N}^n$ be open, let $f : A \to \mathcal{N}$, and let $\boldsymbol{x_0} \in A$ be given. Then, we say that f is WLUD$^\infty$ at $\boldsymbol{x_0}$ if f is WLUD^k at $\boldsymbol{x_0}$ for every $k \in \mathbb{N}$. Moreover, we say that f is WLUD$^\infty$ on A if f is WLUD$^\infty$ at every point in A.

Now, we are ready to state and prove the analogue of Theorem 8 for functions of n variables.

Theorem 8. *Let $A \subseteq \mathcal{N}^n$ be open, let $\boldsymbol{x_0} \in A$, and let $f : A \to \mathcal{N}$ be WLUD$^\infty$ at $\boldsymbol{x_0}$. For each $k \in \mathbb{N}$, let $\delta_k > 0$ in $\mathcal{N}$ correspond to $\epsilon = 1$ in Definition 5. Assume that*

$$\limsup_{\substack{j \to \infty \\ l_1 = 1, \ldots, n \\ \vdots \\ l_j = 1, \ldots, n}} \left(\frac{-\lambda \left(\left. \frac{\partial^j f(\boldsymbol{x})}{\partial x_{l_1} \cdots \partial x_{l_j}} \right|_{\boldsymbol{x} = \boldsymbol{x_0}} \right)}{j} \right) < \infty$$

$$\text{and } \limsup_{k \to \infty} \lambda(\delta_k) < \infty.$$

Then, there exists a neighbourhood U of $\boldsymbol{x_0}$ in A such that, for any $\boldsymbol{\eta} \in U$, we have that

$$f(\boldsymbol{\eta}) = f(\boldsymbol{x_0}) + \sum_{j=1}^{\infty} \frac{1}{j!} \left[(\boldsymbol{\eta} - \boldsymbol{x_0}) \cdot \nabla \right]^j f(\boldsymbol{x_0}).$$

Proof. Let

$$\lambda_0 = \limsup_{\substack{j \to \infty \\ l_1 = 1, \ldots, n \\ \vdots \\ l_j = 1, \ldots, n}} \left(\frac{-\lambda \left(\left. \frac{\partial^j f(\boldsymbol{x})}{\partial x_{l_1} \cdots \partial x_{l_j}} \right|_{\boldsymbol{x} = \boldsymbol{x_0}} \right)}{j} \right).$$

Then, $\lambda_0 \in \mathbb{R}$ and $\lambda_0 < \infty$.

For all $k \in \mathbb{N}$, we have that $B_{\delta_k}(\boldsymbol{x_0}) \subset A$, f is k times differentiable on $B_{\delta_k}(\boldsymbol{x_0})$, and

$$\left| f(\boldsymbol{\eta}) - f(\boldsymbol{x_0}) - \sum_{j=1}^{k} \frac{1}{j!} \left[(\boldsymbol{\eta} - \boldsymbol{x_0}) \cdot \nabla \right]^j f(\boldsymbol{x_0}) \right| \leq |\boldsymbol{\eta} - \boldsymbol{x_0}|^k \text{ for all } \boldsymbol{\eta} \in B_{\delta_k}(\boldsymbol{x_0}).$$

Since $\limsup_{k \to \infty} \lambda(\delta_k) < \infty$, there exists $t > 0$ in $\mathbb{Q}$ such that $\limsup_{k \to \infty} \lambda(\delta_k) < t < \infty$. Thus, there exists $N \in \mathbb{N}$ such that

$$\lambda(\delta_k) < t \text{ for all } k > N. \tag{6}$$

Let $\delta > 0$ in $\mathcal{N}$ be such that $\lambda(\delta) > \max\{\lambda_0, t, 0\}$. It follows from (6) that $\lambda(\delta) > \lambda(\delta_k)$ and hence $0 < \delta \ll \delta_k$ for all $k > N$. Thus, $B_\delta(\boldsymbol{x_0}) \subset A$, f is infinitely often differentiable

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on $B_\delta(\boldsymbol{x_0})$, and

$$\left| f(\boldsymbol{\eta}) - f(\boldsymbol{x_0}) - \sum_{j=1}^{k} \frac{1}{j!}[(\boldsymbol{\eta}-\boldsymbol{x_0})\cdot\nabla]^j f(\boldsymbol{x_0}) \right| \le |\boldsymbol{\eta}-\boldsymbol{x_0}|^k \ \forall \boldsymbol{\eta} \in B_\delta(\boldsymbol{x_0}) \text{ and } \forall k > N. \tag{7}$$

Let $U = B_\delta(\boldsymbol{x_0})$; and let $\boldsymbol{\eta} \in U$ be given. Then, we have that $\lambda(|\boldsymbol{\eta}-\boldsymbol{x_0}|) \ge \lambda(\delta) > \lambda_0$. We will show first that $\sum_{j=1}^{\infty} \frac{1}{j!}[(\boldsymbol{\eta}-\boldsymbol{x_0})\cdot\nabla]^j f(\boldsymbol{x_0})$ converges in $\mathcal{N}$. Since $\lambda(|\boldsymbol{\eta}-\boldsymbol{x_0}|) > \lambda_0$, there exists $q > 0$ in $\mathbb{Q}$ such that $\lambda(|\boldsymbol{\eta}-\boldsymbol{x_0}|) - q > \lambda_0$. Hence, there exists $M \in \mathbb{N}$ such that

$$\lambda(|\boldsymbol{\eta}-\boldsymbol{x_0}|) - q > \frac{-\lambda\left(\left.\frac{\partial^j f(\boldsymbol{x})}{\partial_{x_{l_1}}\cdots\partial_{x_{l_j}}}\right|_{\boldsymbol{x}=\boldsymbol{x_0}} \right)}{j}$$

for all $j > M$ and for $l_1 = 1, \ldots, n$, $l_2 = 1, \ldots, n$, $\ldots$, $l_j = 1, \ldots, n$. It follows that

$$\begin{aligned}
\lambda\left(\left.\frac{\partial^j f(\boldsymbol{x})}{\partial_{x_{l_1}}\cdots\partial_{x_{l_j}}}\right|_{\boldsymbol{x}=\boldsymbol{x_0}} \prod_{m=1}^{j}(\eta_{l_m} - x_{0,l_m}) \right) &\ge \lambda\left(\left.\frac{\partial^j f(\boldsymbol{x})}{\partial_{x_{l_1}}\cdots\partial_{x_{l_j}}}\right|_{\boldsymbol{x}=\boldsymbol{x_0}} |\boldsymbol{\eta}-\boldsymbol{x_0}|^j \right) \\
&= \lambda\left(\left.\frac{\partial^j f(\boldsymbol{x})}{\partial_{x_{l_1}}\cdots\partial_{x_{l_j}}}\right|_{\boldsymbol{x}=\boldsymbol{x_0}} \right) + j\lambda(|\boldsymbol{\eta}-\boldsymbol{x_0}|) \\
&> jq
\end{aligned}$$

for all $j > M$ and for $l_1 = 1, \ldots, n$, $l_2 = 1, \ldots, n$, $\ldots$, $l_j = 1, \ldots, n$. Thus,

$$\begin{aligned}
\lambda([(\boldsymbol{\eta}-\boldsymbol{x_0})\cdot\nabla]^j f(\boldsymbol{x_0})) &= \lambda\left(\sum_{l_1,\ldots,l_j=1}^{n} \left(\left.\frac{\partial^j f(\boldsymbol{x})}{\partial_{x_{l_1}}\cdots\partial_{x_{l_j}}}\right|_{\boldsymbol{x}=\boldsymbol{x_0}} \prod_{m=1}^{j}(\eta_{l_m} - x_{0,l_m}) \right) \right) \\
&> jq
\end{aligned}$$

for all $j > M$; and hence

$$\begin{aligned}
\lim_{j\to\infty} \lambda\left(\frac{1}{j!}[(\boldsymbol{\eta}-\boldsymbol{x_0})\cdot\nabla]^j f(\boldsymbol{x_0}) \right) &= \lim_{j\to\infty} \lambda([(\boldsymbol{\eta}-\boldsymbol{x_0})\cdot\nabla]^j f(\boldsymbol{x_0})) \\
&\ge q \lim_{j\to\infty} j = \infty.
\end{aligned}$$

Thus,

$$\lim_{j\to\infty} \left(\frac{1}{j!} \left[(\boldsymbol{\eta}-\boldsymbol{x_0})\cdot\nabla \right]^j f(\boldsymbol{x_0}) \right) = 0$$

and hence $\sum_{j=1}^{\infty} \frac{1}{j!}[(\boldsymbol{\eta}-\boldsymbol{x_0})\cdot\nabla]^j f(\boldsymbol{x_0})$ converges in $\mathcal{N}$; that is,

$$\lim_{k\to\infty} \sum_{j=1}^{k} \frac{1}{j!} \left[(\boldsymbol{\eta}-\boldsymbol{x_0})\cdot\nabla \right]^j f(\boldsymbol{x_0}) \text{ exists in } \mathcal{N}.$$

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Taking the limit in (7) as $k \to \infty$, we get:

$$0 \le \lim_{k\to\infty} \left| f(\boldsymbol{\eta}) - f(\boldsymbol{x_0}) - \sum_{j=1}^{k} \frac{1}{j!} \left[(\boldsymbol{\eta} - \boldsymbol{x_0}) \cdot \nabla \right]^j f(\boldsymbol{x_0}) \right| \le \lim_{k\to\infty} |\boldsymbol{\eta} - \boldsymbol{x_0}|^k,$$

from which we obtain

$$0 \le \left| f(\boldsymbol{\eta}) - f(\boldsymbol{x_0}) - \lim_{k\to\infty} \sum_{j=1}^{k} \frac{1}{j!} \left[(\boldsymbol{\eta} - \boldsymbol{x_0}) \cdot \nabla \right]^j f(\boldsymbol{x_0}) \right| \le \lim_{k\to\infty} |\boldsymbol{\eta} - \boldsymbol{x_0}|^k.$$

Since $\lambda(|\boldsymbol{\eta} - \boldsymbol{x_0}|) \ge \lambda(\delta) > 0$, we obtain that $\lim_{k\to\infty} |\boldsymbol{\eta} - \boldsymbol{x_0}|^k = 0$. It follows that

$$0 \le \left| f(\boldsymbol{\eta}) - f(\boldsymbol{x_0}) - \sum_{j=1}^{\infty} \frac{1}{j!} \left[(\boldsymbol{\eta} - \boldsymbol{x_0}) \cdot \nabla \right]^j f(\boldsymbol{x_0}) \right| \le 0$$

from which we infer that

$$f(\boldsymbol{\eta}) = f(\boldsymbol{x_0}) + \sum_{j=1}^{\infty} \frac{1}{j!} \left[(\boldsymbol{\eta} - \boldsymbol{x_0}) \cdot \nabla \right]^j f(\boldsymbol{x_0}).$$

□

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