THE EXTREMAL POINTS OF THE RANGE OF A VECTOR-VALUED MEASURE

by I. TWEDDLE

(Received 26 March, 1970)

Recently, several papers have investigated conditions under which the range of a vector-valued measure is a compact convex set (see e.g. [**1**], [**2**], [**3**]). It therefore seems of interest to characterise the extremal points of the range in such cases.

Let $\mathscr{M}$ be a σ-algebra of subsets of a set S and let E be a separated topological vector space. Let $\mathbf{m}: \mathscr{M} \to E$ be a vector-valued measure such that, for each $X \in \mathscr{M}$,

$$R(X) = \{\mathbf{m}(Y): Y \in \mathscr{M},\ Y \subseteq X\}$$

is convex. The range of $\mathbf{m}$ is the set $R = R(S)$.

In the following, it is assumed that any subset of S considered is an element of $\mathscr{M}$. The complement of a subset X of S will be denoted by X', and the set of extremal points of a convex subset A of E will be denoted by $\operatorname{Ext}(A)$.

THEOREM 1. $\mathbf{m}(X) \in \operatorname{Ext}(R)$ *if and only if*

$$R(X) \cap R(X') = \operatorname{Ext}(R(X)) \cap \operatorname{Ext}(R(X')) = \{0\}.$$

THEOREM 2. $\mathbf{m}(X) \in \operatorname{Ext}(R)$ *if and only if* $\mathbf{m}(X) = \mathbf{m}(Y)$ *implies that* $\mathbf{m}(X \cap Z) = \mathbf{m}(Y \cap Z)$ *for each* $Z \in \mathscr{M}$.

Both these results are suggested by the Hahn decomposition theorem for scalar valued measures. Their proofs are divided into several stages.

LEMMA 1. *If* $X \subseteq Y$ *and* $\mathbf{m}(X) \in \operatorname{Ext}(R(Y))$, *then* $\mathbf{m}(Y \backslash X) \in \operatorname{Ext}(R(Y))$.

Proof. Suppose that there exist $W, Z \subseteq Y$ such that $\mathbf{m}(Y \backslash X) = \frac{1}{2}\mathbf{m}(W) + \frac{1}{2}\mathbf{m}(Z)$. Then

$$\begin{aligned}\mathbf{m}(X) &= \mathbf{m}(Y) - \mathbf{m}(Y \backslash X) = \tfrac{1}{2}\{\mathbf{m}(Y) - \mathbf{m}(W)\} + \tfrac{1}{2}\{\mathbf{m}(Y) - \mathbf{m}(Z)\} \\ &= \tfrac{1}{2}\mathbf{m}(Y \backslash W) + \tfrac{1}{2}\mathbf{m}(Y \backslash Z).\end{aligned}$$

Since $\mathbf{m}(X) \in \operatorname{Ext}(R(Y))$, it follows that $\mathbf{m}(Y \backslash W) = \mathbf{m}(Y \backslash Z)$, so that $\mathbf{m}(W) = \mathbf{m}(Z)$, i.e., $\mathbf{m}(Y \backslash X) \in \operatorname{Ext}(R(Y))$.

LEMMA 2. *If* $X \subseteq Y$ *and* $\mathbf{m}(Z) \in \operatorname{Ext}(R(Y))$, $Z \subseteq Y$, *then* $\mathbf{m}(X \cap Z) \in \operatorname{Ext}(R(X))$.

Proof. Suppose that there exist $W_1, W_2 \subseteq X$ such that $\mathbf{m}(X \cap Z) = \frac{1}{2}\mathbf{m}(W_1) + \frac{1}{2}\mathbf{m}(W_2)$. Then

$$\begin{aligned}\mathbf{m}(Z) &= \mathbf{m}(X \cap Z) + \mathbf{m}(Z \backslash X) \\ &= \tfrac{1}{2}\{\mathbf{m}(W_1) + \mathbf{m}(Z \backslash X)\} + \tfrac{1}{2}\{\mathbf{m}(W_2) + \mathbf{m}(Z \backslash X)\} \\ &= \tfrac{1}{2}\mathbf{m}(W_1 \cup (Z \backslash X)) + \tfrac{1}{2}\mathbf{m}(W_2 \cup (Z \backslash X)).\end{aligned}$$

Since $W_1 \cup (Z \backslash X)$ and $W_2 \cup (Z \backslash X)$ are contained in Y, it follows that $\mathbf{m}(W_1 \cup (Z \backslash X)) = \mathbf{m}(W_2 \cup (Z \backslash X))$, which leads as in Lemma 1 to the conclusion that $\mathbf{m}(X \cap Z) \in \operatorname{Ext}(R(X))$.

https://doi.org/10.1017/S0017089500001385 Published online by Cambridge University Press

LEMMA 3. *Let A and B be convex subsets of E. Let $z \in A+B$ and suppose that*

(i) *there exist a unique $x \in A$ and a unique $y \in B$ such that $z = x+y$,*
(ii) $x \in \text{Ext}(A)$, $y \in \text{Ext}(B)$.

Then $z \in \text{Ext}(A+B)$.

Proof. Suppose that $z = \frac{1}{2}(x_1+y_1)+\frac{1}{2}(x_2+y_2)$, where $x_1, x_2 \in A$ and $y_1, y_2 \in B$. Then $z = \frac{1}{2}(x_1+x_2)+\frac{1}{2}(y_1+y_2)$ and, since A and B are convex, $\frac{1}{2}(x_1+x_2) \in A$ and $\frac{1}{2}(y_1+y_2) \in B$. It now follows from (i) that $x = \frac{1}{2}(x_1+x_2)$ and $y = \frac{1}{2}(y_1+y_2)$ and then from (ii) that $x_1 = x_2$ and $y_1 = y_2$. Thus $x_1+y_1 = x_2+y_2$, which gives the required result.

Proof of Theorem 1. If $\mathbf{m}(X) \in \text{Ext}(R)$, so also does $\mathbf{m}(X')$ by Lemma 1. Thus $0 = \mathbf{m}(X \cap X') = \mathbf{m}(X' \cap X) \in \text{Ext}(R(X)) \cap \text{Ext}(R(X'))$ by Lemma 2.

Suppose that $W \subseteq X$, $Z \subseteq X'$ and $\mathbf{m}(W) = \mathbf{m}(Z)$ (*). Then

$$\mathbf{m}(X) = \tfrac{1}{2}\{\mathbf{m}(X)+\mathbf{m}(Z)\}+\tfrac{1}{2}\{\mathbf{m}(X)-\mathbf{m}(W)\} = \tfrac{1}{2}\mathbf{m}(X \cup Z)+\tfrac{1}{2}\mathbf{m}(X \backslash W),$$

so that $\mathbf{m}(X \cup Z) = \mathbf{m}(X \backslash W)$, since $\mathbf{m}(X) \in \text{Ext}(R)$. Thus $\mathbf{m}(Z) = -\mathbf{m}(W)$, which combined with (*) shows that $\mathbf{m}(W) = \mathbf{m}(Z) = 0$, i.e., $R(X) \cap R(X') = \{0\}$. Combining these results, we have

$$\{0\} \subseteq \text{Ext}(R(X)) \cap \text{Ext}(R(X')) \subseteq R(X) \cap R(X') = \{0\},$$

which establishes the necessity of the condition.

Conversely, suppose that $R(X) \cap R(X') = \text{Ext}(R(X)) \cap \text{Ext}(R(X')) = \{0\}$. Since $\mathbf{m}(\emptyset) = 0 \in \text{Ext}(R(X))$, $\mathbf{m}(X) \in \text{Ext}(R(X))$ by Lemma 1. Now $R(S) = R(X)+R(X')$. Choose any $W \subseteq X$, $Z \subseteq X'$ such that $\mathbf{m}(X) = \mathbf{m}(W)+\mathbf{m}(Z)$. Then

$$\mathbf{m}(X \backslash W) = \mathbf{m}(X)-\mathbf{m}(W) = \mathbf{m}(Z),$$

and, since $\mathbf{m}(X \backslash W) \in R(X)$ and $\mathbf{m}(Z) \in R(X')$, it follows that $\mathbf{m}(Z) = 0$ ($\in \text{Ext}(R(X'))$) and $\mathbf{m}(W) = \mathbf{m}(X)$. Thus, by Lemma 3, $\mathbf{m}(X) \in \text{Ext}(R)$.

LEMMA 4. $0 \in \text{Ext}(R(X))$ *if and only if* $Z \subseteq Y \subseteq X$ *and* $\mathbf{m}(Y) = 0$ *imply that* $\mathbf{m}(Z) = 0$.

Proof. Suppose that $Z \subseteq Y \subseteq X$ and $\mathbf{m}(Y) = 0$. Then $0 = \mathbf{m}(Y) = \mathbf{m}(Z)+\mathbf{m}(Y \backslash Z) = \frac{1}{2}\mathbf{m}(Z)+\frac{1}{2}\mathbf{m}(Y \backslash Z)$, and, if $0 \in \text{Ext}(R(X))$, it follows that $\mathbf{m}(Z) = 0$.

Conversely, suppose that the given condition is satisfied, and that $0 = \frac{1}{2}\mathbf{m}(W_1)+\frac{1}{2}\mathbf{m}(W_2)$, where $W_1, W_2 \subseteq X$. Then

$$\begin{aligned} 0 &= \tfrac{1}{2}\{\mathbf{m}(W_1 \cap W_2)+\mathbf{m}(W_1 \backslash W_2)\}+\tfrac{1}{2}\{\mathbf{m}(W_1 \cap W_2)+\mathbf{m}(W_2 \backslash W_1)\} \\ &= \mathbf{m}(W_1 \cap W_2)+\tfrac{1}{2}\mathbf{m}(W_1 \backslash W_2)+\tfrac{1}{2}\mathbf{m}(W_2 \backslash W_1) \\ &= \mathbf{m}(W_1 \cap W_2)+\mathbf{m}(W) \end{aligned}$$

for some $W \subseteq (W_1 \backslash W_2) \cup (W_2 \backslash W_1)$, since $R((W_1 \backslash W_2) \cup (W_2 \backslash W_1))$ is convex. Thus $0 = \mathbf{m}((W_1 \cap W_2) \cup W)$, which, by hypothesis, implies that $\mathbf{m}(W_1 \cap W_2) = 0$. Hence

$$0 = \tfrac{1}{2}\mathbf{m}(W_1 \backslash W_2)+\tfrac{1}{2}\mathbf{m}(W_2 \backslash W_1) = \tfrac{1}{2}\mathbf{m}((W_1 \backslash W_2) \cup (W_2 \backslash W_1)),$$

https://doi.org/10.1017/S0017089500001385 Published online by Cambridge University Press

and, as before, this implies that $\mathbf{m}(W_1\backslash W_2) = \mathbf{m}(W_2\backslash W_1) = 0$. Thus finally, $\mathbf{m}(W_1) = \mathbf{m}(W_2) = 0$, i.e. $0 \in \mathrm{Ext}(R(X))$.

Proof of Theorem 2. Suppose that $\mathbf{m}(X) \in \mathrm{Ext}(R)$ and $\mathbf{m}(Y) = \mathbf{m}(X)$. Then $\mathbf{m}(X) = \frac{1}{2}\mathbf{m}(X \cap Y) + \frac{1}{2}\mathbf{m}(Y \cup (X\backslash Y))$, which implies that $\mathbf{m}(Y \cup (X\backslash Y)) = \mathbf{m}(X)$, so that $\mathbf{m}(X\backslash Y) = 0$. Similarly $\mathbf{m}(Y\backslash X) = 0$.

Now, by Lemmas 2 and 1, 0 is an extremal point of both $R(X)$ and $R(Y)$. Hence

$$\mathbf{m}(X \cap Z) = \mathbf{m}(X \cap Y \cap Z) + \mathbf{m}((X\backslash Y) \cap Z) = \mathbf{m}(X \cap Y \cap Z)$$

and

$$\mathbf{m}(Y \cap Z) = \mathbf{m}(X \cap Y \cap Z) + \mathbf{m}((Y\backslash X) \cap Z) = \mathbf{m}(X \cap Y \cap Z)$$

by Lemma 4. This establishes the necessity of the condition.

Now suppose that the given condition is satisfied. If $Z \subseteq X$ and $\mathbf{m}(Z) = 0$, then $\mathbf{m}(X) = \mathbf{m}(X\backslash Z)$, and, if $W \subseteq Z$, $\mathbf{m}(X \cap W') = \mathbf{m}((X\backslash Z) \cap W') = \mathbf{m}(X\backslash Z) = \mathbf{m}(X)$. Thus $\mathbf{m}(W) = \mathbf{m}(X) - \mathbf{m}(X\backslash W) = 0$. It now follows from Lemma 4 that $0 \in \mathrm{Ext}(R(X))$.

Also, if $\mathbf{m}(X') = \mathbf{m}(Y)$, $\mathbf{m}(X) = \mathbf{m}(Y')$, so that

$$\mathbf{m}(X' \cap Z) = \mathbf{m}(Z) - \mathbf{m}(X \cap Z) = \mathbf{m}(Z) - \mathbf{m}(Y' \cap Z) = \mathbf{m}(Y \cap Z);$$

i.e., X' has the same property as X, so that as before $0 \in \mathrm{Ext}(R(X'))$. The result will follow by Theorem 1 if it is now shown that $R(X) \cap R(X') = \{0\}$.

Suppose that $W \subseteq X, Z \subseteq X'$ and $\mathbf{m}(W) = \mathbf{m}(Z)$. Then $\mathbf{m}(X) = \mathbf{m}(X\backslash W) + \mathbf{m}(W) = \mathbf{m}((X\backslash W) \cup Z)$, so that, by hypothesis,

$$0 = \mathbf{m}(X \cap Z) = \mathbf{m}(((X\backslash W) \cup Z) \cap Z) = \mathbf{m}(Z).$$

This completes the proof.

REFERENCES

1. J. F. C. Kingman and A. P. Robertson, On a theorem of Lyapunov, *J. London Math. Soc.* **43** (1968), 347–351.

2. J. Schmets, Sur une généralisation d'une théorème de Lyapounov, *Bull. Soc. Royale Liège* **35** (1966), 185–194.

3. J. J. Uhl, Jr, The range of a vector-valued measure, *Proc. Amer. Math. Soc.* **23** (1969), 158–163.

UNIVERSITY OF STIRLING

https://doi.org/10.1017/S0017089500001385 Published online by Cambridge University Press