ON FUNCTIONS OF BOUNDED ω-VARIATION, II

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1. Introduction

Let $\omega(x)$ be a non-decreasing function defined in the interval $[a, b]$. We extend the definition to all x by taking $\omega(x) = \omega(a)$ for $x < a$ and $\omega(x) = \omega(b)$ for $x > b$. R. L. Jeffery [2] has denoted by $\mathscr{U}$ the class of functions $F(x)$ defined as follows:

If S denotes the set of points of $[a, b]$ at which $\omega(x)$ is continuous, then $F(x)$ is defined, and continuous over S, at all points of S. At any point of discontinuity x_0 of $\omega(x)$, it is supposed that $F(x)$ tends to a limit as x tends to x_0+ and to x_0- over the points of S. These limits will be denoted by $F(x_0+)$ and $F(x_0-)$. Also for $x < a$, it is assumed that $F(x) = F(a+)$ and for $x > b$, $F(x) = F(b-)$. $F(x)$ may or may not be defined at points of discontinuity of $\omega(x)$.

Jeffery also has introduced the following

Definition. A function $F(x)$ defined on $[a, b]$ and in $\mathscr{U}$ is absolutely continuous relative to ω, $AC-\omega$, if for $\varepsilon > 0$ there exists $\delta > 0$ such that for any set of non-overlapping intervals (x_i, x'_i) on $[a, b]$ with $\sum\{\omega(x'_i+) -\omega(x_i-)\} < \delta$ the relation $\sum |F(x'_i+)-F(x_i-)| < \varepsilon$ is satisfied.

We observe that the above condition for a function to be $AC-\omega$ can be broken up into two parts which, when taken together, become equivalent to that of $AC-\omega$.

Let $a \leqq x_1 < x'_1 \leqq x_2 < x'_2 \leqq \cdots \leqq x_n < x'_n \leqq b$ be any subdivision of $[a, b]$. Following Kennedy [3], we say that the intervals (x_1, x'_1), (x_2, x'_2), $\cdots$, (x_n, x'_n) form an elementary system I in $[a, b]$ which we denote by I: (x_i, x'_i), $i = 1, 2, 3, \cdots, n$. Let

$$\sigma I = \sum_{i=1}^{n} \{F(x'_i+)-F(x_i-)\}, \quad I_\omega = \sum_{i=1}^{n} \{\omega(x'_i+)-\omega(x_i-)\}.$$

Definition. A function $F(x)$ defined on $[a, b]$ and belonging to the class $\mathscr{U}$ is said to be absolutely continuous above relative to ω, $AC-\omega$ above, if for $\varepsilon > 0$ there exists $\delta > 0$ such that for any elementary system I in $[a, b]$ with $I_\omega < \delta$ the relation $\sigma I < \varepsilon$ holds. It is said to be absolutely continuous below relative to ω, $AC-\omega$ below, if the relation $\sigma I > -\varepsilon$ holds whenever $I_\omega < \delta$.

This definition is analogous to the definition in [3] for functions absolutely continuous above and below. Assuming that $\omega(x)$ is not constant in $[a, b]$, let

$$\omega(a) = y_0 < y_1 < y_2 < \cdots < y_n = \omega(b)$$

be any subdivision of $[\omega(a), \omega(b)]$ where $y_i \in \omega(E)$, $E = [a, b]$. For any y_i there is an $x_i \in E$ for which $y_i = \omega(x_i)$. If for any y_i there exist more than one x_i such that $\omega(x_i) = y_i$, we shall take any one x_i. We say that the points $x_0, x_1, x_2, \cdots x_n$ form a subdivision of $[a, b]$ relative to ω or are an ω-subdivision of $[a, b]$. We have introduced in [1] the following

Definition. Let $F(x)$ be defined on $[a, b]$ and be in class $\mathscr{U}$. The least upper bound of

$$V = \sum_{i=1}^{n} |F(x_i+) - F(x_{i-1}-)|$$

for all possible ω-subdivisions $x_0, x_1, \cdots, x_n$ of $[a, b]$ is called the total ω-variation of $F(x)$ and is denoted by $V_\omega(F; a, b)$. If $V_\omega(F; a, b) < \infty$ then $F(x)$ is said to be of bounded variation relative to ω on $[a, b]$.

In [1] we have shown that any function $F(x)$ which is $AC-\omega$ on $[a, b]$ must be $BV-\omega$ on $[a, b]$.

Here we observe that the same result can be proved under weaker conditions on $F(x)$. It is possible to show that if $F(x)$ is $AC-\omega$ above (or below) on $[a, b]$ then it is $BV-\omega$ on $[a, b]$. To prove this, we require some preliminary results for which some further definitions are needed.

Definition. Let $F(x)$ be defined in $[a, b]$ and belong to the class $\mathscr{U}$, and let $I : (x_i, x_i')$, $i = 1, 2, \cdots, n$ be any elementary system in $[a, b]$. The l.u.b. and g.l.b. of the aggregate $\{\sigma I\}$ of sums σI for all possible elementary systems I in $[a, b]$ are called respectively the positive and negative variation of $F(x)$ in $[a, b]$, and are denoted by $V^+(F; a, b)$ and $V^-(F; a, b)$. It is clear that

$$V^+(F; a, b) \geqq 0 \quad \text{and} \quad V^-(F; a, b) \leqq 0.$$

Throughout the paper we shall consider only those functions $F(x)$ of the class $\mathscr{U}$ for which $F(x+)$ and $F(x-)$, $x \in E-S$, are finite.

2. Preliminary lemmas

LEMMA 1. *Let $a < c < b$. If $V^+(F; a, c)$ and $V^+(F; c, b)$ are finite, then so is $V^+(F; a, b)$; further if $F(c-) = F(c+)$ then*

$$V^+(F; a, b) = V^+(F; a, c) + V^+(F; c, b).$$

PROOF. Let $I : (x_i, x_i')$, $i = 1, 2, \cdots, n$ be any elementary system in $[a, b]$. We consider the following cases.

(a) If $x_n' \leqq c$, I becomes an elementary system in $[a, c]$ and so

(1) $$\sigma I \leqq V^+(F; a, c).$$

(b) If $x_1 \geqq c$, I is an elementary system in $[c, b]$, so

(2) $$\sigma I \leqq V^+(F; c, b).$$

(c) If $x'_m \leqq c \leqq x_{m+1}$, $m < n$, I can be exhibited as the sum of two elementary systems, I_1 in $[a, c]$ and I_2 in $[c, b]$ and so,

(3) $$\sigma I = \sigma I_1 + \sigma I_2 \leqq V^+(F; a, c) + V^+(F; c, b).$$

(d) If $x_m < c < x'_m$, $m \leqq n$, then the intervals (x_1, x'_1), $(x_2, x'_2), \cdots$, (x_{m-1}, x'_{m-1}), (x_m, c) and (c, x'_m), $(x_{m+1}, x'_{m+1}), \cdots (x_n, x'_n)$ form elementary systems I_1 and I_2 in $[a, c]$ and $[c, b]$ respectively. Since

$$F(x'_m+) - F(x_m-) = \{F(x'_m+) - F(c-)\} + \{F(c-) - F(c+)\} + \{F(c+) - F(x_m-)\}$$

we have

(4) $$\sigma I = \sigma I_1 + \sigma I_2 + \{F(c-) - F(c+)\} \leqq V^+(F; a, c) + V^+(F; c, b) + K$$

where

$$K = |F(c-) - F(c+)|.$$

Hence from (1), (2), (3), (4) it follows that, in any case

(5) $$\sigma I \leqq V^+(F; a, c) + V^+(F; c, b) + K.$$

Since (5) is true for any elementary system in $[a, b]$ we have

(6) $$V^+(F; a, b) \leqq V^+(F; a, c) + V^+(F; c, b) + K.$$

This proves the first part.

Now suppose that $F(c-) = F(c+)$. Then from (6)

(7) $$V^+(F; a, b) \leqq V^+(F; a, c) + V^+(F; c, b).$$

Let I_1 be any elementary system in $[a, c]$ and I_2 be that in $[c, b]$. I_1 and I_2 together form an elementary system I in $[a, b]$. So

$$\sigma I_1 + \sigma I_2 = \sigma I \leqq V^+(F; a, b).$$

This implies that

(8) $$V^+(F; a, c) + V^+(F; c, b) \leqq V^+(F; a, b).$$

Combining (7) and (8) we obtain

$$V^+(F; a, b) = V^+(F; a, c) + V^+(F; c, b).$$

Proceeding in the same manner as in Lemma 1 we may prove

LEMMA 2. *Let $a < c < b$. If $V^-(F; a, c)$ and $V^-(F; c, b)$ are finite, then so is $V^-(F; a, b)$; further if $F(c-) = F(c+)$ then $V^-(F; a, b) = V^-(F; a, c) + V^-(F; c, b)$.*

LEMMA 3. *Let $x_1, x_2, x_3, \cdots$ be the set of those points in $[a, b]$ for which $F(x_i+) \neq F(x_i-)$. If $V^+(F; a, b)$ {or $V^-(F; a, b)$} is finite, then the series $\sum_i |F(x_i+) - F(x_i-)|$ is convergent.*

PROOF. We suppose that $V^+(F; a, b)$ is finite. The proof in the other case is analogous. Let $\xi_1, \xi_2, \cdots$ be the subset of $x_1, x_2, \cdots$ where $F(\xi_i+) - F(\xi_i-) > 0$. Let n be any positive integer. We arrange $\xi_1, \xi_2, \cdots, \xi_n$ in ascending order and rename them, if necessary, by $\xi'_1, \xi'_2, \cdots, \xi'_n$. It is clear that $\xi'_1 > a$ and $\xi'_n < b$. We now choose the points α_i, α'_i, $\alpha_i < \xi'_i < \alpha'_i$, $i = 2, 3, \cdots, n-1$ in $((\xi'_{i-1} + \xi'_i)/2, (\xi'_i + \xi'_{i+1})/2) \cap S$; α_1, α'_1, $\alpha_1 < \xi'_1 < \alpha'_1$ in $((a + \xi'_1)/2, (\xi'_1 + \xi'_2)/2) \cap S$ and α_n, α'_n, $\alpha_n < \xi'_n < \alpha'_n$ in $((\xi'_{n-1} + \xi'_n)/2, (\xi'_n + b)/2) \cap S$ such that for arbitrary $\varepsilon > 0$,

$$F(\xi'_i+) - F(\xi'_i-) < F(\alpha'_i) - F(\alpha_i) + \varepsilon/2^{i+1}, \quad i = 1, 2, \cdots, n.$$

The intervals (α_i, α'_i), $i = 1, 2, \cdots, n$ form an elementary system I_1 in $[a, b]$ and so $\sigma I_1 \leqq V^+(F; a, b)$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} \{F(\xi_i+) - F(\xi_i-)\} &= \sum_{i=1}^{n} \{F(\xi'_i+) - F(\xi'_i-)\} \\ &\leqq \sigma I_1 + \varepsilon \leqq V^+(F; a, b) + \varepsilon.\end{aligned}$$

Since n may be any positive integer, it follows that the series $\sum_i \{F(\xi_i+) - F(\xi_i-)\}$ is convergent.

Next, let $\eta_1, \eta_2, \cdots$ be the subset of $x_1, x_2, \cdots$ where $F(\eta_i+) - F(\eta_i-) < 0$. For an arbitrary positive integer n, we can choose, as above, an elementary system $I_2 : (\beta_i, \beta'_i)$, $i = 1, 2, \cdots, n$ with $\beta_i, \beta'_i \in S$ and $\beta_1 > a$, $\beta'_n < b$ such that

$$\sum_{i=1}^{n} \{F(\eta_i+) - F(\eta_i-)\} > \sigma I_2 - \varepsilon.$$

Let J denote the elementary system complementary to I_2. Then $\sigma I_2 + \sigma J = F(b-) - F(a+)$. So,

$$\sigma I_2 = F(b-) - F(a+) - \sigma J \geqq F(b-) - F(a+) - V^+(F; a, b).$$

Hence

$$\sum_{i=1}^{n} \{F(\eta_i+) - F(\eta_i-)\} \geqq F(b-) - F(a+) - V^+(F; a, b) - \varepsilon.$$

Since n is any positive integer and since $\sum \{F(\eta_i+) - F(\eta_i-)\} \leqq 0$, the series $\sum_i\{F(\eta_i+)-F(\eta_i-)\}$ therefore converges. The lemma now follows from the fact that

$$\sum_i |F(x_i+)-F(x_i-)| = \sum_i \{F(\xi_i+)-F(\xi_i-)\} - \sum_i \{F(\eta_i+)-F(\eta_i-)\}.$$

LEMMA 4. *If $V^+(F; a, b)$ is finite then so is $V^-(F; a, b)$ and vice versa.*

PROOF. Suppose that $V^+(F; a, b)$ is finite. Let $I : (x_i, x_i')$, $i = 1, 2, \cdots, n$ be any elementary system in $[a, b]$. Then we have

$$\sigma I = \{F(x_n'+)-F(x_1-)\} - \sum_{i=1}^{n-1} \{F(x_{i+1}-)-F(x_i'+)\}.$$

Let $x_1 > a$ and $x_n' < b$. Writing $a = x_0$, $b = x_{n+1}$ we have

$$\sigma I = F(b-)-F(a+) - \sum_{i=0}^{n} \{F(x_{i+1}-)-F(x_i'+)\}.$$

We divide the set of integers $i = 0, 1, 2, \cdots, n$ into two parts A and B such that $i \in A$ if $x_{i+1} = x_i'$ and $i \in B$ if $x_{i+1} > x_i'$. Then

$$\begin{aligned}\sigma I &= F(b-)-F(a+) + \sum_{i\in A} \{F(x_i'+)-F(x_i'-)\} - \sum_{i\in B} \{F(x_{i+1}-)-F(x_i'+)\}\\ &= F(b-)-F(a+)+\textstyle\sum_1-\sum_2.\end{aligned}$$

Let $\xi_1, \xi_2, \cdots$ be the set of points in $[a, b]$ where $F(\xi_i+) \neq F(\xi_i-)$. Then by lemma 3,

$$\sum_i |F(\xi_i+)-F(\xi_i-)| = K \tag{9}$$

is finite. For $i \in B$ and arbitrary $\varepsilon > 0$, we choose the points α_i, $\alpha_i'(> \alpha_i)$ in $(x_i', x_{i+1}) \cap S$ such that

$$F(x_{i+1}-)-F(x_i'+) < F(\alpha_i')-F(\alpha_i)+\varepsilon/2^{i+1}.$$

The intervals (α_i, α_i'), $i \in B$ form an elementary system I_1 in $[a, b]$. So we have

$$\textstyle\sum_2 < \sigma I_1+\varepsilon \leqq V^+(F; a, b)+\varepsilon.$$

Also utilising (9)

$$\textstyle\sum_1 \geqq -\displaystyle\sum_{i\in A} |F(x_i'+)-F(x_i'-)| \geqq -K.$$

Hence

$$\sigma I \geqq F(b-)-F(a+)-V^+(F; a, b)-\varepsilon-K.$$

If $a = x_1, x_n' = b$ or $a = x_1, x_n' < b$ or $a < x_1, x_n' = b$ then it can be similarly shown that $\sigma I \geqq G$, a fixed constant independent of I. Since $V^-(F; a, b) \leqq 0$, it follows that $V^-(F; a, b)$ is finite.

In a similar way it may be shown that if $V^-(F; a, b)$ is finite then $V^+(F; a, b)$ is also finite. This proves the lemma.

3. Theorems and Corollaries

THEOREM 1. *If $F(x)$ is defined in $[a, b]$ and belongs to the class $\mathscr{U}$, then $V_\omega(F; a, b) \leqq V^+(F; a, b) - V^-(F; a, b)$.*

PROOF. If $V^+(F; a, b)$ is infinite, then clearly the theorem holds. Suppose, therefore, that $V^+(F; a, b)$ is finite. By lemma 4, $V^-(F; a, b)$ is then finite.

Let $a \leqq x_0 < x_1 < x_2 < \cdots < x_n \leqq b$ be any ω-subdivision of $[a, b]$. We divide the set of integers $1, 2, 3, \cdots, n$ into two parts P and N such that $F(x_i+) - F(x_{i-1}-) \geqq 0$ for $i \in P$ and $F(x_i+) - F(x_{i-1}-) < 0$ for $i \in N$. The intervals (x_{i-1}, x_i), $i \in P$ and (x_{i-1}, x_i), $i \in N$ form two elementary systems I_1 and I_2 in $[a, b]$. So

$$V = \sum_{i=1}^{n} |F(x_i+) - F(x_{i-1}-)| = \sigma I_1 - \sigma I_2.$$
$$\leqq V^+(F; a, b) - V^-(F; a, b).$$

Since the above inequality is true for any ω-subdivision of $[a, b]$, the theorem follows.

The following example shows that the equality sign need not hold in the relation

$$V_\omega(F; a, b) \leqq V^+(F; a, b) - V^-(F; a, b).$$

Example. Let

$$\omega(x) = \begin{matrix} 0 & \text{for} & 0 \leqq x \leqq \frac{1}{2}, \\ 1 & \text{for} & \frac{1}{2} < x \leqq 1 \end{matrix}$$

and

$$F(x) = \begin{matrix} 4x & \text{for} & 0 \leqq x \leqq \frac{1}{2}, \\ 3-2x & \text{for} & \frac{1}{2} < x \leqq 1. \end{matrix}$$

Then clearly $F(x)$ belongs to the class $\mathscr{U}$, and

$$V^+(F; 0, \tfrac{1}{2}) = 2,\ V^+(F; \tfrac{1}{2}, 1) = 0,\ V^-(F; 0, \tfrac{1}{2}) = 0,\ V^-(F; \tfrac{1}{2}, 1) = -1.$$

Using lemma 1 and lemma 2, we obtain

$$V^+(F; 0, 1) = 2,\ V^-(F; 0, 1) = -1.$$

Any ω-subdivision of $[0, 1]$ consists of only two points x_0, x_1, where $0 \leqq x_0 \leqq \frac{1}{2}$, $\frac{1}{2} < x_1 \leqq 1$. Hence $V = |F(x_1+) - F(x_0-)| = |F(x_1) - F(x_0)|$. Since $0 \leqq F(x_0) \leqq 2$ and $1 \leqq F(x_1) < 2$ we deduce that

$$V_\omega(F; 0, 1) = 2 < V^+(F; 0, 1) - V^-(F; 0, 1).$$

THEOREM 2. *If $F(x)$ is $AC-\omega$ above on $[a, b]$ and $\omega(x)$ is constant in $(\alpha, \beta) \subset [a, b]$, then $F(x)$ is non-increasing in (α, β).*

PROOF. From the definition of $F(x)$, it follows that $F(x)$ is continuous in (α, β). Let $\varepsilon > 0$ be arbitrary. Since $F(x)$ is $AC-\omega$ above on $[a, b]$, there exists a positive number δ such that for every elementary system $I : (x_i, x'_i)$ in $[a, b]$ we have $\sum_i\{F(x'_i+)-F(x_i-)\} < \varepsilon$ whenever $\sum_i\{\omega(x'_i+)-\omega(x_i-)\} < \delta$. Let x_1 and $x_2(> x_1)$ be any two points in (α, β). Then $\{\omega(x_2+)-\omega(x_1-)\} < \delta$, and it follows that $F(x_2)-F(x_1) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $F(x_2) \leqq F(x_1)$ which proves the theorem.

COROLLARY. *If $F(x)$ is $AC-\omega$ on $[a, b]$ and $\omega(x)$ is constant in $(\alpha, \beta) \subset [a, b]$, then $F(x)$ is constant in (α, β).*

THEOREM 3. *If $F(x)$ is $AC-\omega$ above on $[a, b]$, then $F(x)$ is $BV-\omega$ on $[a, b]$.*

PROOF. Since $F(x)$ is $AC-\omega$ above on $[a, b]$ there exists a number $\delta > 0$ such that for every elementary system I in $[a, b]$ we have

(10) $$\sigma I < 1 \quad \text{whenever} \quad I_\omega < \delta.$$

We consider the following cases.

(I). *The saltus of $\omega(x)$ at every point of $[a, b]$ is less than $\frac{1}{2}\delta$.*

In this case $[a, b]$ can be divided into a finite number of subintervals

$$[c_0, c_1], [c_1, c_2], \cdots [c_{N-1}, c_N] \; (a = c_0 < c_1 < \cdots < c_N = b)$$

such that

(11) $$\{\omega(c_r+)-\omega(c_{r-1}-)\} < \tfrac{1}{2}\delta, \qquad r = 1, 2, \cdots, N.$$

Let $I : (x_i, x'_i)$, $i = 1, 2, \cdots, n$ be any elementary system in $[c_{r-1}, c_r]$. Then by (11), $I_\omega < \delta$ and so by (10), $\sigma I < 1$. This implies that

$$V^+(F; c_{r-1}, c_r) \leqq 1, \qquad r = 1, 2, \cdots, N.$$

By lemma 1, it follows, therefore, that $V^+(F; a, b)$ is finite.

(II). *There exist points in $[a, b]$ at which the saltus of $\omega(x)$ is $\geqq \frac{1}{2}\delta$.*

It is known [4] that these points are finite in number. Let them be $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. In $[\alpha_{r-1}, \alpha_r]$ we choose points $\alpha, \beta(> \alpha)$ of S such that

(12) $$\omega(\alpha)-\omega(\alpha_{r-1}+) < \tfrac{1}{2}\delta \quad \text{and} \quad \omega(\alpha_r-)-\omega(\beta) < \tfrac{1}{2}\delta.$$

At each point in $[\alpha, \beta]$ the saltus of $\omega(x)$ is less than $\frac{1}{2}\delta$. So, by Case (I), $V^+(F; \alpha, \beta)$ is finite.

Let $I' : (x_i, x'_i)$, $i = 1, 2, \cdots, n$ be any elementary system in $[\alpha_{r-1}, \alpha]$. If $\alpha_{r-1} < x_1$ then by (12), $I'_\omega < \delta$ and so by (10), $\sigma I' < 1$.

If $\alpha_{r-1} = x_1$ we choose a point ξ in $(\alpha_{r-1}, x_1') \cap S$ such that

$$|F(\xi)-F(\alpha_{r-1}+)| < 1.$$

The intervals $(\xi, x_1'), (x_2, x_2'), \cdots, (x_n, x_n')$ form an elementary system I'' in $[\alpha_{r-1}, \alpha]$. By (12), $I''_\omega < \delta$ and so $\sigma I'' < 1$. Now

$$\begin{aligned}\sigma I' &= \{F(x_1'+)-F(x_1-)\}+\sum_{i=2}^{n}\{F(x_i'+)-F(x_i-)\}\\ &= \{F(\alpha_{r-1}+)-F(\alpha_{r-1}-)\}+\{F(\xi)-F(\alpha_{r-1}+)\}+\sigma I''\\ &< 2+K, \quad \text{where} \quad K = |F(\alpha_{r-1}+)-F(\alpha_{r-1}-)|.\end{aligned}$$

So, in any case $\sigma I' < 2+K$. Since this is true for every elementary system I' in $[\alpha_{r-1}, \alpha]$, it follows that $V^+(F; \alpha_{r-1}, \alpha)$ is finite. Similarly it can be shown that $V^+(F; \beta, \alpha_r)$ is finite and consequently by lemma 1, it follows that $V^+(F; a, b)$ is finite. The proof of the theorem is, therefore, complete because by lemma 4, $V^-(F; a, b)$ is finite and so by theorem 1, $F(x)$ is $BV-\omega$ on $[a, b]$.

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