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Trees and amenable equivalence relations

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Abstract. Let R be a Borel equivalence relation with countable equivalence classes on a measure space M. Intuitively, a 'treeing' of R is a measurably-varying way of makin each equivalence class into the vertices of a tree. We make this definition rigorous. We prove that if each equivalence class becomes a tree with polynomial growth, then the equivalence relation is amenable. We prove that if the equivalence relation is finite measure-preserving and amenable, then almost every tree (i.e., equivalence class) must have one or two ends.

0. *Introduction*

Classically, ergodic theory has studied actions of the reals or of the integers on measure spaces. More recently, it has broadened to include actions of Lie groups, algebraic groups and discrete groups. The most recent trend [**6**, **7**, **9**, **11**] has been to consider any kind of groupoid structure, where the unit space is a measure space. This includes equivalence relations on measure spaces (the case where the groupoid is principal), for which many of the fundamental techniques were developed in [**3**-**5**].

At a conference in Santa Barbara in the late 1970s, Alain Connes suggested the study of equivalence relations with an additional piece of data: a measurably-varying simplicial complex structure on each equivalence class (see Definition 1.7). In this paper, we take up a special case of this, where the simplicial complex is a tree (see Definition 1.5).

The simplest example of such a 'treed' equivalence relation comes from considering the orbits of the action of a finitely-generated free group F acting freely on a measure space. A tree structure is then obtained on each orbit by saying that two points x and y are 'adjacent' if there is some generator $g \in F$ such that $gx = y$ or $gy = x$. The resulting tree is homogeneous, i.e., has a vertex transitive automorphism group. Thus, in a sense, treed equivalence relations are a groupoid analogue of free groups. In another sense (see Example 1.6.3), treed equivalence relations are analogous to foliations by manifolds of negative curvature, in the same way as a homogeneous tree is analogous to hyperbolic space (see [**8**, Chapter II]).

Amenability is another property of equivalence relations; it is the groupoid analogue of the property of amenability for groups. It has been studied in detail by R. Zimmer [**10**], by Connes et al. [**2**] and by others. The purpose of this paper is to study how amenability and treeings interact. One immediate consequence of [**2**] is that any amenable equivalence relation is treeable, where each equivalence class becomes tree isomorphic to the Cayley graph of $\mathbf{Z}$ (with generating set $\{\pm 1\}$), i.e., to the tree with two edges belonging to every vertex.

In this paper we prove two less immediate results. *First*, we show, in the presence of a finite invariant measure, that if the equivalence relation is amenable, then a.e. equivalence class has a very simple tree-structure:

THEOREM 5.1. *Let (M, R) be an amenable equivalence space with finite R-invariant measure. Let S be a treeing of (M, R). Then, for* a.e. $x \in M$, *the tree $R(x)$ has one or two ends.*

We preface with a few remarks about an arbitrary tree with one end. So let T be any (locally finite) tree with exactly one end. Note that any directed edge is either directed toward the end or away from it. If $e_1, \ldots, e_{2n}$ are the directed edges of some even length geodesic between two vertices, then we will say that the geodesic is balanced if $e_1, \ldots, e_n$ are directed toward the end, while $e_{n+1}, \ldots, e_{2n}$ are directed away. Two vertices will be called equivalent if the geodesic connecting one to the other is balanced. (Thus equivalent implies separated by an even distance.) The equivalence classes of this equivalence relation are called *horocycles.* There is a linear ordering on horocycles: if H_1 and H_2 are horocyles, then we will write $H_1 < H_2$ if there exists a geodesic connecting a vertex of H_1 to a vertex of H_2 which is always directed away from the end of T. It is easy to show from the local finiteness and one-endedness of T that there exists a unique maximal horocycle $H(T)$ under this ordering. Note that since $H(T)$ is a horocycle, then any two vertices in $H(T)$ are separated by an even distance. Note also that half that distance is exactly the number of horocycles (other than $H(T)$ itself) met by the geodesic between them.

Returning to the theorem at hand, let P denote the union of the sets $H(R(x))$ as x ranges over M'. Because $P \cap R(x) = \varnothing$, for a.e. $x \in M$, quasi-invariance implies that P is of positive measure. As remarked above, we need only verify that $(P, R \mid P)$ is amenable. Further, by [**10**, Theorem 3.6, p. 29], we need only show that $R \mid P$ is hyperfinite, i.e., is a countable union of equivalence relations each of which has finite equivalence classes. We define the equivalence relation R_i on P by: $(x, y) \in R_i$ iff the distance from x to y in the tree $R(x)$ is $\leq 2i$. (Transitivity follows from the last sentence of the preceding paragraph.) Then each R_i has finite equivalence classes and $R = \bigcup_{i=1}^{\infty} R_i$, as needed.

Thus we may assume that M' is null, i.e., that a.e. $R(x)$ has more than one end.

This result and its proof are similar to the result on foliations by manifolds of negative curvature in Zimmer [**12**].

A pendant vertex is a vertex that belongs to only one edge. A tree with only one end must have pendant vertices. Further, there is only one tree with two ends which has no pendant vertices: the tree with exactly two edges belonging to each vertex. We call this tree the integer tree and obtain an analogue of [**2**]:

COROLLARY. *Let (M, R) be an amenable equivalence space with R-invariant measure. Let S be a treeing of (M, R). Assume, for* a.e. $x \in M$, *that the tree $R(x)$ has no pendant vertices. Then, for* a.e. $x \in M$, *$R(x)$ is the integer tree.*

Second, we show that if the trees are sufficiently uncomplicated, then the equivalence relation is amenable.

THEOREM 5.2. *Suppose S is a treeing of the equivalence space (M, R). For* a.e. $x \in M$, *assume either that $R(x)$ has finitely many ends or that $R(x)$ has polynomial growth. Then (M, R) is amenable.*

A. Kechris has pointed out to me that 'polynomial growth' may be replaced by 'subexponential growth' with no significant change in the proof.

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1. *Basic definitions*

All measures are assumed σ-additive and σ-finite. All Borel spaces are assumed to be standard. We define $\mathbf{Z}_+$ to be the positive integers, $\mathbf{Z}_+^0$ to be the nonnegative integers and $\mathbf{R}$ to be the real numbers. If S is any set, then we denote the number of elements in S by $|S|$.

If R is an equivalence relation on a set X and $x \in X$, then we denote the R-equivalence class of x by $R(x)$. The *R-saturation of* a subset $W \subseteq X$ is the union of all R-equivalence classes that intersect W. If $K \subseteq X$ and R is an equivalence relation on X then we denote $R \cap (K \times K)$ by $R|K$.

Let B be a Borel space and let R be an equivalence relation on B whose graph is a Borel subset of $B \times B$ and whose equivalence classes are countable. An *automorphism of* (B, R) is an automorphism $f: B \to B$ of the Borel space B such that $(x, y) \in R \Rightarrow (f(x), f(y)) \in R$. We denote the group of all automorphisms of (B, R) by $\mathrm{Aut}\,(B, R)$. A measure μ on B is said to be *R-quasi-invariant* if the R-saturation of any μ-null set is again μ-null. A measure μ on B is said to be *R-invariant* if every automorphism of (B, R) leaves μ invariant.

Let M denote the measure space (B, μ). Assuming that R has countable equivalence classes and that μ is quasi-invariant, then we say that (M, R) is an *equivalence space.* If (M, R) is an equivalence space, and G is a group, then a measurable map $\alpha: R \to G$ is called a *cocycle* if, for a.e. $x \in M$, for all $y, z \in R(x)$, we have $\alpha(x, y)\alpha(y, z) = \alpha(x, z)$.

If X is a locally compact, countably based topological space, then we denote the sup-norm Banach space of continuous real-valued vanishing-at-∞ functions on X by $C_0(X)$. If $K \subseteq X$ is compact, then we let $M(K)$ be the space all Borel probability measures on K, viewed as a weak-$*$ compact, convex subset of the unit ball in $C_0(X)^*$.

We now define amenability of an equivalence space as in [**10**, Definition 3.1, p. 27]: Let E be a separable Banach space and let $\alpha: R \to \mathrm{Iso}\,(E)$ be a measurable cocycle, where $\mathrm{Iso}\,(E)$ denotes the topological space of all isometric automorphisms of E with the strong operator topology. (We will denote the actions of $\mathrm{Iso}\,(E)$ on E and on E^* on the right.) Let $\alpha^*: R \to \mathrm{Iso}\,(E^*)$ denote the adjoint cocycle, $\alpha^*(x, y) := (\alpha(x, y)^*)^{-1}$. A mapping $x \mapsto A_x$ which associates to each $x \in M$ a weak-$*$ compact, convex subset A_x of the unit ball of E^* is said to be *Borel* if $\{(x, a) \in M \times E^* \mid a \in A_x\}$ is a Borel subset of $M \times E^*$. It is said to be *α^*-invariant* if, for a.e.

$x \in M$, for all $y \in R(x)$, $A_x\alpha^*(x, y) = A_y$. We will commonly denote $x \mapsto A_x$ by $(A_x)_{x\in M}$.

Definition 1.1. Let (M, R) be an equivalence space. Let $\alpha: R \to \mathrm{Iso}(E)$ be a measurable cocycle, and let $(A_x)_{x\in M}$ be an α^*-invariant Borel field of weak-* compact, convex subsets of the unit ball of E^*. Then the ordered pair $(\alpha, (A_x)_{x\in M})$ is called an *affine space over* (M, R).

A measurable mapping $f: M \to E^*$ is said to be a *section of* $(A_x)_{x\in M}$ if $f(x) \in A_x$, for a.e. $x \in M$. A section f is said to be *α^*-invariant* (or, simply *invariant*) if, for a.e. $x \in M$, for all $y \in R(x)$, we have $f(x)\alpha^*(x, y) = f(y)$.

Definition 1.2. We say that an equivalence space is *amenable* if any affine space over it has an invariant section.

For more information on amenability and its properties, see [**13**, § 4.3, especially pp. 82–84].

All trees are assumed to have countable vertex sets and edge sets. (For the basic terminology concerning trees, we cite [**8**].) Let $\mathcal{U}$ denote the tree with infinitely many vertices, and infinitely many edges belonging to each vertex. We call this the *universal tree.* A tree is said to be *locally finite* if every vertex belongs to only finitely many edges. If T is a tree, then we denote the vertex set and (undirected) edge set of T by $V(T)$ and $E(T)$, respectively. If, in addition $v \in V(T)$, then we denote the set of edges in T belonging to v by $E_T(v)$. If $v \in V(T)$, then we call $|E_T(v)|$ the *valence* of v.

A *directed edge of* T is an ordered pair consisting of two adjacent vertices. The set of directed edges of T is denoted $DE(T)$. If $(v, w) \in DE(T)$, then we define $s(v, w) := v$, $r(v, w) := w$ and $(v, w)' := (w, v)$. If $v \in V(T)$, then we define

$$DE_T(v) := \{e \in DE(T) \mid s(e) = v\}.$$

Let T be a tree with distance function $d: V(T) \times V(T) \to Z^0_+$. If $v \in V(T)$, $n \in Z^0_+$, then we let $B_T(v, n) = \{w \in V(T) \mid d(v, w) \leq n\}$. We define the *ends of* T as in [**8**, Exercise 1), pp. 20–21]. We denote the topological space of ends of T by ∂T. We define (non-standard) the *trunk vertices of* T to be the collection of vertices of T which lie on a geodesic connecting two ends. The *trunk of* T is the subtree of T whose vertex set consists of the trunk vertices of T.

We will also need

Definition 1.3. Let T be a tree. We say that T *has polynomial growth* if for some (equivalently, any) $v \in V(T)$, the function $n \mapsto |B_T(v, n)|: \mathbf{Z}_+ \to \mathbf{Z}_+$ is bounded above by some polynomial in n.

Next, we head toward the definition of a 'treed equivalence space', which is the basic object of study in this work. As a first approximation, we define a 'graphed equivalence space':

Definition 1.4. Let (M, R) be an equivalence space. A *graphing of* (M, R) is a symmetric relation $S \subset R$ on B such that S is a Borel subset of $M \times M$. We then say that (M, R, S) is a *graphed equivalence space.*

(Generally, S is neither reflexive nor transitive; it is not an equivalence relation.)

We will abuse notation and use $R(x)$ to refer also to the graph with vertex set $R(x)$ and directed edge set $S\cap[R(x)\times R(x)]$.

The following definition makes rigorous what is meant by 'measurably putting a tree structure on each equivalence class' of an equivalence relation.

Definition 1.5. Let (M, R, S) be a graphed equivalence space. If for a.e. $x\in M$, the graph $R(x)$ is a locally finite tree (i.e., it is connected with no circuits, cf [**8**, Definition 6, p. 17]), then we say that S is a *treeing of* (M, R) or that (M, R, S) is a *treed equivalence space.*

If $x\in M$, then we define the *S-valence of* x to be the valence of x as a vertex in the tree $R(x)$.

We now give some examples of treed equivalence spaces.

Example 1.6.1 (*Free group action*). Let F be a free group with finite free generating set K. Let M be an essentially free right F-space. Then the orbits of F on M give an equivalence relation which is treeable: Define $S:=\{(x,y)\in M\times M\mid y\in x\cdot K$ or $x\in y\cdot K\}$. Each equivalence class is then tree-isomorphic to the homogeneous tree with $2|K|$ edges belonging to each vertex.

Example 1.6.2 (*Treed bundle construction*). Let T be a tree, G a group acting by automorphisms on T. (Let this action be on the left.) Let M be an essentially free G-space (with G acting on the right). Define a G-action on $M\times V(T)$ by $g\cdot(x,t)=(x\cdot g^{-1}, g\cdot t)$. And let $M':=G\backslash(M\times V(T))$. Then, for a.e. $x\in M$, the map $\{x\}\times V(T)\to M'$ is injective and thus we can transfer the tree-structure from T to the image of this map. Define an equivalence relation R whose equivalence classes are the images of the various maps $\{x\}\times V(T)\to M'$, with x ranging through M. We then have a treed equivalence space.

Example 1.6.3 ($SL(2,\mathbf{Q}_p)$*-action*). Let $\mathbf{Q}_p$ denote the p-adic rational numbers, $\mathbf{Z}_p$ the p-adic integers, $G:=SL(2,\mathbf{Q}_p)$, $\Gamma:=SL(2,\mathbf{Z}_p)$. Then G acts naturally on a homogeneous tree [**8**, § II.1.1, pp. 69–74], call it T, with Γ equal to the stabilizer of some vertex. Let M be an essentially free left G-space. Then $M':=\Gamma\backslash M$ is an equivalence space, where the equivalence classes are a.e. the images of G-orbits under the natural map $M\to M'$. So a.e. equivalence class can be identified with $\Gamma\backslash G$, which we can identify with T. This gives a treed equivalence space. (Note: This construction can be seen as a special case of the treed bundle construction above.)

Given a tree T, we say that two vertices of T *have the same vertex type* if they are conjugate under Aut (T). We now wish to construct a treed equivalence space with a tree that has infinitely many vertex types. This example indicates that the trees which can occur in treed equivalence relations can be more complicated than the ones indicated in the previous examples. It also shows that a tree with one end can occur in every equivalence class of a treed equivalence space, cf. Theorems 5.1 and 5.2.

Example 1.6.4. Let $\mathbf{Z}_+$ denote the measure space consisting of the positive integers with counting measure. Let A be a two point measure space containing the two points 0 and 1, both the measure $\frac{1}{2}$. Let $B:=\mathbf{Z}_+\times A\times A\times\cdots$, where we are taking a countable product with the product measure. Given a point $(n, a_1, a_2, \ldots) \in M$, we will call n the 0th coordinate, a_1 the first coordinate, a_2 the second, etc.

For $k = 1, 2, \ldots$, let B_k denote the set of all points in B whose 0th coordinate is k and whose first through kth coordinates are all 0. We now define a relation S on the measure space $M := B_1 \cup B_{1+2} \cup B_{1+2+3} \cup \cdots$. If $x \in B_{1+\cdots+i}$ and $y \in B_{1+\cdots+(i+1)}$, then we say that x is i-adjacent to y if all coordinates of x and y agree except perhaps coordinates 0 through $0+\cdots+(i+1)$. We let S be the union of all sets $\{(x, y), (y, x)\}$, where x is i-adjacent to y, for some i. Define $R \subseteq M \times M$ to be the totality of all (x, y) such that, for some $z_1, \ldots, z_n \in M$, we have (x, z_1), $(z_1, z_2), \ldots, (z_{n-1}, z_n)$, $(z_n, y) \in S$. Let $x_0 := (1, 0, 0, \ldots)$. Let T be the tree whose vertex set is $R(x_0)$ and whose directed edge set is $S \cap [R(x_0) \times R(x_0)]$. Say that a vertex in T is a k-vertex if the value of its 0th coordinate is $1+\cdots+k$. Then T has the property that the valence a k-vertex is 1 if $k = 1$ and $2^k + 1$ if $k > 1$. Since vertices of different valence are not conjugate under Aut (T), it follows that T has infinitely many vertex types. The tree T is isomorphic to any $R(x)$, $x \in M$.

Now let (M, R, S) be a treed equivalence space and let $M_1 \subseteq B$ be a Borel subset. Let $R_1 := R \mid M_1$. We now define a subset $S_1 \subseteq R_1$ on (M_1, R_1) by letting S_1 be the totality of all $(x, y) \in R_1$ such that the geodesic (in the tree $R(x)$) from x to y does not pass through any vertices inside M_1, except for x and y. We call S_1 the *restriction of S to M_1* and denote S_1 by $S \mid M_1$.

Let (M, R, S) be a treed equivalence space. Let M_0 be the set consisting of those $x \in M$ such that x is a trunk vertex of the tree $R(x)$. Then M_0 is a measurable subset of M and we call $(M_0, R \mid M_0, S \mid M_0)$ the *trunk of* (M, R, S).

We conclude this section by remarking that the definition of a graphed equivalence space has a natural generalization:

Definition 1.7. Let (M, R) be an equivalence space. For each $n = 1, 2, \ldots$, let

$$R^n := \{(x_0, \ldots, x_n) \mid (x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n) \in R\}.$$

Let $S^1 \subseteq R^1, \ldots, S^n \subseteq R^n$ be Borel subsets. Assume that each S_i is invariant under permutation of coordinates. We say that $(M, R, S^1, \ldots, S^n)$ is a *simplical complex space* if, for every $i = 2, \ldots, n$, for a.e. $x_0 \in M$, for all $x_1, \ldots, x_i \in R(x_0)$, we have

$$(x_0, \ldots, x_i) \in S^i \;\Rightarrow\; (x_1, \ldots, x_i) \in S^{i-1}.$$

2. *Existence of simplifications*

Recall that $\mathcal{U}$ denotes the universal tree (see § 1). By abuse of notation, we will also use $V(\mathcal{U})$ to denote the measure space of counting measure on the set $V(\mathcal{U})$. Recall from § 1 that if T is a locally finite tree and $v \in V(T)$, then $E_T(v)$ denotes the set of edges belonging to v and $DE_T(v)$ denotes the set of directed edges which start at v.

Definition 2.1. Let (M, R, S) be a treed equivalence relation and let $D \subseteq M \times V(\mathcal{U})$ be measurable. Let $\Phi: D \to M$ be a measurable map. Assume, for every $x \in M$, that $D_x := \{\mathcal{V} \in V(\mathcal{U}) \mid (x, \mathcal{V}) \in D\}$ is the set of vertices of some locally finite subtree of $\mathcal{U}$. By abuse of notation, we will denote this subtree by D_x as well. Assume further, for a.e. $x \in M$, that the map $\Phi_x := \mathcal{V} \mapsto \Phi(x, \mathcal{V})$ is a tree isomorphism from D_x to

$R(x)$. Then we say that (D, Φ) is a *simplification of* (M, R, S). If there exists $\mathcal{V}_0 \in V(\mathcal{U})$ such that $\Phi(x, \mathcal{V}_0) = x$, for a.e. $x \in M$, then we say that (D, Φ) is *level at* $\mathcal{V}_0$.

Note that in the following definition the action of $\mathrm{Aut}(M, R)$ on M is denoted as a *right* action.

Definition 2.2. Let (D, Φ) be a simplification of a treed equivalence space (M, R, S). Let $\mathcal{V}_0 \in V(\mathcal{U})$. Let $\alpha : R \to \mathrm{Aut}(\mathcal{U})$ be a cocycle. Then we say that α is *compatible with* (D, Φ) if, for a.e. $x \in M$, for all $y \in R(x)$, for all $\mathcal{V} \in D_x$, we have $\mathcal{V}\alpha(x, y) \in D_y$ and $\Phi_y(\mathcal{V}\alpha(x, y)) = \Phi_x(\mathcal{V})$.

THEOREM 2.3. *Let* (M, R, S) *be a treed equivalence space and let* $\mathcal{V}_0 \in V(\mathcal{U})$. *Then there exists*

(1) *a simplification* (D, Φ) *of* (M, R, S) *which is level at* $\mathcal{V}_0$ *and*

(2) *a cocycle* $\alpha : R \to \mathrm{Aut}(\mathcal{U})$ *which is compatible with* (D, Φ).

We preface the proof with a few definitions and remarks.

Definition 2.4. Let T be a tree. An *edge-labeling of* T is defined to be a map $c : DE(T) \to \mathbf{Z}_+$ such that for every $v \in V(T)$, the restriction $c \mid DE_T(v)$ is one-to-one. If $\mathcal{C}$ *is an edge-labeling of* $\mathcal{U}$ *and, for every* $\mathcal{V} \in V(\mathcal{U})$, we have that $\mathcal{C} \mid DE_{\mathcal{U}}(\mathcal{V}) : DE_{\mathcal{U}}(\mathcal{V}) \to \mathbf{Z}_+$ is bijective, then we say that $\mathcal{C}$ is *perfect.*

It is not hard to show that any tree admits an edge-labeling and that $\mathcal{U}$ admits a perfect edge-labeling.

Now let D, E be locally finite subtrees of $\mathcal{U}$. Assume that D and E are isomorphic and that $\alpha : D \to E$ is an isomorphism between them. We claim that α can be extended to an automorphism of $\mathcal{U}$.

To prove this, choose a perfect edge-labeling $\mathcal{C}$ of $\mathcal{U}$, and choose a vertex $v \in V(D)$. Let $X := DE_{\mathcal{U}}(v)$, $Y := DE_D(v)$. Now α is defined on Y and we extend it to X as follows. The edge-labeling $\mathcal{C}$ defines a well-ordering on X and on $\alpha(X)$. (More generally, any embedding of a set X into a well-ordered set like $\mathbf{Z}_+$ gives a well-ordering on X.) Map the first edge in $X \setminus Y$ to the first edge in $\alpha(X) \setminus \alpha(Y)$, the second to the second, the third to the third, etc. Now extend α to the vertices adjacent to v by setting $\alpha(r(x)) := r(\alpha(x))$, for all $x \in X$. Then extend α to the directed edges ending at v by setting $\alpha(x') := (\alpha(x))'$, for all $x \in X$. (Recall from § 1 that $r(v, w) := w$ and $(v, w)' := (w, v)$, for any directed edge (v, w).) We now repeat this process for each of the vertices adjacent to v and continue inductively.

Definition 2.5. Let D, E be locally finite subtrees of $\mathcal{U}$ and let α denote an isomorphism between them. Let $\mathcal{C}$ be a perfect edge-labeling of $\mathcal{U}$ and let $v \in V(D)$. The process described above defines an extension of α to an automorphism of $\mathcal{U}$. We call this the *standard extension of* α *with respect to* $\mathcal{C}$ *and* v.

Next we claim that any locally finite tree can be embedded as a subtree of $\mathcal{U}$. Let T denote a locally finite tree, let c be an edge-labeling of T and let $\mathcal{C}$ be a perfect edge-labeling of $\mathcal{U}$. Let $v \in V(T)$, $\mathcal{V} \in V(\mathcal{U})$. We define an embedding ϕ as follows. Define $\phi(v) = \mathcal{V}$. As before, the edge-labelings define a well-ordering on $DE_T(v)$ and $DE_{\mathcal{U}}(\mathcal{V})$. Let ϕ map the first edge to the first edge, the second to the

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second, etc. As above, define $\phi(r(e)) := r(\phi(e))$ and $\phi(e') := (\phi(e))'$, for all $e \in DE_T(v)$. For each vertex w adjacent to v, we repeat the process on the edges starting at w except for the edge (w, v). Continuing inductively, we obtain an embedding.

Definition 2.6. Let T be a locally finite tree, c an edge-labeling of T, $\mathscr{C}$ a perfect edge-labeling of $\mathscr{U}$. Let $v \in V(T)$, $\mathscr{V} \in V(\mathscr{U})$. Then the process described above gives an embedding of T in $\mathscr{U}$, which we call the *standard embedding of T with respect to v, $\mathscr{V}$, c and $\mathscr{C}$.*

In the following proof, we let $\mathrm{Aut}(M, R)$ act on M on the right.

Proof of Theorem 2.3. Let $\mathscr{C}: DE(\mathscr{U}) \to \mathbf{Z}_+$ be a perfect edge-labeling of $\mathscr{U}$. Choose a vertex $\mathscr{V}_0 \in V(\mathscr{U})$. Let $G \subseteq \mathrm{Aut}(M, R)$ be a countable subgroup such that the equivalence classes of R are exactly the orbits of G. (See [**4**, Theorem 1, p. 291].) Let $g_1, g_2, \ldots$ be a listing of the elements of G. Define $N: S \to \mathbf{Z}_+$ by $N(x, y) := \min\{i \mid xg_i = y\}$. Then, for a.e. $x \in M$, the function $N_x := N \mid S \cap [R(x) \times R(x)]$ is an edge-labeling of the tree $R(x)$. Let $e_x : R(x) \to \mathscr{U}$ be the standard embedding of $R(x)$ with respect to x, $\mathscr{V}_0$, N_x and $\mathscr{C}$. Let $D_x := e_x(R(x))$. Let $\Phi_x : D_x \to R(x)$ be the inverse of e_x.

Let $D := \{(x, \mathscr{V}) \in M \times V(\mathscr{U}) \mid \mathscr{V} \in D_x\}$ and let $\Phi: D \to M$ be defined by $\Phi(x, v) := \Phi_x(v)$. For $(x, y) \in R$, let $\alpha(x, y)$ be the standard extension of $\Phi_y^{-1} \circ \Phi_x : D_x \to D_y$ with respect to $\mathscr{C}$ and $\mathscr{V}$.

It is now straightfoward to check that (D, Φ) and $\alpha : R \to \mathrm{Aut}(\mathscr{U})$ satisfy the required properties. □

3. *Analysis of a single tree*

Many of the arguments for Theorems 5.1 and 5.2 can be stated and proved as results about a single tree, rather than about a treed equivalence space. In this section, we separate out these parts of the proof.

Let T denote a locally finite tree. Let d denote the distance function on T. Recall that $V(T)$, $E(T)$, $DE(T)$ and ∂T denote the vertices, edges, directed edges and ends of T, respectively. Recall also from § 1 that $s(v, w) := v$, $r(v, w) := w$, $(v, w)' := (w, v)$, for all $(v, w) \in DE(T)$.

A *simple flow on* T is a map $f: V(T) \to V(T)$ such that $d(v, f(v)) = 1$, for all $v \in V(T)$. An *edge function on* T is a bounded function $k: DE(T) \to \mathbf{R}$ such that $k(e') = -k(e)$, for all $e \in DE(T)$. We say that k is *decisive* if, for all $v \in V(T)$, there exists $e \in DE_T(v)$ such that $k(e) > 0$.

If k is a decisive edge function on T and f is a simple flow on T, then we say that f *decides by* k if, for all $v \in V(T)$, we have $k(v, f(v)) = \max k(DE_T(v))$.

An edge function k is said to be *increasing* if, for any geodesic segment t, u, v (i.e., t adjacent to u, u adjacent to v and $t \neq v$), we have $k(t, u) \leq k(u, v)$.

If f is a simple flow on T and k is an edge function on T, then we define $s_f(k): V(T) \to \mathbf{R}$ by $s_f(k)(v) := k(v, f(v))$. Further, we define $r_f(k): V(T) \to \mathbf{R}$ by $r_f(k)(v) := 0$ if $v \neq f(V(T))$ and $r_f(k)(f(v)) := k(v, f(v))$, for all $v \in V(T)$. We call $s_f(k)$ and $r_f(k)$ the *source* and *range of k with respect to f*, respectively.

LEMMA 3.1. *Say k is an increasing, decisive edge function and f is a simple flow which decides by k. Then $r_f(k)(v) \leq s_f(k)(v)$, for all $v \in V(T)$ and $r_f(k)(v) < s_f(k)(v)$, for all $v \in V(T) \backslash f(V(T))$.*

Proof. Case 1. $v \in f(V(T))$. Define u by $f(u) = v$ and set $w := f(v)$. Because k is decisive and f decides by k, we see that $k(u, v) > 0$ and that $k(v, w) > 0$. So $k(v, u) < 0$, which shows that $u \neq w$. So then u, v, w are consecutive vertices of a geodesic segment. Since k is increasing, $k(u, v) \leq k(v, w)$. But $r_f(k)(v) = k(u, v)$ and $s_f(k)(v) = k(v, w)$, so we are done.

Case 2. $v \notin f(V(T))$. Let $w := f(v)$. Since k is decisive and f decides by k, we see that $k(v, w) > 0$. But $r_f(k)(v) = 0$ and $s_f(k)(v) = k(v, w)$. So we are done. □

Let $\mu \in M(\partial T)$ be a Borel probability measure on ∂T, the compact topological space of ends of T. For $e \in DE(T)$, let $\partial_e T$ denote the set of ends of T whose geodesic to $s(e)$ passes through $r(e)$. We can then define an edge function k on T by $k(e) := \mu(\partial_{e'} T) - \mu(\partial_e T)$. Further:

LEMMA 3.2. *Suppose every vertex of T has valence ≥3. Let $\mu \in M(\partial T)$. Then $k(e) := \mu(\partial_{e'} T) - \mu(\partial_e T)$ defines an increasing, decisive edge function on T. Further, if f is a simple flow which decides by k, then there exists $v \in V(T)$, such that $r_f(k)(v) < s_f(k)(v)$.*

Proof. First, we prove that k is increasing. Let u, v, w be consecutive vertices of a geodesic segment. Let $d := (u, v)$, $e := (v, w)$ be the directed edges in the segment. Then $\partial_e T \subseteq \partial_d T$ and $\partial_{d'} T \subseteq \partial_{e'} T$. So $k(d) \leq k(e)$.

Next, we show that k is decisive. Let $v \in V(T)$. Let $e_1, \ldots, e_s$ be the (distinct) elements of $DE_T(v)$. We must show that $k(e_i) > 0$, for some i. Note that $\partial T = \partial_{e_1} T \cup \cdots \cup \partial_{e_s} T$. We know that $s \geq 3$, so, for some i, $\mu(\partial_{e_i} T) \leq \frac{1}{3}$. But $\partial T = \partial_{e_i} T \cup \partial_{e_i'} T$, so $\mu(\partial_{e_i'} T) \geq \frac{2}{3}$. So $k(e_i) \geq \frac{2}{3} - \frac{1}{3} > 0$.

Last, we prove the statement about the simple flow f. Assume *for a contradiction* that $r_f(k)(v) \geq s_f(k)(v)$, for all $v \in V(T)$. Then, by the first part of Lemma 3.1, $r_f(k)(v) = s_f(k)(v)$, for all $v \in V(T)$. So, by the second part of Lemma 3.1, $f(V(T)) = V(T)$.

Choose $u \in V(T)$ and set $v := f(u)$, $w := f(v)$. Let $X := DE_T(v) \backslash \{(v, u), (v, w)\}$. We *claim* that $\mu(\partial_e T) = 0$, for all $e \in X$. Since

$$r_f(k)(v) = \mu(\partial_{(v,u)} T) - \mu(\partial_{(v,w)} T) - \sum_{e \in X} \mu(\partial_e T)$$

and

$$s_f(k)(v) = \mu(\partial_{(v,u)} T) - \mu(\partial_{(v,w)} T) + \sum_{e \in X} \mu(\partial_e T),$$

it follows that $\sum_{e \in X} \mu(\partial_e T) = 0$. *This establishes the claim.*

Since v has valence ≥3, $X \neq \emptyset$, so choose $e \in X$. Let $w' := r(e)$. Now $w' \in V(T) = f(V(T))$ and $w' \neq w = f(v)$, so there exists a vertex $v' \in V(T)$ adjacent to w' such that $v' \neq v$ and such that $f(v') = w'$. Let $e' := (v', w')$. Then $\partial_{e''} T \subseteq \partial_e T$, so the claim made above shows that $\mu(\partial_{e''} T) = 0$. But then $k(v', f(v')) = k(v', w') = k(e') = \mu(\partial_{e''} T) - \mu(\partial_{e'} T) \leq 0$, so k is not decisive, a *contradiction.* □

If $a_1, a_2, \ldots$ is a sequence in some topological space, then we will adopt the notation that $a_n \to a$ as $n \to \infty$ (edz) means that there exists a subset $Z \subseteq \mathbf{Z}_+$ of (upper) density zero such that $a_n \to a$ as $n \to \infty$, $n \notin Z$. (The acronym 'edz' stands for 'except density zero'.)

LEMMA 3.3. *Let $f: \mathbf{Z}_+ \to \mathbf{Z}_+$ be an increasing function bounded above by some polynomial. Then $f(n+1)/f(n) \to 1$ as $n \to \infty$ (edz).*

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Proof. We give here only a heuristic proof: If (for a contradiction) $\varepsilon > 0$ and $f(n+1)/f(n) > 1+\varepsilon$ for, say, 1% of all integers, then 1% of the entries in $f(1), f(2), \ldots$ are bounded below by some exponential function. This is a contradiction, since f is bounded above by a polynomial. □

Let Δ denote symmetric difference of sets: $A\Delta B := (A\backslash B) \cup (B\backslash A)$.

COROLLARY 3.4. *If T is a tree of polynomial growth, and $v, w \in V(T)$, then:*

$$\frac{|B(v, n)|}{|B(w, n)|} \to 1 \quad \text{and} \quad \frac{|B(v, n)\Delta B(w, n)|}{|B(w, n)|} \to 0$$

as $n \to \infty(\text{edz})$.

Proof. Let d denote the distance from v to w in T. Since edz-limits distribute over (finite) multiplication, it follows from Lemma 3.3 that $|B(w, n+d)/B(w, n)| \to 1$ as $n \to \infty(\text{edz})$ and that $|B(w, n-d)/B(w, n)| \to 1$ as $n \to \infty(\text{edz})$. Now use the facts that $B(v, n) \subseteq B(w, n+d)$ and that $B(v, n)\Delta B(w, n) \subseteq B(w, n+d)\backslash B(w, n-d)$. □

4. *Cocycle invariance*

Recall that $\mathscr{U}$ is the universal tree, defined in § 1. By abuse of notation, we use $V(\mathscr{U})$ to denote the measure space of counting measure on the set $V(\mathscr{U})$.

Let (D, Φ) be a simplifiction of the treed equivalence space (M, R, S) which is level at some vertex $\mathscr{V}_0 \in V(\mathscr{U})$. Let $\alpha : R \to \text{Aut}(\mathscr{U})$ be a cocycle compatible with (D, Φ). Recall, for $x \in M$, that $D_x := \{\mathscr{V} \in V(\mathscr{U}) \mid (x, \mathscr{V}) \in D\}$. We also use D_x to refer to the subtree of $\mathscr{U}$ with vertex set D_x. Recall that $\Phi_x(\mathscr{V}) := \Phi(x, \mathscr{V})$, for $x \in M$ and $\mathscr{V} \in V(\mathscr{U})$. As usual, we let $\text{Aut}(\mathscr{U})$ act on the right of $V(\mathscr{U})$.

For $x \in M$, let $f_x : D_x \to D_x$ be a simple flow. We say that $(f_x)_{x\in M}$ is an *α-invariant family of flows* if

(1) $(x, v) \mapsto f_x(v) : D \to V(\mathscr{U})$ is measurable and

(2) $f_x(v)\alpha(x, y) = f_y(v\alpha(x, y))$, for a.e. $x \in M$, for all $y \in R(x)$, for all $v \in V(\mathscr{U})$.

For $x \in M$, let $k_x : DE(D_x) \to \mathbf{R}$ be an edge function. Define

$$DE(D) := \{(x, v, w) \in M \times V(\mathscr{U}) \times V(\mathscr{U}) \mid v, w \in D_x \text{ and } v \text{ is adjacent to } w\},$$

with the inherited measure space structure from $M \times V(\mathscr{U}) \times V(\mathscr{U})$. We say that $(k_x)_{x\in M}$ is an *α-invariant family of edge functions* if

(1) $(x, v, w) \mapsto k_x(v, w) : DE(D) \to \mathbf{R}$ is measurable and

(2) $(k_x(v, w))\alpha(x, y) = k_y(v\alpha(x, y), w\alpha(x, y))$, for a.e. $x \in M$, for all $y \in R(x)$, for all $(v, w) \in DE(\mathscr{U})$.

If $r, s : M \to \mathbf{R}$ are two Borel functions, then we say that s is an *R-translate* of r if there exists a partition $\{M_1, M_2, \ldots\}$ of M and a sequence $g_1, g_2, \ldots$ of automorphisms of (M, R) such that

(1) $\{M_1g_1, M_2g_2, \ldots\}$ is a pairwise disjoint collection of measurable subsets of M,

(2) $r(x) = s(xg_i)$, for a.e. $x \in M_i$, for all $i = 1, 2, \ldots$ and

(3) $s(x) = 0$, for a.e. $x \in M\backslash(\bigcup_i M_ig_i)$.

LEMMA 4.1. *Let $(f_x)_{x\in M}$ be an α-invariant family of flows and let $(k_x)_{x\in M}$ be an α-invariant family of edge functions. Define $s(x) := s_{f_x}(k_x)(\mathscr{V}_0)$ and $r(x) := r_{f_x}(k_x)(\mathscr{V}_0)$ to be the source and range of k_x with respect to f_x (see § 3). Then r is an R-translate of s.*

Proof. Define $F : M \to M$ by $F(x) := \Phi_x(f_x(\mathscr{V}_0))$. It is clear that $F(x) \in R(x)$, for a.e.

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$x \in M$. Using [**4**, Theorem 1, p. 291], we may choose a countable subgroup $G \subseteq \mathrm{Aut}(M, R)$ such that the equivalence classes of R are the orbits of G. Let $g_1, g_2, \ldots$ be a listing of the elements of G. Let

$$M_i := \{x \in M \mid F(x) \neq xg_1, \ldots, F(x) \neq xg_{i-1} \quad \text{and} \quad F(x) = xg_i\}.$$

(As usual, G acts on M on the right.) It is now straightforward to verify conditions (1), (2), (3) above. □

5. *Proofs of the main theorems*

THEOREM 5.1. *Let (M, R) be an amenable equivalence space with finite R-invariant measure. Let S be a treeing of (M, R). Then, for* a.e. $x \in M$, *the tree $R(x)$ has one or two ends.*

Proof. Assume *for a contradiction* that there exists a non-neglible subset $N \subseteq M$ such that for all $x \in N$, the tree $R(x)$ has more than two ends. Let M' be the set of elements in N which have S-valence ≥ 3. Since any tree with three or more ends must have a vertex of valence ≥ 3, we see that every R-equivalence class in N meets M'. Thus, by quasi-invariance of μ, we have $\mu(M') > 0$.

Let $R' := R \mid M'$ and let $S' := S \mid M'$ be the restriction of S to M', as defined toward the end of § 1. It is not hard to see that the S'-valence of any element of M' is ≥ 3. It is also easy to see that (M', R') is amenable. (It is possible to prove this last fact without appealing to anything substantial. However the result of [**2**] makes the result obvious: If T is an endomorphism of a measure space M and $P \subseteq M$ has positive measure, then there exists an endomorphism of P whose orbits are a.e. the intersections with P of the orbits of T.) Thus we may replace (M, R) by (M', R') and assume that the S-valence of a.e. element of M is ≥ 3.

Recall that $\mathcal{U}$ denotes the universal tree (§ 1). Choose any $\mathcal{V}_0 \in V(\mathcal{U})$. By Theorem 2.3, there exists (1) a simplification (D, Φ) of (M, R, S) which is level at $\mathcal{V}_0$ and (2) a cocycle $\alpha : R \to \mathrm{Aut}(\mathcal{U})$ which is compatible with (D, Φ). Note that there is a natural action of $\mathrm{Aut}(\mathcal{U})$ on $C_0(\partial\mathcal{U})$, hence on $C_0(\partial\mathcal{U})^*$. We will denote this action as a right action.

As usual, let $D_x := \{\mathcal{V} \in V(\mathcal{U}) \mid (x, \mathcal{V}) \in D\}$, for $x \in M$. We abuse notation and let D_x also refer to the subtree of $\mathcal{U}$ with vertex set D_x. We identify each ∂D_x with a compact subset of $\partial\mathcal{U}$. Note that $(M(\partial D_x))_{x \in M}$ is an α^*-invariant field of weak-$*$ compact, convex subsets of the unit ball of $C_0(\mathcal{U})^*$. By Definition 1.2, we find an α^*-invariant section $x \mapsto \mu_x \in M(\partial D_x)$. By Lemma 3.2, $k_x(e) := \mu_x(\partial_{e'}T) - \mu_x(\partial_e T)$ defines an increasing, decisive edge functon on D_x, for a.e. $x \in M$. It is easily verified that $(k_x)_{x \in M}$ is an α-invariant family of edge functions.

Next, we wish to define an α-invariant family of flows. Given $x \in M$, we need to define a flow $f_x : V(D_x) \to V(D_x)$. Given $v \in V(D_x)$, let $e_1, \ldots, e_s$ be the (distinct) elements of $DE_{D_x}(v)$, ordered so that $k_x(e_1) \geq \cdots \geq k_x(e_s)$. Choose r so that

$$k_x(e_1) = \cdots = k_x(e_r) > k_x(e_{r+1}).$$

Using [**4**, Theorem 1, p. 291], choose a countable subgroup $G \subseteq \mathrm{Aut}(M, R)$ such that the orbits of G are the equivalence classes of R. As ususal, we denote the action of G on M on the right. Let $g_1, g_2, \ldots$ be a listing of the elements of G. For $i = 1, \ldots, r$, let $n_i := \min\{n \mid \Phi_x(v)g_n = \Phi_x(e_i)\}$. Now reorder $e_1, \ldots, e_r$ so that $n_1 < \cdots < n_r$. Define $f_x(v) := r(e_1)$.

It is straightforward to verify that $(f_x)_{x\in M}$ defines an α-invariant family of flows and, for all $x\in M$, that f_x decides by k_x. As in Lemma 4.1, we define

$$s(x) := s_{f_x}(k_x)(\mathcal{V}_0), \quad r(x) := r_{f_x}(k_x)(\mathcal{V}_0).$$

By Lemma 4.1, r is an R-translate of s. Thus $\int r\,d\mu = \int s\,d\mu$. (Note that r and s are bounded and nonnegative and that μ is finite and R-invariant.)

By Lemma 3.1, for a.e. $x\in M$ and all $\mathcal{V}\in V(\mathcal{U})$, we have

$$r_{f_x}(k_x)(\mathcal{V}) \le s_{f_x}(k_x)(\mathcal{V}).$$

Specializing to $\mathcal{V}=\mathcal{V}_0$, we get $r(x)\le s(x)$, for a.e. $x\in M$. Further, the last statement of Lemma 3.2 shows that for a.e. $x\in M$, there exist $\mathcal{V}_x\in D_x$ such that

$$r_{f_x}(k_x)(\mathcal{V}_x) < s_{f_x}(k_x)(\mathcal{V}_x).$$

Then, for a.e. $x\in M$, $r(\Phi_x(\mathcal{V}_x)) < s(\Phi_x(\mathcal{V}_x))$. Since $(x, \Phi_x(\mathcal{V}_x))\in R$, this shows that the R-saturation of $P := \{x\in M \mid r(x) < s(x)\}$ is a conull subset of M. By quasi-invariance of μ, $\mu(P)>0$. Thus $\int r\,d\mu < \int s\,d\mu$. But we showed above that $\int r\,d\mu = \int s\,d\mu$, *contradiction.* □

We remark that R-invariance is a necessary hypothesis in Theorem 5.1: Let F denote a subgroup of finite index in $SL_2(\mathbf{Z})$ which is isomorphic to a (finitely generated) free group. Let $G := SL_2(\mathbf{R})$ and let P denote the subgroup of upper triangular matrices in G. Then the action of F on G/P is amenable, by amenability of the group P (see [**13**, Corollary 4.3.7, p. 80]). It is also essentially free, hence treeable, by Example 1.6.1. But in this case a.e. tree is isomorphic to the homogeneous tree with ≥ 4 edges belonging to each vertex. This tree has uncountably many ends.

THEOREM 5.2. *Suppose S is a treeing of the equivalence space (M, R). For a.e. $x\in M$, assume either that $R(x)$ has finitely many ends or that $R(x)$ has polynomial growth. Then (M, R) is amenable.*

Proof. By replacing the measure on M by a different measure μ in the same measure class, we may assume that $\mu(M)<\infty$.

By passing to ergodic components, we may assume that R is ergodic (i.e., that every R-invariant measurable set is null or conull). We remark that, since amenability is an invariant of stable orbit equivalence, it suffices to show, for some Borel set $P\subseteq M$ of positive measure, that $(P, R\mid P)$ is amenable.

Let M' be the set of $x\in M$ such that the tree $R(x)$ has only one end. By ergodicity, M' is null or conull. We first handle the case where M' is conull.

We preface with a few remarks about an arbitrary tree with one end. So let T be any (locally finite) tree with exactly one end. Note that any directed edge is either directed toward the end or away from it. If $e_1, \ldots, e_{2n}$ are the directed edges of some even length geodesic between two vertices, then we will say that the geodesic is balanced if $e_1, \ldots, e_n$ are directed toward the end, while $e_{n+1}, \ldots, e_{2n}$ are directed away. Two vertices will be called equivalent if the geodesic connecting one to the other is balanced. (Thus equivalent implies separated by an even distance.) The equivalence classes of this equivalence relation are called *horocycles.* There is a linear ordering on horocycles: if H_1 and H_2 are horocycles, then we will write $H_1 < H_2$ if there exists a geodesic connecting a vertex of H_1 to a vertex of H_2 which is always directed away from the end of T. It is easy to show from the local finiteness and one-endedness of T that there exists a unique maximal horocycle $H(T)$ under this ordering. Note that since $H(T)$ is a horocycle, then any two vertices in $H(T)$ are separated by an even distance. Note also that half that distance is exactly the number of horocycles (other than $H(T)$ itself) met by the geodesic between them.

Returning to the theorem at hand, let P denote the union of the sets $H(R(x))$ as x ranges over M'. Because $P \cap R(x) \neq \emptyset$, for a.e. $x \in M$, quasi-invariance implies that P is of positive measure. As remarked above, we need only verify that $(OP, R \mid P)$ is amenable. Further, by [**10**, Theorem 3.6, p. 29], we need only show that $R|P$ is hyperfinite, i.e., is a countable union of equivalence relations each of which has finite equivalence classes. We define the equivalence relation R_i on P by: $(x, y) \in R_i$ iff the distance from x to y in the tree $R(x)$ is $\leq 2i$. (Transitivity follows from the last sentence of the preceding paragraph.) Then each R_i has finite equivalence classes and $R = \bigcup_{i=1}^{\infty} R_i$, as needed.

Thus we may assume that M' is null, i.e., that a.e. $R(x)$ has more than one end.

Clearly the trunk of a tree with polynomial growth is again a tree with polynomial growth. Almost as easy is the fact that the trunk of a tree with finitely many ends is a tree with polynomial growth (and with finitely many ends as well, although we do not need this). Thus we may replace (M, R, S) by its trunk (see the end of § 1) and assume that a.e. $R(x)$ has polynomial growth.

We now verify the definition of amenability in Definition 1.2. So let E be a separable Banach space and let $\alpha : R \to \operatorname{Iso}(E)$ be a (measurable) cocycle, where $\operatorname{Iso}(E)$ is given the strong operator topology. (We will denote the actions of $\operatorname{Iso}(E)$ on E and E^* as right actions.) Let α^* denote the adjoint cocycle to α. Let $(A_x)_{x \in M}$ be an an α^*-invariant measurable field of weak-* compact, convex subsets of the unit ball of E^*. Let $\mathscr{A}$ denote the subset of $L^{\infty}(M, E^*)$ consisting of maps $f : M \to E^*$ such that $f(x) \in A_x$, for a.e. $x \in M$. We now need to construct an α^*-invariant section $f \in \mathscr{A}$.

By the selection theorem [**1**, Theorem 3.4.3, p. 77], $\mathscr{A} \neq 0$, so choose $f_0 \in \mathscr{A}$. For $x \in M$ and $n = 0, 1, 2, \ldots$, let $B(x, n)$ denote the ball of radius n around the vertex x of the tree $R(x)$. Let $f_n \in \mathscr{A}$ be defined by

$$f_n(x) := \frac{1}{|B(x, n)|} \sum_{z \in B(x,n)} f(z)\alpha^*(z, x).$$

Recall the notational convention '(edz)' introduced in Lemma 3.3. We claim that $f_n(x)\alpha^*(x, y) - f_n(y) \to 0$ in the norm topology of E^* as $n \to \infty$(edz), for a.e. $x \in M$, for all $y \in R(x)$. Define

$$E_n := \left[\frac{1}{|B(x, n)|} - \frac{1}{|B(y, n)|}\right] \sum_{z \in B(x, n)} f_n(z)\alpha^*(z, y),$$

$$F_n := \frac{1}{|B(y, n)|} \sum_{z \in B(x,n)} f_n(z)\alpha^*(z, y) - \frac{1}{|B(y, n)|} \sum_{z \in B(y,n)} f_n(z)\alpha^*(z, y).$$

Note that $f_n(x)\alpha^*(x, y) - f_n(y) = E_n + F_n$. By Corollary 3.4,

$$\frac{|B(x, n)|}{|B(y, n)|} \to 1 \quad \text{as } n \to \infty(\text{edz}),$$

$$\frac{|B(x, n) \Delta B(y, n)|}{|B(y, n)|} \to 0 \quad \text{as } n \to \infty(\text{edz}).$$

Since $\|f_n(z)\| \le 1$ and since $\alpha^*(z, y)$ is an isometry of E^*, the former limit implies $\|E_n\| \to 0$ as $n \to \infty(\text{edz})$, while the latter gives

$$\|F_n\| \le \frac{1}{|B(y, n)|} \left[\sum_{z \in B(x,n) \Delta B(y,n)} \|f_n(z)\alpha^*(z, y)\| \right] \to 0$$

as $n \to \infty(\text{edz})$. *This establishes the claim.*

As usual, let Aut (M, R) act on M on the right. Define a right action of Aut (M, R) on $\mathcal{A}$ by $(a \cdot g)(x) := a(xg^{-1})\alpha^*(xg^{-1}, x)$. From the claim above, it follows that if $g \in \text{Aut}\,(M, R)$, then $f_n \cdot g - f_n \to 0$ pointwise a.e. as $n \to \infty(\text{edz})$. Since $\|f_n \cdot g - f_n\|_{L^\infty} \le 2\mu(M)$, for all n, the dominated convergence theorem implies that $f_n \cdot g - f_n \to 0$ as $n \to \infty(\text{edz})$ in the weak-* topology of $L^\infty(M, E^*) \cong L^1(M, E)^*$.

Let $g_1, g_2, \ldots$ be a listing of the elements of G. Let $n'_1, n'_2, \ldots$ be a co-density-zero increasing sequence of integers (meaning that $\mathbf{Z}_+ \backslash \{n'_1, n'_2, \ldots\}$ is of density zero in $\mathbf{Z}_+$) such that $f_{n'_k} \cdot g_1 - f_{n'_k} \to 0$ weak-* as $k \to \infty$. Now take a co-density-zero subsequence $n''_1, n''_2, \ldots$ of $n'_1, n'_2, \ldots$ such that $f_{n''_k} \cdot g_2 - f_{n''_k} \to 0$ weak-*. Continue in this way, then take the diagonal, i.e., let $n_1 := n'_1$, $n_2 := n''_2, \ldots$. Now replace $f_1, f_2, \ldots$ by $f_{n_1}, f_{n_2}, \ldots$ and assume that $f_n \cdot g - f_n \to 0$ weak-* as $n \to \infty$, for all $g \in G$. (Note the lack of an 'edz'-qualifier in this last limit.)

Since $\mathcal{A} \subseteq L^\infty(M, E^*)$ is weak-* compact, pass to a subsequence again and assume that $f_n \to f$, in the weak-* topology, for some $f \in \mathcal{A}$. Then $f \cdot g = f$ a.e., for all $g \in \text{Aut}\,(M, R)$. This is equivalent to α^*-invariance of f. □

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