SPACETIME COORDINATES IN THE GEOCENTRIC REFERENCE FRAME

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ABSTRACT. A geocentric relativistic reference frame is established which is close to the conventional non-relativistic equatorial frame of reference. Within post-Newtonian approximation *the worldline of the geocentre* is used to connect points by spacelike geodesics on the equal proper time hypersurface and to establish a properly chosen tetrad reference frame. Points on the earth surface and near the earth-space are coordinated making use of the *Frobenius matrix* of integrating factors which connects the geocentric orthonormal tetrad with the tangent spacetime of relativistic pseudo-Riemann geometry. The gravity field of the earth and its relative velocity with respect to the solar system barycentre cause coordinate effects of the order of 10 cm for topocentric point positioning.

1. GEODETIC SPACETIME POSITIONING IN NEWTON APPROXIMATION

The *observer's proper reference frame* with respect to Newton gravitation, also referred to as the topocentric frame of reference, is defined by the *orthonormal triad*

$$\left.\begin{array}{lll} \underset{\sim}{c}_3 := -\dfrac{\underset{\sim}{y}}{\|\underset{\sim}{y}\|} & (\text{"vertical"}) \\ \underset{\sim}{c}_2 := -\dfrac{\underset{\sim}{y} \wedge \underset{\sim}{\omega}}{\|\underset{\sim}{y} \wedge \underset{\sim}{\omega}\|} & (\text{"east"}) \\ \underset{\sim}{c}_1 := \underset{\sim}{c}_2 \wedge \underset{\sim}{c}_3 \, , & (\text{"south"}) \end{array}\right] \qquad 1(1)$$

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where $\underset{\sim}{y}$ denotes the topocentric gravity vector, ω the terrestrial rotation vector and $\|\cdot\|$, "$\wedge$", respectively, indicates the norm of a vector, the vector product of the two vectors, respectively. To the triplet $\{\underset{\sim}{c}_1,\underset{\sim}{c}_2,\underset{\sim}{c}_3\}$ we also refer to as the *moving triad* or "repere mobil" according to E. Cartan. *Figure 1.1* illustrates the movement of the topocentric triad with respect to the *geocentric proper reference frame*, the *equatorial* orthonormal triad $\{\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$ defined by

$$\left.\begin{aligned} \underset{\sim}{e}_3 &:= \frac{\underset{\sim}{w}}{\|\underset{\sim}{w}\|} \\ \underset{\sim}{e}_2 &:= \frac{\underset{\sim}{\psi} \wedge \underset{\sim}{\omega}}{\|\underset{\sim}{\psi} \wedge \underset{\sim}{\omega}\|} \\ \underset{\sim}{e}_1 &:= \underset{\sim}{e}_2 \wedge \underset{\sim}{e}_3 , \end{aligned}\right] \qquad 1(2)$$

where ψ denotes the instantaneous ecliptic normal vector. For more details we refer to E. Grafarend (1979).

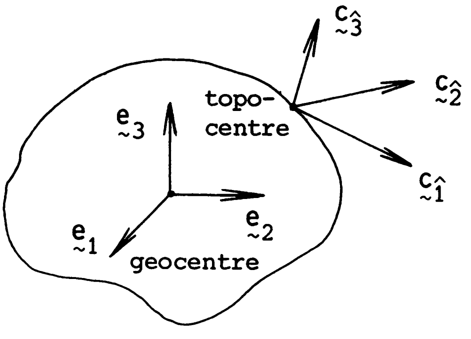

Figure 1.1 : Movement of the topocentric triad relative to the geocentric triad

Two points in infinitesimal neighbourhood are connected by the infinitesimal displacement vector dx on the earth surface, a twodimensional Riemann space V^2. The displacement vector dx has to be represented in the two orthonormal triads $\{\underset{\sim}{c}_1,\underset{\sim}{c}_2,\underset{\sim}{c}_3\}$, $\{\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3\}$, respectively, namely by

1st representation (non-holonomic basis)

$$d\underset{\sim}{x} = \bar{d}y^{\hat{1}}\, \underset{\sim}{c}_{\hat{1}} + \bar{d}y^{\hat{2}}\, \underset{\sim}{c}_{\hat{2}} + \bar{d}y^{\hat{3}}\, \underset{\sim}{c}_{\hat{3}} = \omega^{\hat{i}}\, \underset{\sim}{c}_{\hat{i}} = \underline{\omega c} \qquad 1(3)$$

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2nd representation (holonomic basis)

$$\underset{\sim}{dx} = dx^1 \underset{\sim 1}{e} + dx^2 \underset{\sim 2}{e} + dx^3 \underset{\sim 3}{e} = dx^i \underset{\sim i}{e} = \underline{dx}\ \underline{e} \ . \qquad 1(4)$$

$\{đ y^{\hat{1}}, đ y^{\hat{2}}, đ y^{\hat{3}}\} = \{\omega^{\hat{1}}, \omega^{\hat{2}}, \omega^{\hat{3}}\}$ indicates the imperfect differentials or differential 1-form. (đ Planck notation). Note that the exterior differential $d\underline{w} \neq 0$. $\{dx^1, dx^2, dx^3\}$ denotes the perfect differentials in the coordinate base $\{\underset{\sim 1}{e}, \underset{\sim 2}{e}, \underset{\sim 3}{e}\}$.

$$\underline{dx} = \underline{\omega}\ F \xleftrightarrow[|F| \neq 0]{\text{dual}} \underline{e} = F^{-1}\ \underline{c} \qquad 1(5)$$

The vector $\underline{\omega}$ of differential forms and the vector $\underline{dx}$ of coordinate differentials is related by the 3x3 *Frobenius matrix* of integrating factors. (C.F. Gauß uses for a 2x2 matrix of integrating factors the elements {a,b,c,d}; C.W. Misner et al (1973 p. 1087) call the corresponding 4x4 matrix of integrating factors A; A. Einstein conventionally writes $h_{\alpha a}$.) It is wellknown that ($\underline{e}$, $\underline{c}$ or $\underline{dx}$, $\underline{\omega}$ are related by the *Frobenius matrix of rotational type*

$$F = R(\Lambda,\Phi,0) = R_3(0)\ R_2(\tfrac{\pi}{2} - \Phi) R_3(\Lambda) \ , \qquad 1(6)$$

where Λ,Φ denote astronomical longitude, astronomical latitude, $R_3(\Lambda)$ the 3x3 rotation matrix around the 3-axis etc.

2. GEODETIC SPACETIME POSITIONING IN POST-NEWTON APPROXIMATION

In general relativity the spacetime geometry is the fourdimensional pseudo-Riemann space V^4 which can be embedded into a tendimensional Euclid space E^{10}. For more details we refer to D. Kramer et al (1980 p. 354). Terrestrial mass points form a *worldline tube*. Along the worldline of the geocentre - for the notion of the mass centre we refer to C.M. Will (1981) - we introduce the 4-velocity vector

$$\underset{\sim}{t} := \frac{d\underset{\sim}{x}}{d\tau} \ , \qquad 2(1)$$

where τ denotes proper time. Orthogonal to the tangent vector of the geocentre's worldline the space is filled by *spacelike geodesics*. Again the fourdimensional displacement vector dx has to be represented in two tetrads $\{\underset{\sim 0}{c}, \underset{\sim 1}{c}, \underset{\sim 2}{c}, \underset{\sim 3}{c}\}$, $\{\underset{\sim 0}{g}, \underset{\sim 1}{g}, \underset{\sim 2}{g}, \underset{\sim 3}{g}\}$, respectively, namely by

1st representation (non-holonomic basis)

$$\underset{\sim}{dx} = \bar{d}y^{\hat{0}}\, \underset{\sim}{c}_{\hat{0}} + \ldots + \bar{d}y^{\hat{3}}\, \underset{\sim}{c}_{\hat{3}} = \bar{d}y^{\hat{\alpha}}\, \underset{\sim}{c}_{\hat{\alpha}} = \underline{\omega}\ \underline{c} \qquad 2(2)$$

2nd representation (holonomic basis)

$$\underset{\sim}{dx} = dx^{0}\, \underset{\sim}{g}_{0} + \ldots + dx^{3}\, \underset{\sim}{g}_{3} = dx^{\mu}\, \underset{\sim}{g}_{\mu} = \underline{dx}\ \underline{g} \qquad 2(3)$$

$\{\bar{d}y^{\hat{0}}, \bar{d}y^{\hat{1}}, \bar{d}y^{\hat{2}}, \bar{d}y^{\hat{3}}\} = \{\omega^{\hat{0}}, \omega^{\hat{1}}, \omega^{\hat{2}}, \omega^{\hat{3}}\}$ indicates the imperfect differentials or differential 1-forms with respect to the *orthonormal geocentre's proper reference frame*. ($\bar{d}$ Planck notation). Note that the exterior differential $d\underline{\omega} \neq 0$. $\{dx^0, dx^1, dx^2, dx^3\}$ denotes the perfect differentials in the Gauss coordinate base

$$\underset{\sim}{g}_{\mu} := \frac{\partial \underset{\sim}{x}}{\partial x^{\mu}}\ , \quad (\mu = 0,1,2,3) \qquad 2(4)$$

a system of base vectors which span the local tangent space of V^4. See *Figure 2.1*.

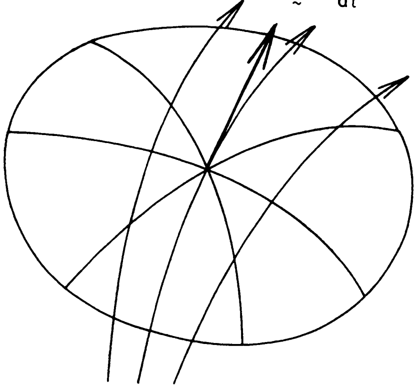

Figure 2.1: $\mathbb{V}^4 \subset \mathbb{E}^{10}$

While we can materialize an orthonormal geocentre's proper reference frame easily, we are left with the problem to construct the coordinate bases $\underset{\sim}{g}_{\mu}$.

$$\underline{dx} = \underline{\omega}\ F \ \underset{|F| \neq 0}{\overset{\text{dual}}{\longleftrightarrow}}\ \underline{g} = F^{-1}\ \underline{c} \qquad 2(5)$$

The vector $\underline{\omega}$ of differential forms and the vector $\underline{dx}$ of coordinate differentials is related by the 4x4 *Frobenius matrix* of integrating factors. (C.W. Misner et al (1973 p. 1087) call the 4x4 matrix of integrating factors $A \sim A^{\alpha}{}_{\hat{\beta}}$). In general, the matrix F accounts for the Lorentz boost (pure Lorentz) and the gauge adjustment (post-Galilei transformation). A standard procedure of group theory suggests to decompose the *Frobenius matrix* into *stretch* (symmetric matrix) and *rotation* (orthogonal matrix), a result known as the *polar decomposition theorem:*

$$F = S_{\ell} R \qquad 2(6)$$

S_{ℓ} denotes the 4x4 left stretch matrix (symmetric matrix) and the 4x4 rotation matrix (orthogonal matrix). *Obviously here we have to find the 4x4 Frobenius matrix* $F \sim F^{\mu}{}_{\circ\hat{\alpha}}$! The various connections will be a first-hand aid:

$$d\underset{\sim}{g}_{\mu} = \Gamma^{\gamma}_{\mu\bullet}\underset{\sim}{g}_{\gamma} \quad \text{versus} \quad d\underset{\sim}{c}_{\hat{\alpha}} = \Omega_{\hat{\alpha}\bullet}{}^{\hat{\beta}}\underset{\sim}{c}_{\hat{\beta}} \qquad 2(7)$$

or

$$\underline{dg} = \underset{4x4}{\Gamma}\,\underline{g} \quad \text{versus} \quad \underline{dc} = \underset{4x4}{\Omega}\,\underline{c} \qquad 2(8)$$

The orthonormal frame $\underline{c}$ allows a connection of rotational type, namely

$$\Omega = dR\, R^* = -\Omega^* \,, \quad \text{(antisymmetry)} \qquad 2(9)$$

where R* announces the transpose of R.

$$c = Fg \Longrightarrow dc = dFg + Fdg \underset{2(5),2(8)}{} \Longrightarrow$$

$$\Longrightarrow \Omega c = \Omega F g = dFg = F\Gamma g$$

$$\boxed{dF + F\Gamma = \Omega F} \qquad 2(10)$$

is a first order system for the *unknown* matrix F and *given* the connection matrices Γ, Ω. In terms of differential geometry the identity 2(10) represents the connection Ω *pulled back* from the connection Γ .

Let us have a closer look to the Ω connection: For the geocentre's proper reference frame 2(7) is structured according to

$$\begin{aligned} d\underset{\sim}{c}_{\hat{o}} &= \Omega_{\hat{o}\bullet}{}^{\hat{j}}\underset{\sim}{c}_{\hat{j}} \\ d\underset{\sim}{c}_{\hat{i}} &= \Omega_{\hat{i}\bullet}{}^{\hat{o}}\underset{\sim}{c}_{\hat{o}} + \Omega_{\hat{i}\bullet}{}^{\hat{j}}\underset{\sim}{c}_{\hat{j}} \end{aligned} \qquad 2(11)$$

Note that for *Fermi-Walker transport* holds

$$\Omega_{\hat{i}\cdot}^{\;\hat{o}} \sim u^{\hat{o}}\, a^{\hat{i}} \;, \qquad 2(12)$$

where $u^{\hat{o}}$ is the velocity, $a^{\hat{i}}$ the acceleration. The connection matrix Γ is computed from the metric tensor for events near the geocentre in post-Newton approximation.

$$g_{oo} = -1 + \frac{1}{c^2}\, U_{,\hat{\ell}\hat{m}}\, y^{\hat{\ell}} y^{\hat{m}} + \frac{2}{c^2}\, U_{\oplus} + o(c^{-4})$$

$$g_{ij} = (1 + \frac{2}{c^2}\, U_{\oplus})\delta_{ij} - \frac{1}{3c^2}(U_{,i\hat{k}}\delta_{\hat{\ell}j} + \ldots)y^{\hat{k}} y^{\hat{\ell}} \qquad 2(13)$$

$$g_{oj} = o(c^{-3}) \doteq 0 \;,$$

where $U_{\oplus}$ denotes the Newton gravitational potential of the *earth* of type $gm_{\oplus}r^{-1}$. In contrast $U_{,\hat{\ell}\hat{m}}$ are the coordinates of the *tensor of second derivatives* of the gravitational potential U of the *external* masses (sun, moon etc.) with respect to the frame $\tilde{c}_{\alpha}$. c characterizes the vacuum velocity of light. 2(13) originates from A. Einstein, L. Infeld and B. Hoffmann (1938) and A. Eddington and G. Clark (1938). For more details we refer to C.W. Misner et al (1973). Here the metric tensor $g_{\mu\gamma}$ for *events near the geocentre* is the *superposition* of the second order Taylor expansion of the gravitational potentials of the celestial bodies except for the earth *and* the gravitational potential of the earth. The computation of $\Gamma_{\mu\cdot\times}^{\;\gamma}(g_{\alpha\beta})$ is standard. In contrast, for the selection of the connection $\Omega_{\hat{\alpha}\cdot\hat{\gamma}}^{\;\hat{\beta}}$ we have a certain freedom. Traditionally $\tilde{c}_{\alpha}$ is chosen in such a way that the *spatial rotation* $\Omega_{\hat{i}}^{\;\hat{j}} = 0$. Instead M. Fujimoto and T. Fukushima have proposed at this Leningrad conference to select a so-called *natural reference frame* such that $\Omega_{\hat{i}}^{\;\hat{j}} \neq 0$, but leading to a *Frobenius matrix* which is *symmetric* or of *stretch type*. Then no rotations within the polar decomposition $F = S_{\ell}R$ are permitted. For simplicity we have chosen here the traditional way by allowing no spatial rotations $\Omega_{\hat{i}}^{\;\hat{j}}$, that is $\Omega_{\hat{i}}^{\;\hat{j}} = 0$. In solving the system of first order differential equations 2(10) we arrive at the Frobenius matrix (lengthy computation)

$$F^{\mu}_{\;\cdot\hat{\alpha}} = \left[\begin{array}{c|c} F^{o}_{\;\cdot\hat{o}} & F^{o}_{\;\cdot\hat{k}} \\ \hline F^{j}_{\;\cdot\hat{o}} & F^{j}_{\;\cdot\hat{k}} \end{array}\right] = \qquad 2(14)$$

$$= \left[\begin{array}{c|c} 1 + \frac{U}{c^2} + \frac{v^2}{2c^2} & \frac{v^k}{c} \\ \hline \frac{v^j}{c} & (1 - \frac{U}{c^2})\delta^{j}_{\;\cdot\hat{k}} + \frac{1}{c^2}\, v^j v^k + \ldots \end{array}\right]$$

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-e.g. see C.W. Misner et al (1973 p 1087) -

where for $F^{j}_{\cdot\hat{k}}$ "..." accounts for (*i*) geodetic precession, (*ii*) the Lense-Thirring term and (*iii*) the term for Thomas precession. For more details we refer to M. Fujimoto et al (1982).

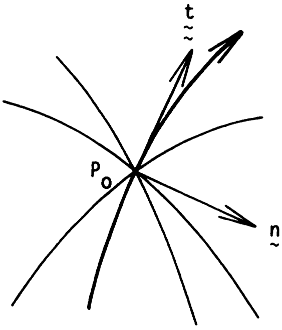

Figure 2.2: Worldline of the geocentre

Earlier we had introduced the worldline of the geocentre and its orthogonal trajectories, the bundle of spacelike geodesics. See *Figure 2.2* . The spacelike geodesics form an equal time surface. Let us denote the vector n as being normal to the tangent vector t 2(1), the vector of 4-velocity. Then the differential equation of the geodesic

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\rho\cdot\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \qquad 2(15)$$

parameterized by the conformal factor λ, will be solved by Taylor series expansion - *Legendre series* - up to second order, namely by

$$x^\mu(\lambda) = x^\mu(0) + \frac{dx^\mu}{d\lambda}(0)\lambda + \frac{1}{2}\frac{d^2x^\mu}{d\lambda^2}(0)\lambda^2 + o(\lambda^3) = \qquad 2(16)$$

$$= x^\mu(0) + n^\mu\lambda - \frac{1}{2}\Gamma^{\mu}_{\rho\cdot\sigma}(0)n^\rho n^\sigma\lambda^2 + o_3(\lambda^3)$$

or

$$x^\mu(\lambda) - x^\mu(0) = {}_oF^{\mu}_{\hat{j}\cdot}\,y^{\hat{j}} - \frac{1}{2}\,{}_o\Gamma^{\mu}_{\rho\cdot\sigma}\,F^{\sigma}_{\hat{j}\cdot}\,F^{\sigma}_{\hat{k}\cdot}\,y^{\hat{j}}y^{\hat{k}} + o_3(\lambda^3) \qquad 2(17)$$

The first term of the right-hand side which accounts for the *non-holonomity of the frame of reference* is of the order of 10^{-8} or 10 cm! In contrast, the second term on the right-hand side which reflects the *curvature of spacetime near the earth* is of the order of 10^{-16}. Finally for the geodesists' use we transform the spatial coordinates $\{y^{\hat{1}},y^{\hat{2}},y^{\hat{3}}\}$ of the geocentre's proper reference frame which is *nonrotating* into the earth-fixed rotating triad by

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$$\begin{bmatrix} y^{\hat{1}} \\ y^{\hat{2}} \\ y^{\hat{3}} \end{bmatrix}_{\text{earth-fixed rotating triad}} = R_3(-\text{GMST})R_1(y)R_2(x) \begin{bmatrix} y^{\hat{1}} \\ y^{\hat{2}} \\ y^{\hat{3}} \end{bmatrix}_{\text{non-rotating equatorial triad}} \quad 2(18)$$

where GMST denotes *Greenwich mean siderial time* angle, (x,y) the coordinates of *polar motion.*

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