108.22 The *n* days of Christmas and other series

Trivia Quiz Question: In the song 'The Twelve Days of Christmas' [1] what is the total number of gifts given over the 12 days?

Anyone can find the answer using paper, pencil and simple addition but the mathematician expects a formula for n days. Its quest leads to a simple result which may be used to generate expressions for the sum of powers of natural numbers.

A general formula

For the sequence consisting entirely of ones denote the sum of its first n terms by $T_0(n) = \sum_{i=1}^{n} 1 = n$ and define further series for $m \geqslant 0$ by

$$T_{m+1}(n) = \sum_{i=1}^{n} T_m(i). \tag{1}$$

Hence $T_0(n)$ is the number of new gifts added to the list on day n, $T_1(n) = 1 + 2 + 3 + \ldots + n$ is the number of gifts given on day n and $T_2(n)$ is the total number of gifts given by the n-th day of Christmas.

The results for these series up to $T_8(n)$ have been displayed in Table 1.

	$T_0(n)$	$T_1(n)$	$T_2(n)$	$T_3(n)$	$T_4(n)$	$T_5(n)$	$T_6(n)$
1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8
1	3	6	10	15	21	28	36
1	4	10	20	35	56	84	120
1	5	15	35	70	126	210	330
1	6	21	56	126	252	462	792
1	7	28	84	210	462	924	1716
1	8	36	120	330	792	1716	3432
1	9	45	165	495	1287	3003	6435
1	10	55	220	715	2002	5005	11440
1	11	66	286	1001	3003	8008	19448
1	12	78	364	1365	4368	12376	31824

TABLE 1. The series $T_0(n)$ to $T_6(n)$ for $1 \leqslant n \leqslant 12$.

In particular we see that 78 gifts were given on day 12 making a total of 364 gifts on all 12 days of Christmas.

Equation (1) shows

$$T_{m+1}(n) = \sum_{i=1}^{n} T_m(i) = \sum_{i=1}^{n-1} T_m(i) + T_m(n) = T_{m+1}(n-1) + T_m(n) \tag{2}$$

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so that each total is the sum of two previous totals in a similar way that Pascal's triangle for binomial coefficients is generated from

$$\binom{p}{0} = \binom{p}{p} = 1 \qquad \text{and} \qquad \binom{p+1}{r+1} = \binom{p}{r} + \binom{p}{r+1}. \tag{3}$$

Table 2 shows how we can identify the rows of Pascal's triangle as the diagonals of Table 1 and hence evaluate $T_m(n)$. For example, $T_2(7) = 84$, on row 7 of the T_2 column, is on the $p = 9$ diagonal in the $r = 3$ column corresponding to the binomial coefficient

$$\binom{9}{3} = \frac{9!}{3!\,6!} = \frac{9 \times 8 \times 7}{1 \times 2 \times 3} = 84.$$

In general $T_m(n)$ is found on row n of the T_m column or the $r = m + 1$ entry in the $p = n + m$ diagonal so that

$$T_m(n) = \binom{n+m}{m+1} = \frac{n(n+1)(n+2)\dots(n+m)}{(m+1)!}. \tag{4}$$

Using (4), $T_{m+1}(n) = T_m(n) + T_{m+1}(n-1)$ becomes

$$\binom{n+m+1}{m+2} = \binom{n+m}{m+1} + \binom{n+m}{m+2}$$

consistent with (3) and explaining the appearance of Pascal's triangle in the tables.

		$T_0(n)$	$T_1(n)$	$T_2(n)$	$T_3(n)$	$T_4(n)$	$T_5(n)$	$T_6(n)$
	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$
	1	1	1	1	1	1	1	1
$p=0$	1	2	3	4	5	6	7	8
$p=1$	1	3	6	10	15	21	28	36
$p=2$	1	4	10	20	35	56	84	120
$p=3$	1	5	15	35	70	126	210	330
$p=4$	1	6	21	56	126	252	462	792
$p=5$	1	7	28	84	210	462	924	1716
$p=6$	1	8	36	120	330	792	1716	3432
$p=7$	1	9	45	165	495	1287	3003	6435
$p=8$	1	10	55	220	715	2002	5005	11440
$p=9$	1	11	66	286	1001	3003	8008	19448
$p=10$	1	12	78	364	1365	4368	12376	31824
$p=11$	1	13	91	455	1820	6188	18654	50388
$p=12$	1	14	105	560	2380	8568	27132	77520
$p=13$	1	15	120	680	3060	11628	38760	116280
$p=14$	1	16	136	816	3876	15504	54264	170544

TABLE 2: The Pascal's triangle diagonals with associated values of p and r

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Hence the solution to the original problem is that the total number of gifts given on all the n days of Christmas is

$$T_2(n) = \binom{n+2}{3} = \frac{1}{6}n(n+1)(n+2).$$

Applications to sums of powers of natural numbers

Defining $S_m(n) = \sum_{i=1}^{n} i^m$ as the sum of the m-th powers of the first n natural numbers, we can recognise that $S_0(n) = T_0(n) = \binom{n}{1} = n$ and $S_1(n) = T_1(n) = \binom{n+1}{2} = \frac{1}{2}n(n+1)$. We may also use (4) to find the result for other values of m. By first expressing i^m from $T_{m-1}(i)$ in terms of $T_0(i)$ to $T_{m-2}(i)$ we can then find $S_m(n)$ by summing for $i = 1$ to n using (1). For example, consider the case $m = 4$.

For $m = 0, 1, 2, 3$ from (4)

$$\begin{aligned}
T_0(i) &= i && = i \\
2T_1(i) &= i(i+1) && = i^2 + 1 \\
6T_2(i) &= i(i+1)(i+2) && = i^3 + 3i^2 + 2i \\
24T_3(i) &= i(i+1)(i+2)(i+3) && = i^4 + 6i^3 + 11i^2 + 6i.
\end{aligned}$$

Expressing i^4 in terms of $T_m(i)$

$$\begin{aligned}
i^4 &= 24T_3(i) - 6i^3 - 11i^2 - 6i && = 24T_3(i) - 6\left(6T_2(i) - 3i^2 - 2i\right) - 11i^2 - 6i \\
&= 24T_3(i) - 36T_2(i) + 7i^2 + 6i && = 24T_3(i) - 36T_2(i) + 7\left(2T_1(i) - i\right) + 6i \\
&= 24T_3(i) - 36T_2(i) + 14T_i(i) - i && = 24T_3(i) - 36T_2(i) + 14T_1(i) - T_0(i).
\end{aligned}$$

Summing for $i = 1$ to n

$$\begin{aligned}
S_4(n) &= 24T_4(n) - 36T_3(n) + 14T_2(n) - T_1(n) \\
&= 24\binom{n+1}{5} - 36\binom{n+1}{4} + 14\binom{n+1}{3} - \binom{n+1}{2}
\end{aligned}$$

which, making use of the factor $n(n+1)$ common to all $T_m(n)$, simplifies to

$$S_4(n) = \frac{1}{30}n(n+1)(2n+1)\left(3n^2 + 3n + 1\right).$$

Further results may be found in the same way and may be confirmed in [2] for $1 \leqslant m \leqslant 10$.

Acknowledgements

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References

1. The twelve days of Christmas - BBC Teach
2. Eric W. Weisstein "Power Sum." From *MathWorld*–A Wolfram Web resource. https://mathworld.wolfram.com/PowerSum.html

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108.23 A recurrence relation derived graphically

The n-th triangular number, T_n, is defined as $\frac{1}{2}n(n + 1)$ for integer $n > 0$. We shall find a recurrence relation which gives setS of three consecutive triangle numbers summing to a triangle number. The first two examples are:

$$T_1 + T_2 + T_3 = T_4, \qquad T_8 + T_9 + T_{10} = T_{16}.$$

Let

$$T_k + T_{k+1} + T_{k+2} = T_l. \tag{1}$$

We shall use the fact that the suffix to the symbol for a triangular number is the side length of its figurate representation. This will enable us to transform a two-dimensional problem to one of a single dimension.

In Figure 1 we have drawn a triangle for T_l containing T_k, T_{k+1}, T_{k+2}. For equality, it must be the case that the regions of overlap sum to the central uncovered region. We can find the dimensions of these regions from the figure, leading to the reduced equation

$$T_{2k+1-l} + T_{2k+2-l} + T_{2k+3-l} = T_{2l-3k-4}. \tag{2}$$

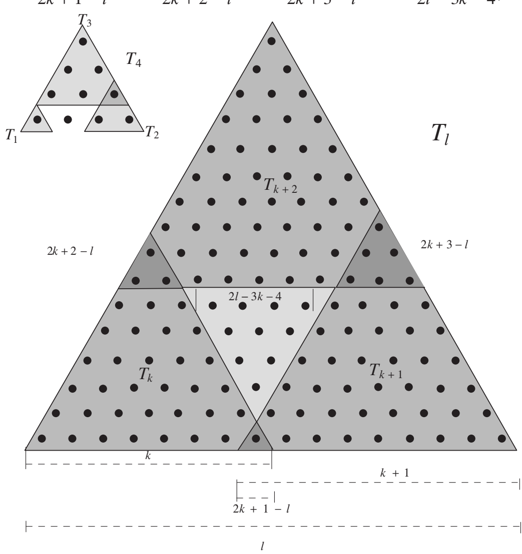

FIGURE 1