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MINIMAL RATIONAL CURVES ON COMPLETE TORIC MANIFOLDS AND APPLICATIONS

YIFEI CHEN[1], BAOHUA FU[1] AND JUN-MUK HWANG[2]

[1] *Institute of Mathematics, AMSS, 55 Zhongguancun East Road, Beijing 100190, People's Republic of China* (yifeichen@amss.ac.cn; bhfu@math.ac.cn)

[2] *Korea Institute for Advanced Study, Hoegiro 87, Seoul 130-722, Republic of Korea* (jmhwang@kias.re.kr)

Dedicated to Professor V. V. Shokurov

Abstract We show that minimal rational components on a complete toric manifold X correspond bijectively to some special primitive collections in the fan defining X, and the associated varieties of minimal rational tangents are linear subspaces. Two applications are given: the first is a classification of n-dimensional toric Fano manifolds with a minimal rational component of degree n, and the second shows that any complete toric manifold satisfying certain combinatorial conditions on the fan has the target rigidity property.

Keywords: toric varieties; rational curves; varieties of minimal rational tangents

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1. Introduction

We work over the complex number field in this paper. For a complete uniruled smooth variety X, let $\mathrm{RatCurves}^n(X)$ be the normalized space of rational curves on X (see [**13**, Chapter II, Definition 2.11]). For an irreducible component $\mathcal{K}$ of $\mathrm{RatCurves}^n(X)$, let $\rho\colon \mathcal{U} \to \mathcal{K}$ and $\mu\colon \mathcal{U} \to X$ be the associated universal family morphisms. An irreducible component $\mathcal{K}$ of $\mathrm{RatCurves}^n(X)$ is called *a dominating component* if μ is dominant, and *a minimal component* if, furthermore, for a general point $x \in X$, the variety $\mu^{-1}(x)$ is complete. Members of a minimal component are called *minimal rational curves*. Note that a minimal rational curve through a general point $x \in X$ does not deform to a reducible curve through x. The *degree* of $\mathcal{K}$ is the degree of the intersection of $-K_X$ with any member in $\mathcal{K}$. For a fixed minimal component $\mathcal{K}$ and a general point $x \in X$, we define the tangent map $\tau_x\colon \mathcal{K}_x := \rho(\mu^{-1}(x)) \dashrightarrow \mathbb{P}(T_xX)$ by $\tau_x(\alpha) = \mathbb{P}(T_xC)$, where $C = \mu(\rho^{-1}(\alpha))$ is smooth at x. We denote by $\mathcal{C}_x$ the closure of the image of τ_x in $\mathbb{P}(T_xX)$, which is called the *variety of minimal rational tangents* (VMRT) of $\mathcal{K}$ at the point $x \in X$. We recommend [**9**] for a general introduction to VMRT.

It turns out that the projective geometry of $\mathcal{C}_x \subset \mathbb{P}(T_xX)$ encodes a lot of the geometrical properties on X, which can be a useful tool in solving a number of problems on uniruled varieties (see the surveys [**9**, **14**]). Thus, for a given X, it is worthwhile to determine $\mathcal{C}_x$. This has been worked out for many examples when X has Picard number 1 (see [**9**, **14**]). However, not many cases with large Picard number have been investigated. The first main result of this paper, Corollary 2.5, gives a description of $\mathcal{C}_x$ for a complete toric manifold. This implies that the minimal components of $\mathrm{RatCurves}^n(X)$ correspond bijectively to some special primitive collections (see Proposition 3.2), which can be easily read off from the fan defining X. As an application, we get examples of complete toric manifolds that do not have any minimal component (see Example 3.4). We are also able to classify toric Fano manifolds admitting a minimal component of degree $n = \dim X$ (see Proposition 3.8). Motivated by a conjecture of Mukai (see [**7**]), we propose in § 3 a conjectural upper bound for $\rho_X(\dim \mathcal{C}_x + 1)$ when X is a toric Fano manifold.

The last section applies these results to study deformation rigidity of surjective morphisms to some toric manifolds. Recall that a compact complex manifold X is said to have the *target rigidity property* (TRP) (see [**10**]) if, for any surjective morphism $f: Y \to X$ from a compact complex manifold Y, every deformation $f_t: Y \to X$, $t \in \mathbb{C}$, $|t| < 1$, of f comes from automorphisms of X, i.e. there exists a family of automorphisms $\phi_t: X \to X$ such that $\phi_0 = \mathrm{Id}_X$ and $f_t = \phi_t \circ f$. If X is simply connected and not uniruled, then it has the TRP (see [**12**]). Conjecturally, all Fano manifolds with Picard number 1, except projective spaces, have the TRP (see [**10**, Conjecture 1.1]). On the other hand, one can construct many uniruled manifolds with arbitrarily large Picard numbers that do not have the TRP (see, for example, [**11**]). Very little is known about uniruled manifolds with large Picard number having the TRP. The only known case is in [**11**], where the TRP is proven for the blow-ups of $d \geqslant 3$ distinct points in $\mathbb{P}^2$. Even in dimension 2, a complete classification of uniruled surfaces with the TRP is still unknown.

Following an idea of [**11**], we show that if a complete toric manifold satisfies some combinatorial conditions, then it has the TRP (see Theorem 4.4). As a consequence, any surjective morphism from a toric manifold to such varieties is automatically a toric morphism. Examples of toric varieties satisfying our combinatorial conditions include those associated with Weyl chambers (see [**16**]). As an application, we show that every projective variety of dimension greater than or equal to 2 is birational to a variety with the TRP.

2. Varieties of minimal rational tangents on a complete toric manifold

We begin with some preliminary results. The first one is more or less obvious.

Lemma 2.1. *Let X be a complete variety on which a connected algebraic group G acts with an open orbit $X_0 \subset X$. Suppose that the stabilizer $\mathrm{Stab}_G(x) \subset G$ of a point $x \in X_0$ is connected. For any dominating component $\mathcal{K}$ of $\mathrm{RatCurves}^n(X)$, the subvariety $\mathcal{K}_x \subset \mathcal{K}$ is then irreducible.*

Proof. The group G acts on the universal family $\mu: \mathcal{U} \to X$ of $\mathcal{K}$. This action descends to the finite morphism $\mu': \mathcal{U}' \to X$ obtained by the Stein factorization of μ. Since the stabilizer $\mathrm{Stab}_G(x)$ is connected, it fixes a point $y \in \mu'^{-1}(x)$, i.e. $\mathrm{Stab}_G(y) = \mathrm{Stab}_G(x)$.

It follows that $\mathcal{U}'$ contains an open subset isomorphic to X_0, i.e. μ' is birational. This means that $\mathcal{K}_x$ is irreducible. □

For the next result, we need to define some notation. Let G be a connected algebraic group. For an irreducible closed algebraic subvariety $S \subset G$ that contains the identity $e \in G$, let $[S]$ be the subgroup of G generated by elements in S and let $\langle S \rangle$ be the smallest closed algebraic subgroup of G containing S. Clearly, $[S] \subset \langle S \rangle$. The following is from [**2**, Proof of Corollary 1].

Proposition 2.2. *Let G be a connected algebraic group over the complex numbers. Let S be a closed irreducible algebraic subvariety of G containing the identity $e \in G$. Then $[S] = \langle S \rangle$. In particular, if there exists a Lie subalgebra $\mathfrak{h}$ of the Lie algebra of G such that $S \subset \exp(\mathfrak{h})$, then $\dim\langle S \rangle \leqslant \dim \mathfrak{h}$.*

From now on, let X be a complete toric manifold of dimension n and let $\boldsymbol{T} \subset X$ be the open orbit of the torus $(\mathbb{C}^*)^n$. By a toric subvariety of X we mean a subvariety that is toric under the induced action of a subtorus. A rational curve $C \subset X$ is called *a standard curve* if, under the normalization $\nu\colon \mathbb{P}^1 \to C$, we have that $\nu^*(TX) \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-p-1}$ for some integer p. It is easy to see that the deformations of a standard curve C correspond to a dominating component $\mathcal{K}^C$ of $\mathrm{RatCurves}^n(X)$. As in § 1, for $x \in \boldsymbol{T}$, we denote by $\mathcal{K}_x^C$ the collection of members of $\mathcal{K}^C$ passing through x. Then, $\dim \mathcal{K}_x^C = p$ and $\mathcal{K}_x^C$ is irreducible by Lemma 2.1. Just as before, we can define a variety of tangents $\mathrm{VT}_x^C \subset \mathbb{P}(T_x X)$ by taking the closure of tangents at x of curves in $\mathcal{K}_x^C$ that are smooth at x.

Theorem 2.3. *Let X be a complete toric manifold and let $C \subset X$ be a standard curve. The variety of tangents VT_x^C of $\mathcal{K}^C$ at a point $x \in \boldsymbol{T}$ is then a linear subspace in $\mathbb{P}(T_x X)$.*

Proof. Let $D = X \setminus \boldsymbol{T}$ be the boundary divisor and let $\Omega_X(\log D)$ be the sheaf of germs of 1-forms with logarithmic poles along D. It is well known that $\Omega_X(\log D) \simeq \mathcal{O}_X^{\oplus n}$ (see [**15**, Proposition 3.1]). Given $x \in \boldsymbol{T}$, we may assume that x is the identity of the group $\boldsymbol{T}$ by using the torus action. For an irreducible and reduced curve C passing through x, let $\langle C \rangle$ be the smallest toric subvariety of X containing C. Then,

$$\langle C \rangle \cap \boldsymbol{T} = \langle C \cap \boldsymbol{T} \rangle,$$

where the right-hand side is in the sense of Proposition 2.2. Let V be the subspace of $H^0(X, \Omega_X(\log D)) \simeq \mathbb{C}^n$ consisting of vectors that annihilate the tangent vectors TC along C. From $\Omega_X(\log D) \simeq \mathcal{O}_X^{\oplus n}$, the vector space V contains the space

$$\begin{aligned} H^0(C, T_X^*|_C) \subset H^0(C, \Omega_X(\log D)|_C) &= H^0(C, \mathcal{O}_C^{\oplus n}) \\ &= H^0(X, \mathcal{O}_X^{\oplus n}) \\ &= H^0(X, \Omega_X(\log D)). \end{aligned}$$

As C is a standard curve, the space $H^0(C, T_X^*|_C)$ has dimension $n - p - 1$, with $p = \dim \mathcal{K}_x^C$; thus, we get that $\dim V \geqslant n - p - 1$. By Proposition 2.2 (applied to $\mathbb{C}^n \xrightarrow{\exp} \boldsymbol{T}$

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and with $\mathfrak{h}$ the vector space generated by vectors tangent to $C \cap \boldsymbol{T}$), this implies that $\dim\langle C\rangle = \dim\langle C \cap \boldsymbol{T}\rangle \leqslant n - \dim V \leqslant p+1$.

Let $\mathcal{K}^o \subset \mathcal{K}_x^C$ be the open subset consisting of standard curves. Denote by $\operatorname{locus}(\mathcal{K}^o)$ the closure of the union of members of $\mathcal{K}^o$. Then, $\operatorname{locus}(\mathcal{K}^o)$ is a $(p+1)$-dimensional subvariety of X. Consider the p-dimensional family of toric subvarieties $\{\langle C_t\rangle\}_{t\in\mathcal{K}^o}$ that pass through the fixed point x. Since there is no positive-dimensional family of toric subvarieties fixing a point, we have that $\langle C_t\rangle = \langle C_{t'}\rangle$ for two general points $t, t' \in \mathcal{K}^o$. Thus, $\operatorname{locus}(\mathcal{K}^o) \subset \langle C_t\rangle$ for some general $t \in \mathcal{K}^o$. Since $\dim(\operatorname{locus}(\mathcal{K}^o)) = p+1$, while $\dim(\langle C_t\rangle) \leqslant p+1$, we have that $\langle C_t\rangle = \operatorname{locus}(\mathcal{K}^o)$. In particular, we have $\mathrm{VT}_x^C = \mathbb{P}(T_x\langle C_t\rangle)$, which is linear of dimension p since the toric subvariety $\langle C_t\rangle$ is smooth at x. □

Remark 2.4. The same argument works for singular complete toric varieties if one assumes that the standard curve C is contained in the smooth locus of X.

Corollary 2.5. *Let X be a complete toric manifold and let $\mathcal{K}$ be a minimal component of degree $p+2$. The variety of minimal rational tangents $\mathcal{C}_x$ at a general point $x \in X$ is a linear subspace. The locus of curves in $\mathcal{K}$, $\operatorname{locus}(\mathcal{K}_e)$, passing through the identity $e \in \boldsymbol{T}$ is a toric subvariety in X, which is isomorphic to $\mathbb{P}^{p+1}$ with trivial normal bundle.*

Proof. As a general member of $\mathcal{K}$ is a standard curve (see [**9**, Theorem 1.2]), Theorem 2.3 implies that $\mathcal{C}_x$ is a linear subspace for general $x \in X$. As shown in [**1**, Lemma 3.3], $\operatorname{locus}(\mathcal{K}_x)$ is an immersed $\mathbb{P}^{p+1}$ with trivial normal bundle. By the proof of Theorem 2.3, the subvariety $\operatorname{locus}(\mathcal{K}_e)$ is equal to $\langle C\rangle$ for a general curve C in $\mathcal{K}_e$; thus, it is a toric subvariety in X. Being a toric subvariety, the variety $\operatorname{locus}(\mathcal{K}_e)$ is itself normal; thus, it is smooth and isomorphic to $\mathbb{P}^{p+1}$. □

Corollary 2.6. *Let X be a complete toric manifold of dimension n and let $\mathcal{K}$ be a minimal component of $\operatorname{RatCurves}^n(X)$ of degree $p+2$. There then exists an open dense subset X_0 of X isomorphic to $\mathbb{P}^{p+1} \times (\mathbb{C}^*)^{n-p-1}$ as toric varieties, and any member of $\mathcal{K}$ meeting X_0 is a line on a fibre of the natural projection $\phi_0 \colon \mathbb{P}^{p+1} \times (\mathbb{C}^*)^{n-p-1} \to (\mathbb{C}^*)^{n-p-1}$.*

Proof. By [**1**, Theorem 1.1] (the assumption of the projectivity of X is unnecessary in this theorem), there exists an open subset U of X that has a $\mathbb{P}^{p+1}$-bundle structure. By Corollary 2.5, $\operatorname{locus}(\mathcal{K}_e)$ is isomorphic to $\mathbb{P}^{p+1}$. The open subset U contains the image of the orbit of $\operatorname{locus}(\mathcal{K}_e)$ under the torus action, which gives a projective bundle over $(\mathbb{C}^*)^{n-p-1}$. The claim now follows from the fact that any toric projective bundle over the torus $(\mathbb{C}^*)^{n-p-1}$ is trivial (as a toric bundle). □

3. Combinatorial description of minimal rational curves

We now relate minimal components on X to combinatorial data of the fan corresponding to X. The basic results on toric varieties can be found in [**15**]. Recall that X is described by a finite fan $\varSigma$ in the vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$, where N is a free abelian group of rank $n = \dim X$. As X is smooth and complete, the support of $\varSigma$ is the whole space $N_{\mathbb{Q}}$, and every cone in $\varSigma$ is generated by a part of a basis of N. For any i, we denote by $\varSigma(i)$

the set of all i-dimensional cones in Σ. For each $\sigma \in \Sigma(1)$, we take a primitive generator of $\sigma \cap N$. We denote by $G(\Sigma)$ the set of all such generators, which is in bijection with the set of all one-dimensional cones in Σ. The Picard number ρ_X of X is given by $\sharp G(\Sigma) - n$ (see [**15**, Corollary 2.5]).

Definition 3.1 (Batyrev [3]).

(i) A non-empty subset $\mathfrak{P} = \{x_1, \dots, x_k\}$ of $G(\Sigma)$ is called a *primitive collection* if, for any i, the elements of $\mathfrak{P} \setminus \{x_i\}$ generate a $(k-1)$-dimensional cone in Σ, while $\mathfrak{P}$ does not generate a k-dimensional cone in Σ.

(ii) For a primitive collection $\mathfrak{P} = \{x_1, \dots, x_k\}$ of $G(\Sigma)$, let $\sigma(\mathfrak{P})$ be the unique cone in Σ that contains $x_1 + \cdots + x_k$ in its interior. Let $y_1, \dots, y_m$ be generators of $\sigma(\mathfrak{P})$; there then exists a unique equation with $a_i \in \mathbb{Z}_{>0}$:

$$x_1 + \cdots + x_k = a_1 y_1 + \cdots + a_m y_m.$$

This is the *primitive relation* associated with $\mathfrak{P}$. The *degree* of $\mathfrak{P}$ is $\deg(\mathfrak{P}) = k - \sum_i a_i$. The *order* of $\mathfrak{P}$ is k.

Proposition 3.2. *Let X be a complete toric manifold of dimension n. There then exists a bijection between minimal components of degree k on X and primitive collections $\mathfrak{P} = \{x_1, \dots, x_k\}$ of $G(\Sigma)$ such that $x_1 + \cdots + x_k = 0$.*

Proof. If $\mathcal{K}$ is a minimal component in $\mathrm{RatCurves}^n(X)$ of degree k, by Corollary 2.6, there exists an open dense toric subvariety $X_0 \simeq \mathbb{P}^{k-1} \times (\mathbb{C}^*)^{n+1-k}$ such that lines in the factor $\mathbb{P}^{k-1}$ give general members of $\mathcal{K}$. The fan defining X_0 is the fan of $\mathbb{P}^{k-1}$ but viewed as a fan in $\mathbb{R}^n$. This gives a collection $\mathfrak{P} = \{x_1, \dots, x_k\}$ of elements in $G(\Sigma)$ such that, for any $x_i \in \mathfrak{P}$, the elements in $\mathfrak{P} \setminus \{x_i\}$ generate a $(k-1)$-dimensional cone. Moreover, we have that $x_1 + \cdots + x_k = 0$, which implies that $\mathfrak{P}$ does not generate a k-dimensional cone in Σ, since every cone of Σ is generated by a part of a basis of N. We conclude that $\mathfrak{P}$ is a primitive collection of $G(\Sigma)$.

Now, assume that $\mathfrak{P} = \{x_1, \dots, x_k\}$ is a primitive collection such that $x_1 + \cdots + x_k = 0$. Let Σ' be the subfan of Σ determined by $\mathfrak{P}$, i.e. Σ' is the collection of all cones in Σ generated by subsets of $\{x_1, \dots, x_k\}$. Let $U_{\Sigma'}$ be the toric variety associated with Σ'; then $U_{\Sigma'}$ is isomorphic to $\mathbb{P}^{k-1} \times (\mathbb{C}^*)^{n-k+1}$. On the other hand, $U_{\Sigma'}$ is an open subset of X. Take a line C in $\mathbb{P}^{k-1}$; its deformations then form a minimal component in $\mathrm{RatCurves}^n(X)$. $\square$

When X is a projective toric manifold, there always exists a minimal component in $\mathrm{RatCurves}^n(X)$ (for example, we can take a dominant family of rational curves that has the minimal degree with respect to an ample line bundle on X). Proposition 3.2 has the following corollary, which has been proved by Batyrev (see [**3**, Proposition 3.2]) in a completely different way.

Corollary 3.3. *Let Σ be a fan that defines a projective toric manifold X. There then exists a primitive collection $\mathfrak{P} = \{x_1, \dots, x_k\}$ such that $x_1 + \cdots + x_k = 0$.*

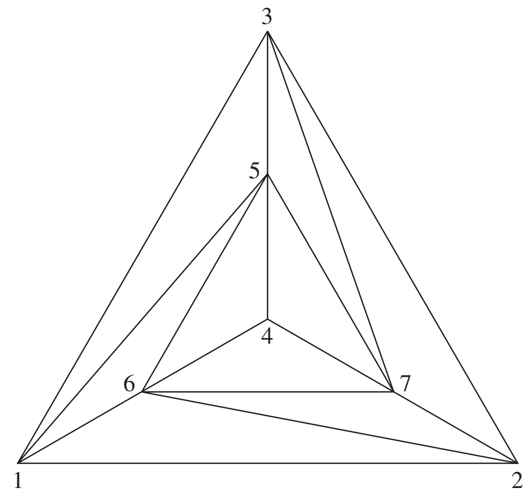

Figure 1. Complete non-projective toric variety.

Example 3.4. The assumption of projectivity of X in the previous corollary is important, as shown by the following example of [**15**, § 2.3]. Let e_1, e_2, e_3 be a basis of $\mathbb{Z}^3$. Let

$$e_4 = -e_1 - e_2 - e_3, \qquad e_5 = -e_1 - e_2, \qquad e_6 = -e_2 - e_3, \qquad e_7 = -e_1 - e_3.$$

Let Σ be the complete regular fan in $\mathbb{R}^3$ obtained by joining 0 with the simplices of the triangulated tetrahedron in Figure 1. We have that $e_1 + e_2 + e_5 = 0$; however, the set $\{e_1, e_2, e_5\}$ is not a primitive collection, since the cone generated by e_2, e_5 is not in Σ. Similarly, we see that $\{e_2, e_3, e_6\}$, $\{e_1, e_3, e_7\}$ are not primitive collections. This implies that there is no minimal component in $\mathrm{RatCurves}^n(X)$. This is another way to see that the toric variety X defined by Σ is smooth complete but non-projective. The subfan in Σ generated by e_1, e_2, e_5 gives a toric subvariety that is isomorphic to $\mathbb{P}^2 \setminus \{pt\}$. If we denote by C the invariant curve corresponding to the cone generated by e_1, e_3, then its cohomology class is given by $e_1 + e_2 + e_5 = 0$, which implies that its normal bundle in X is given by $\mathcal{O}(1) \oplus \mathcal{O}$, i.e. C is a standard curve. By Theorem 2.3, the variety of tangents VT_x^C determined by C is isomorphic to $\mathbb{P}^1$, while all members of $\mathcal{K}_x^C$ lie in an open set in $\mathbb{P}^2 \setminus \{pt\}$. If we denote by $\pi\colon Y \to X$ the blow-up of X along the invariant curve C, then Y is still non-projective since C deforms in X (see [**5**, Proposition 2]). The fan $\Sigma(Y)$ of Y has a new element $e_0 = e_1 + e_3 = -e_7$. Thus, the non-projective variety Y has a unique minimal component, and its VMRT is just one point.

If one blows up X along the invariant curve corresponding to the cone generated by e_3 and e_7, one obtains a projective variety X' (see [**15**, § 2.3]). The fan Σ' of X' has a new element $e_8 = -e_1$ in $G(\Sigma')$. This implies that there exists a unique minimal component in $\mathrm{RatCurves}^n(X')$, and its VMRT is just a point.

As an application of Proposition 3.2, we give an upper bound for the number of minimal components in $\mathrm{RatCurves}^n(X)$.

Proposition 3.5. *Let X be a complete toric manifold of dimension n and Picard number ρ_X. For an integer p, we denote by n_p the number of minimal components in* $\mathrm{RatCurves}^n(X)$ *of degree $p+2$. We then have that*

(i)
$$\sum_{p=0}^{n-1} n_p(p+2) \leqslant n + \rho_X;$$

(ii) *if n_p and n_q are non-zero for some integers p and q, then $p+q \leqslant n-2$;*

(iii) *if $p \geqslant (n-1)/2$, then $n_p \leqslant 1$.*

Proof. Suppose that we have two primitive collections $\mathfrak{P}_1 = \{x_1, \dots, x_{k+1}\}$ and $\mathfrak{P}_2 = \{y_1, \dots, y_{h+1}\}$ such that $x_1 + \dots + x_{k+1} = y_1 + \dots + y_{h+1} = 0$. If $\mathfrak{P}_1 \cap \mathfrak{P}_2$ is non-empty, we may assume that $x_{k+1} = y_{h+1}$; we then get that $x_1 + \dots + x_k = y_1 + \dots + y_h$. As $x_1, \dots, x_k$ and $y_1, \dots, y_h$ generate cones in Σ, this implies that the two cones are the same; thus, $\mathfrak{P}_1 = \mathfrak{P}_2$. Now (i) follows from Proposition 3.2 and the fact that the number of elements in $G(\Sigma)$ is equal to $n + \rho_X$.

The other two statements follow from the proof of [**10**, Proposition 2.2], where it was shown that two linear subspaces in $\mathbb{P}(T_xX)$, which are components of VMRTs, have an empty intersection in $\mathbb{P}(T_xX)$ for $x \in X$ general. □

Remark 3.6. Even in dimension 2, one can construct many examples where the inequality in (i) is an equality. If one restricts the problem to toric Fano manifolds, the inequality in (i) becomes an equality for products of copies of S_3 with projective spaces, where S_3 is the blow-up of $\mathbb{P}^2$ at three general points. In [**17**, Example 4.7], Sato constructed a toric Fano 4-fold with $\rho = 5$ by blowing up $\mathbb{P}^2 \times \mathbb{P}^2$, for which the inequality (i) becomes an equality. It seems a subtle problem to classify cases where (i) becomes an equality.

Another application of Proposition 3.2 is the following. Recall that if X has a minimal component of degree $n+1$, then $X \simeq \mathbb{P}^n$ (see [**8**]). The following proposition settles the next case, when X is a toric Fano manifold. Recall that, for a toric Fano manifold, every element in $G(\Sigma)$ is a primitive vector in N, $G(\Sigma)$ is the set of vertices of a polytope Q, and each facet of Q is the convex hull of a basis of N.

Lemma 3.7 (Casagrande [6, Lemma 3.3]). *Assume that X is a toric Fano manifold. If Σ has two different primitive relations $x+y=z$ and $x+w=v$, then $w=-x-y$ and $v=-y$. Therefore, there exist at most two primitive collections of order 2 and degree 1 containing x, and the associated primitive relations are $x+y=(-w)$ and $x+w=(-y)$.*

Proposition 3.8. *Let X be a toric Fano manifold of dimension $n \geqslant 3$ that admits a minimal component of degree n. Then, X is isomorphic to $\mathbb{P}^{n-1} \times \mathbb{P}^1$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ or a blow-up of $\mathbb{P}^{n-2}$ on $\mathbb{P}^{n-1} \times \mathbb{P}^1$. In particular, we have that $\rho_X \leqslant 3$.*

Proof. Note that if $z_1, z_2 \in G(\Sigma)$ are two elements such that the two-dimensional cone generated by them is not in Σ, then $\{z_1, z_2\}$ is a primitive collection. The Fano condition on X implies that either $z_1 + z_2 = 0$ or there exists another element $z \in G(\Sigma)$ such that $z_1 + z_2 = z$ (see, for example, [**6**, p. 1480]).

By Proposition 3.2, the assumption implies that there exists a primitive collection $\mathfrak{P} = \{x_1, \dots, x_n\}$ such that $x_1 + \cdots + x_n = 0$. The vector space $H := \mathbb{R}x_1 + \cdots + \mathbb{R}x_n$ divides $N \otimes_{\mathbb{Z}} \mathbb{R}$ into two sides. Assume that, on one side, we have at least three elements y_1, y_2, y_3 in $G(\Sigma)$. Take an element $z \in G(\Sigma)$ on the other side of H; then each of $\{z, y_1\}$, $\{z, y_2\}$, $\{z, y_3\}$ is a primitive collection. By Lemma 3.7, up to reordering, we have that $z + y_1 = 0$, $z + y_2 = (-y_3)$ and $-y_2$, $-y_3$ are all elements in $G(\Sigma)$. This gives that $y_1 = y_2 + y_3$. We consider the primitive collections $\{-y_2, y_1\}$, $\{-y_2, y_3\}$. By Lemma 3.7, we obtain that $-y_2 + y_1 = -y_3$, which contradicts $y_1 = y_2 + y_3$. Thus, there exist at most two elements on each side of H. Let y_1, y_2 be the two elements on one side of H and let z_1, z_2 be those on the other side. If z_i is not in $\{-y_1, -y_2\}$, then we may apply Lemma 3.7, which shows that $-y_1$, $-y_2$ are in $G(\Sigma)$, a contradiction. Up to reordering, we may assume that $z_1 = -y_1$ and $z_2 = -y_2$. Consider the primitive collections $\{-y_1, y_2\}$ and $\{-y_2, y_1\}$. Their primitive relations are $-y_1 + y_2 = x_i$, $-y_2 + y_1 = x_j$ for some i, j. This implies that $x_i + x_j = 0$, which is not possible, since $n \geqslant 3$. In conclusion, the set $G(\Sigma)$ has at most $n + 3$ elements, while $\rho_X = \sharp G(\Sigma) - n$. As a consequence, we have that $\rho_X \leqslant 3$.

If $\rho_X = 3$, there exist two elements y_1, y_2 on one side of H and an element z on the other side. Up to reordering, the previous argument shows that $z = -y_1$ and $-y_1 + y_2 = x_1$, i.e. $y_1 + x_1 = y_2$. This shows that X is the blow-up of $\mathbb{P}^{n-1} \times \mathbb{P}^1$ along the invariant subvariety isomorphic to $\mathbb{P}^{n-2}$ corresponding to the cone generated by x_1, y_1.

If $\rho_X = 2$, i.e. $G(\Sigma)$ has $n + 2$ elements, on each side of H, there exists exactly one element in $G(\Sigma)$, say, y or z. As $\{y, z\}$ is a primitive collection, one has that either $y + z = 0$ or $y + z = x_i$ for some i. The first case corresponds to $\mathbb{P}^{n-1} \times \mathbb{P}^1$, while the second fan corresponds to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n-1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. □

For a toric Fano manifold X of dimension n, the pseudo-index ι_X is the smallest intersection number $-K_X \cdot C$ among all rational curves on X. In [**7**, Theorem 1] it was proven that $\rho_X \leqslant 2n$ and $\rho_X(\iota_X - 1) \leqslant n$, which confirms a conjecture of Mukai. As an analogue of this, we would like to propose the following conjecture.

Conjecture 3.9. *For a toric Fano manifold X^n, with $n \geqslant 3$, if there exists a minimal component $\mathcal{K}$ of degree $p + 2$, then $\rho_X \cdot (p + 1) \leqslant n(n + 1)/2$.*

Note that the equality holds if $X \simeq (S_3)^d \times \mathbb{P}^{2d}$ or $(S_3)^d \times \mathbb{P}^{2d+1}$, where S_3 is the blow-up of $\mathbb{P}^2$ along three general points. In dimension 3, the equality also holds for the blow-up of $\mathbb{P}^2 \times \mathbb{P}^1$ along a $\mathbb{P}^1$ contained in $\mathbb{P}^2$.

Since $\rho_X \leqslant 2n$ by [**7**], we may assume that $p + 1 > (n + 1)/4$, to check Conjecture 3.9. When $n = 3$, this implies that $p \geqslant 1$; thus, the minimal component $\mathcal{K}$ has degree greater than or equal to 3. Hence, Conjecture 3.9 is immediate from Proposition 3.8. To check Conjecture 3.9 for $n = 4$, note that if $p \geqslant 2$, then the minimal component $\mathcal{K}$ has degree greater than or equal to 4, and we are done by Proposition 3.8. Hence, we need only show that if $p = 1$, then $\rho_X \leqslant 5$. We can use the classification of four-dimensional toric Fano manifolds in [**4**] to check that if $\rho_X \geqslant 6$, then $p = 0$, which shows that our conjecture holds for $n = 4$.

4. Deformation rigidity of morphisms onto some toric manifolds

Recall that a *web* (of rank 1) on a complex manifold U is a submanifold $W \subset \mathbb{P}T(U)$ with finitely many connected components, each of which is biholomorphic to U by the natural projection $\mathbb{P}T(U) \to U$. A web W on a manifold U is *equivalent* to a web W' on a manifold U' if there exists a biholomorphic map $\varphi\colon U \to U'$ such that its differential $\mathrm{d}\varphi\colon \mathbb{P}T(U) \to \mathbb{P}T(U')$ sends W bijectively to W'. Given a web W on U, a holomorphic vector field v on U is an *infinitesimal automorphism* of W if, for any relatively compact domain $U_0 \subset U$, the one-parameter family of biholomorphic maps generated by v,

$$\{\exp(tv)\colon U_0 \to U_t := \exp(tv)(U_0),\ t \in \mathbb{C},\ |t| < \epsilon\}$$

for sufficiently small ϵ, defines an equivalence of webs $W|_{U_0}$ and $W|_{U_t}$ for each t.

Proposition 4.1 (Hwang [11, Proposition 3.1]). *Let U be a complex manifold with pairwise pointwise independent holomorphic vector fields $v_1, \dots, v_d$. A holomorphic vector field v on U is an infinitesimal automorphism of the web defined by $v_1, \dots, v_d$ if and only if, for each $i = 1, \dots, d$, there exists a holomorphic function h_i on U such that $[v, v_i] = h_i v_i$, where the bracket denotes the Lie bracket of vector fields.*

Let U be a complex manifold of dimension n. Recall (see [**11**]) that a *d-web of fibrations* on U is a collection of Zariski open subsets $U_1, \dots, U_d$ of U and surjective proper holomorphic maps $p_i\colon U_i \to V_i$ for some $(n-1)$-dimensional complex manifolds V_i such that, for each $i \neq j$, the fibres of p_i, p_j through a general point of U are distinct. Note that the kernel of the differential $\mathrm{d}p_i\colon T(U_i) \to T(V_i)$ defines a subvariety W_i in $\mathbb{P}T(U)$, and the map $W_i \to U$ is birational. Let $W = W_1 \cup \cdots \cup W_d$; there then exists a unique maximal Zariski open subset in U, denoted by $\mathrm{Dom}(W)$, over which W defines a web.

The following proposition was essentially proved by Hwang [**11**, Proposition 4.5].

Proposition 4.2. *Let X be a smooth complete variety with a web W of fibrations and let $f\colon Y \to X$ be a generically finite morphism. The Kodaira–Spencer class $\tau \in H^0(Y, f^*T(X))$ of any deformation of f then defines a multi-valued vector field on X, which is locally an infinitesimal automorphism of the web $W|_{\mathrm{Dom}(W)}$.*

Recall that a complete manifold X is said to have the *target rigidity property* (TRP) if, for any surjective morphism $f\colon Y \to X$, every deformation of f with Y and X fixed comes from automorphisms of X. The following simple proposition is one of the motivations for introducing this property.

Proposition 4.3. *Let X be a complete manifold having the TRP. Let $f\colon Y \to X$ be a surjective morphism from a smooth complete variety Y. Any holomorphic vector field on Y then descends to a holomorphic vector field on X such that f is equivariant with*

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respect to the one-parameter groups of automorphisms of Y and X generated by the holomorphic vector fields. In particular, if Y is toric, then X is a toric manifold and f is a toric morphism.

Proof. Let v be a vector field on Y and let ϕ_t be the one-parameter subgroup of automorphisms generated by v. The map $f \circ \phi_t \colon Y \to X$ gives a deformation of f. As X has the TRP, there exist automorphisms $\psi_t \colon X \to X$ such that $f \circ \phi_t = \psi_t \circ f$, which gives the claim. □

The main result of this section is the following theorem.

Theorem 4.4. *Let X be a complete toric manifold of dimension n defined by a fan $\Sigma \subset N_{\mathbb{Q}}$. Let $G(\Sigma)$ be the set of primitive vectors generating one-dimensional cones in Σ. Assume that there exist $n+1$ vectors $e_1, \dots, e_{n+1}$ in $G(\Sigma)$ such that*

(i) *every n vectors of these e_i are linearly independent,*

(ii) *for any $i = 1, \dots, n+1$, the vector $-e_i$ is also in $G(\Sigma)$.*

Then, X has the TRP and every surjective morphism from a toric manifold to X is a toric morphism.

Proof. The proof is a modification of the proof of [**11**, Main Theorem]. By Proposition 3.2, the collections $\{e_i, -e_i\}$, $i = 1, \dots, n+1$, correspond to the minimal components $\mathcal{K}_1, \dots, \mathcal{K}_{n+1}$ in $\mathrm{RatCurves}^n(X)$. By Corollary 2.6, these collections define an $(n+1)$-web of fibrations, say W_{n+1} on X. The key point is that, for each $\mathcal{K}_i$, the tangent vector field v_i to the foliation of curves in $\mathcal{K}_i$, which is defined *a priori* only on an open set of X, comes from a $\mathbb{C}^*$-action on X; thus, v_i is a well-defined vector field on X. In particular, it is an infinitesimal automorphism of any web of fibrations on X (see [**11**, Proposition 4.4]). We now need some explicit computations on infinitesimal automorphisms of W_{n+1}.

There exists a Zariski open subset U of X with analytic coordinates $x_1, \dots, x_n$, on which the web W_n formed by $\mathcal{K}_1, \dots, \mathcal{K}_n$ is analytically equivalent to the web defined by the vector fields $\partial_1, \dots, \partial_n$, where $\partial_i = \partial/\partial x_i$. The vector field corresponding to $\mathcal{K}_{n+1}$ can be written as $v_{n+1} = \sum_i f_i \partial_i$ for some analytic function f_i on U. By our assumption, each f_i is not identically 0 on U. As v_{n+1} is an infinitesimal automorphism of the web W_n, by Proposition 4.1, there exist holomorphic functions h_j, $j = 1, \dots, n$, such that

$$\left[\sum_{i=1}^{n} f_i \partial_i, \partial_j\right] = [v_{n+1}, v_j] = h_j v_j = h_j \partial_j,$$

which implies that $\partial_j f_i = 0$ for all $i \neq j$; thus, the function f_i depends only on x_i. Now assume that we have an infinitesimal automorphism $v := \sum_j g_j \partial_j$ of the web W_{n+1} on an analytic open subset U_0, where the g_j are analytic functions on U_0. Arguing as before then shows that the function g_j depends only on x_j. By Proposition 4.1, there exists a holomorphic function h such that

$$\left[\sum_{j=1}^{n} g_j \partial_j, \sum_{i=1}^{n} f_i \partial_i\right] = h \cdot \sum_{i=1}^{n} f_i \partial_i.$$

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As f_i, g_i depend only on x_i, this gives that $\sum_{i=1}^{n}(g_i f_i' - g_i' f_i)\partial_i = \sum_{i=1}^{n} h f_i \partial_i$. In other words, we have that

$$\frac{f_i g_i' - g_i f_i'}{f_i} = \frac{f_j g_j' - g_j f_j'}{f_j}.$$

As the right-hand side depends only on x_j, while the left-hand side depends only on x_i, it is equal to a constant, say b. We obtain that $(g_i/f_i)' = b/f_i$ for all i. Thus, there exists a constant a_i such that

$$g_i = a_i f_i + b f_i \int \frac{1}{f_i}.$$

This equation shows that we can extend g_j to the Zariski open subset U of X by analytic continuation as multi-valued functions. Moreover, these functions are either univalent or of infinite monodromy, since the integral yields a logarithm (see [**11**, Proposition 3.4]). In particular, any infinitesimal automorphism of W_{n+1} is either univalent or of infinite monodromy.

To complete the proof we now proceed as in the proof of [**11**, § 6, Main Theorem]. Assume that we have a surjective morphism $f\colon Y \to X$. By an argument using the Stein factorization (see, for example, [**12**, § 2.2]), we may assume that f is generically finite. By Proposition 4.2, the Kodaira–Spencer class of any deformation of f defines an infinitesimal automorphism τ of the web W_{n+1} with finite local monodromy. The above calculation now implies that the multi-valued vector field τ is in fact univalent. This gives $\tau \in f^* H^0(X, TX)$, i.e. this deformation comes from automorphisms of X. The second statement follows from Proposition 4.3. □

We consider toric manifolds associated with a root system (see [**16**]). Let R be a reduced root system in a Euclidean space E. Let $M(R)$ be the lattice in E generated by the roots of R and let $N(R)$ be the lattice dual to $M(R)$. For any set of simple roots S we define the Weyl chamber of S by $\sigma_S := \{v \in N(R)_{\mathbb{Q}} \mid \forall u \in S,\ \langle u, v\rangle \geqslant 0\}$. Let $\Sigma(R)$ be the fan in the lattice $N(R)$ that consists of all Weyl chambers of R and all their faces. Let $X(R)$ be the toric variety associated with the fan $\Sigma(R)$. It is projective and smooth.

Corollary 4.5. *The toric manifold $X(R)$ has the TRP if and only if R contains no irreducible component isomorphic to the root system A_1.*

Proof. Note that $X(R_1 \times R_2) \simeq X(R_1) \times X(R_2)$ and $X(A_1) \simeq \mathbb{P}^1$, which does not have the TRP. It is easy to see that if X and Y have the TRP, so does $X \times Y$. Thus, we can assume that R is an irreducible root system of rank greater than or equal to 2. Note that, for any cone σ_S, its opposite $-\sigma_S$ is again a cone in $\Sigma(R)$. In particular, $-G(\Sigma(R)) = G(\Sigma(R))$. Let $e_1, \ldots, e_n$ be primitive vectors on one-dimensional cones of σ_S and take any other vector $e_{n+1} \in G(\Sigma(R))$ outside $\sigma_S \cup -\sigma_S$. The condition of Theorem 4.4 is then satisfied. □

As an application, we have the following characterization of toric morphisms onto a projective space.

Corollary 4.6. *Let Y be a smooth complete toric variety and let $f: Y \to \mathbb{P}^n$ be a surjective morphism. Let $\{p_1, \ldots, p_{n+1}\}$ be the $n+1$ fixed points by the torus action on $\mathbb{P}^n$. If $f^{-1}(p_i)$ is a toric subvariety in Y for all i, then f is a toric morphism.*

Proof. Let $\mathrm{Bl}(\mathbb{P}^n) \to \mathbb{P}^n$ be the blow-up of the $n+1$ fixed points and let $\mathrm{Bl}(Y) \to Y$ be the blow-up along $\bigcup_i f^{-1}(p_i)$; we then have a surjective morphism $\tilde{f}\colon \mathrm{Bl}(Y) \to \mathrm{Bl}(\mathbb{P}^n)$. As $f^{-1}(p_i)$ is toric, $\mathrm{Bl}(Y)$ is a toric variety. By Theorem 4.4, $\mathrm{Bl}(\mathbb{P}^n)$ has the TRP, which implies that $\tilde{f}$ is a toric morphism. This shows that f is a toric morphism (possibly with respect to another toric structure on $\mathbb{P}^n$). □

As another application of the ideas of [**11**], we show that every projective variety of dimension greater than or equal to 2 is birational to a variety with the TRP.

Proposition 4.7. *Let X be a projective variety of dimension $n \geqslant 2$. There then exists a composition of successive blow-ups $Z \to X$ such that Z has the TRP.*

Proof. Let $\mathrm{Bl}(\mathbb{P}^n) \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along $n+2$ points $\{p_1, \ldots, p_{n+2}\}$ of general position. By considering the strict transforms of lines through one of these points, we obtain a linear $(n+2)$-web W of rank 1 on $\mathrm{Bl}(\mathbb{P}^n)$. Through a general point of $\mathrm{Bl}(\mathbb{P}^n)$, the web W is locally equivalent to the web generated by the vector fields:

$$\partial_i,\ i = 1, \ldots, n,\ \sum_j (x_j - a_j)\partial_j,\ \sum_j (x_j - b_j)\partial_j.$$

By a similar argument to that in [**11**, Proposition 3.5] (see also the proof of Theorem 4.4), one shows that W has no non-zero infinitesimal automorphism. By Proposition 4.2, this implies that any generically finite surjective morphism to $\mathrm{Bl}(\mathbb{P}^n)$ has no non-trivial deformation.

For any projective variety X, we now fix a finite surjective morphism $g: X \to \mathbb{P}^n$, and denote by Z the composition of blow-ups of X along $g^{-1}(p_i)$, $i = 1, \ldots, n+2$. We then get a generically finite surjective morphism $h: Z \to \mathrm{Bl}(\mathbb{P}^n)$. For any generically finite surjective morphism $f: Y \to Z$ and its deformation $f_t: Y \to Z$, the composition $h \circ f_t$ gives a deformation of the generically finite morphism $h \circ f: Z \to \mathrm{Bl}(\mathbb{P}^n)$. By the rigidity of said morphism onto $\mathrm{Bl}(\mathbb{P}^n)$, we deduce that $h \circ f_t = h \circ f$ for all t, which shows that Z has the TRP. □

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