Bull. Aust. Math. Soc. **108** (2023), 205–216
doi:10.1017/S0004972722001575

ON THE NUMBER OF ALGEBRAIC POINTS ON THE GRAPH OF THE WEIERSTRASS SIGMA FUNCTIONS

GOREKH PRASAD SENA ✉ and K. SENTHIL KUMAR

(Received 11 October 2022; accepted 9 November 2022; first published online 13 January 2023)

Abstract

Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ with invariants g_2, g_3 and $\sigma_\Omega(z)$ the associated Weierstrass σ-function. Let η_1 and η_2 be the quasi-periods associated to ω_1 and ω_2, respectively. Assuming η_2/η_1 is a nonzero real number, we give an upper bound for the number of algebraic points on the graph of $\sigma_\Omega(z)$ of bounded degrees and bounded absolute Weil heights in some unbounded region of $\mathbb{C}$ in the following three cases: (i) ω_1 and ω_2 algebraic; (ii) g_2 and g_3 algebraic; (iii) the algebraic points are far from the lattice points.

2020 *Mathematics subject classification*: primary 11J89; secondary 11J82, 33E05.

Keywords and phrases: Weierstrass σ-function, algebraic points, quasi-periods, Weil height, Mahler measure.

1. Introduction

In 2011, Masser [6] proved that for any positive integer d, there exists an effective constant $C > 0$ such that for all $H > e^e$, there are at most $C(\log H/\log\log H)^2$ algebraic numbers $\alpha \in (2, 3)$, with $\zeta(\alpha)$ also algebraic, such that both $\alpha, \zeta(\alpha)$ have degrees at most d and multiplicative heights at most H, where $\zeta(z)$ is the Riemann zeta function. Recall that for an algebraic number α of degree d, its multiplicative height is defined by $H(\alpha) = (M(\alpha))^{1/d}$, where $M(\alpha)$ is its Mahler measure. In the same paper, Masser suggested some possible extensions of his method to other classes of functions. There have been several results already published based on his suggestions of which a recent result by Boxall *et al.* [2] is closely related to our work. To state the main results of [2], we need to introduce some notation.

Let Ω be a lattice in $\mathbb{C}$. Throughout our discussion, we fix a $\mathbb{Z}$-basis $\{\omega_1, \omega_2\}$ of Ω such that $\tau = \omega_2/\omega_1$ lies in the upper half plane $\mathbb{H}$ of $\mathbb{C}$ with $|\tau| \geq 1$ and the real part of τ lies in the interval $[-\frac{1}{2}, \frac{1}{2}]$. Such a basis always exists. Let η_1 and η_2 be the quasi-periods associated to ω_1 and ω_2, respectively. For a pair α, β of algebraic numbers, we put $H(\alpha, \beta) = \max\{H(\alpha), H(\beta)\}$. In [2], the authors proved two results.

This research is partially supported by the MATRICS grant MTR/2021/000476 and the second author is thankful to the SERB, India.

The first [2, Theorem 1.1] is for lattices Ω for which ω_1, ω_2 are algebraic: *if* $\mathrm{Im}(\tau) \le 1.9$ *and* ω_1 *and* ω_2 *are both algebraic, where* $\mathrm{Im}(z)$ *is the imaginary part of* z*, then there exists a constant* $C_1 = C_1(\Omega) > 0$ *such that for all* $d \ge e$ *and* $H \ge e^e$*, there are at most*

$$C_1 d^6 (\log d)(\log H)^2 \log\log H \tag{1.1}$$

algebraic numbers z *such that* $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d, H(z, \sigma_\Omega(z)) \le H$ *and* $z \notin \Omega$. Their second result [2, Theorem 1.2] deals with the case in which the invariants

$$g_2 = 60 \sum_{\omega \in \Omega'} \omega^{-4}, \quad g_3 = 140 \sum_{\omega \in \Omega'} \omega^{-6}$$

are both algebraic (ω_1, ω_2 need not be algebraic) and here also $\mathrm{Im}(\tau) \le 1.9$, where $\Omega' = \Omega \setminus \{0\}$. In this case, instead of the bound (1.1), they give the bound

$$C_2 d^{20} (\log d)^5 (\log H)^2 (\log\log H)$$

for the number of algebraic numbers z such that $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d, H(z, \sigma_\Omega(z)) \le H$. Here, C_2 is a constant depending only on Ω. The importance of the results in [2] is that they count algebraic points of bounded degrees and heights on the entire graph of $\sigma_\Omega(z)$. Earlier, Besson [1] also proved similar results for the number of algebraic points on the graph of the Weierstrass σ-function, but his results are restricted to bounded domains. One of the main ingredients in [2] is the lower bound of $\sigma_\Omega(z)$ in terms of the exponential function e^z and the values of $\sigma_\Omega(z)$ on the fundamental domain P enclosed by the parallelogram with vertices $\frac{1}{2}(\pm\omega_1 \pm \omega_2)$. In this paper, we extend the main results of [2] to a general $\tau \in \mathbb{H}$ under the assumption that $\rho = \eta_2/\eta_1$ is a nonzero real number. With this assumption, we are only able to count the algebraic points of $\sigma_\Omega(z)$ in an unbounded subset $\mathcal{A}_\rho$ of $\mathbb{C}$ defined as follows. First, put

$$\Omega^+ = \{m\omega_1 + n\omega_2 : mn \ge 0\} \quad \text{and} \quad \Omega^- = \{m\omega_1 + n\omega_2 : mn \le 0\},$$

$$\mathcal{A}^+ = \{z \in \mathbb{C} : \text{there exists } z_0 \in P \text{ such that } z - z_0 \in \Omega^+\},$$

and

$$\mathcal{A}^- = \{z \in \mathbb{C} : \text{there exists } z_0 \in P \text{ such that } z - z_0 \in \Omega^-\}.$$

Finally, define

$$\mathcal{A}_\rho = \begin{cases} \mathcal{A}^+ & \text{if } \rho > 0, \\ \mathcal{A}^- & \text{if } \rho < 0. \end{cases}$$

Our first result is an analogue of [2, Theorem 1.1].

THEOREM 1.1. *Let* $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *be a lattice in* $\mathbb{C}$ *such that* ω_1 *and* ω_2 *are both algebraic. Assume that* $\rho = \eta_2/\eta_1$ *is a nonzero real number. Then there exists a constant* $C_3 = C_3(\Omega) > 0$ *such that for all* $d \ge e$ *and* $H \ge e^e$*, there are at most*

$$C_3 d^6 (\log d)(\log H)^2 \log\log H$$

algebraic numbers z *such that* $z \in \mathcal{A}_\rho$, $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d$ *and* $H(z, \sigma_\Omega(z)) \le H$.

Our second result is for g_2, g_3 algebraic, analogous to [2, Theorem 1.2].

THEOREM 1.2. *Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ such that g_2 and g_3 are both algebraic. Assume that $\rho = \eta_2/\eta_1$ is a nonzero real number. Then there exists a constant $C_4 = C_4(\Omega) > 0$ such that for all $d \geq e$ and $H \geq e^e$, there are at most*

$$C_4 d^{20}(\log d)^5(\log H)^2 \log\log H$$

algebraic numbers z such that $z \in \mathcal{A}_\rho$, $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \leq d$ and $H(z, \sigma_\Omega(z)) \leq H$.

We also prove the following more general result with no assumptions on the quantities ω_1, ω_2, g_2 and g_3. In this case, we are only able to count the algebraic points of $\sigma_\Omega(z)$ which are not close to the lattice points.

THEOREM 1.3. *Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$. Assume that $\rho = \eta_2/\eta_1$ is a nonzero real number. Let $0 < \delta < \min\{1, |\omega_1 + \omega_2|/2, |\omega_1 - \omega_2|/2\}$. Then there exists a constant $C_5 = C_5(\delta, \Omega) > 0$ such that for all $d \geq e$ and $H \geq e^e$, there are at most*

$$C_5 d^4(\log d)(\log H)^2 \log\log H$$

algebraic numbers z such that $z \in \mathcal{A}_\rho, [\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \leq d$, $H(z, \sigma_\Omega(z)) \leq H$, dist$(z, \Omega) \geq \delta$, where

$$\mathit{dist}(z, \Omega) = \min_{w \in \Omega} |z - w|.$$

Like [2], in all our results, we count algebraic points of $\sigma_\Omega(z)$ on some unbounded regions of $\mathbb{C}$. Since $\mathbb{R}$ is unbounded, by a result of Heins [4, page 114], we deduce that there are uncountably many lattices Ω with $|\tau| > 1.9$ and $\eta_2/\eta_1 \in \mathbb{R} \setminus \{0\}$. However, despite Theorem 1.2, we do not know a single example of a lattice Ω with algebraic invariants g_2, g_3 for which there exists a nonzero algebraic number α such that $\sigma_\Omega(\alpha)$ is algebraic. It is, in fact, expected that there is no such α. However, the number of algebraic numbers α with degrees at most d and multiplicative heights at most H is at most $4^d H^d$, which means that such α are very rare.

We end this section with three results which are needed for the proof of our theorems.

PROPOSITION 1.4 [1, Théorème 1.2]. *Let $T \geq 1$ be an integer and $R \geq 2$ be a real number. Consider any nonzero polynomial $P(X, Y) \in \mathbb{C}[X, Y]$ of degree at most T in each variable. Then there exists an effective constant $C_6 > 0$ such that the function $P(z, \sigma_\Omega(z))$ has at most*

$$C_6 T\left(R + \sqrt{T}\right)^2 \log(R + T)$$

zeros in the disk $|z| \leq R$.

PROPOSITION 1.5 [6, Proposition 2]. *Fix integers $d \geq 1, T \geq \sqrt{8d}$ and positive real numbers A, Z, M and H with $H \geq 1$. Let f_1, f_2 be two analytic functions on a neighbourhood of the disk $|z| \leq 2Z$. Suppose that $|f_1(z)| \leq M, |f_2(z)| \leq M$ for all $|z| \leq 2Z$. Let $\mathcal{Z} \subseteq \mathbb{C}$ be finite and such that for all $z, z' \in \mathcal{Z}$:*

(1) $|z| \leq Z$;
(2) $|z - z'| \leq 1/A$;

https://doi.org/10.1017/S0004972722001575 Published online by Cambridge University Press

(3) $[\mathbb{Q}(f_1(z), f_2(z)) : \mathbb{Q}] \le d$;
(4) $H(f_1(z), f_2(z)) \le H$.

If

$$(AZ)^T > (4T)^{96d^2/T}(M+1)^{16d}H^{48d^2}, \tag{1.2}$$

then there exists a nonzero polynomial $P \in \mathbb{Z}[X, Y]$ *of total degree at most* T *such that* $P(f_1(z), f_2(z)) = 0$ *for all* $z \in \mathcal{Z}$.

PROPOSITION 1.6 [5, Lemma 7.1]. *For any Weierstrass* σ*-function* $\sigma_\Omega(z)$, *there exists a constant* $C_7 = C_7(\Omega)$ *such that for any* $R \ge 1$,

$$|\sigma_\Omega(z)| \le C_7^{R^2} \quad \textit{for all } |z| \le R.$$

Our paper is organised as follows. In Section 2, we prove an analogue of [2, Proposition 2.1] for $z \in \mathcal{A}_\rho$. Then we prove Theorem 1.1 in Section 3, Theorem 1.2 in Section 4 and Theorem 1.3 in Section 5.

2. Lower bound

Recall that P is the fundamental domain of the lattice $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ enclosed by the parallelogram with vertices $\frac{1}{2}(\pm\omega_1 \pm \omega_2)$.

PROPOSITION 2.1. *Let* $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ *be a lattice in* $\mathbb{C}$ *with* $\rho = \eta_2/\eta_1$ *a nonzero real number. Then there exist positive constants* r *and* C *depending only on* Ω *such that for all* $z \in \mathcal{A}_\rho$ *with* $|z| \ge r$, *there exists* $z_0 \in P$ *with*

$$|\sigma_\Omega(z)| \ge |\sigma_\Omega(z_0)|e^{C|z|^2}.$$

PROOF. As in the proof of [2, Proposition 2.1], we may assume that $\Omega = \mathbb{Z} + \mathbb{Z}\tau$. Let $z \in \mathcal{A}_\rho$ and $z_0 \in P$ be such that $z = z_0 + m + n\tau$ for some integers m and n. Then

$$\sigma_\Omega(z_0 + m + n\tau) = (-1)^{m+n+mn}\sigma_\Omega(z_0)e^{(m\eta_1+n\eta_2)(z_0+m/2+(n/2)\tau)}$$

(see [7, page 255]). Hence,

$$|\sigma_\Omega(z_0 + m + n\tau)| = |\sigma_\Omega(z_0)|e^{R(m,n,z_0)},$$

where $R(m, n, z_0) = \mathrm{Re}[(m\eta_1 + n\eta_2)(z_0 + m/2 + (n/2)\tau)]$. Note that

$$R(m, n, z_0) = \mathrm{Re}\Big(\frac{\eta_1}{2}\Big)m^2 + \mathrm{Re}\Big(\frac{\eta_1\tau + \eta_2}{2}\Big)mn + \mathrm{Re}\Big(\frac{\eta_2\tau}{2}\Big)n^2 + \mathrm{Re}(\eta_1 z_0)m + \mathrm{Re}(\eta_2 z_0)n.$$

Further, from Legendre's relation $\eta_1\tau - \eta_2 = 2\pi i$, we obtain

$$\mathrm{Re}(\eta_1\tau) = \mathrm{Re}(\eta_2) \quad \text{and} \quad \mathrm{Re}\Big(\frac{\eta_1\tau + \eta_2}{2}\Big) = \mathrm{Re}(\eta_2).$$

Moreover,

$$\mathrm{Re}(\eta_2\tau) = \mathrm{Re}(\eta_1\rho\tau) = \rho\,\mathrm{Re}(\eta_1\tau) = \rho\,\mathrm{Re}(\eta_2) = \rho^2\,\mathrm{Re}(\eta_1).$$

Therefore,

$$\operatorname{Re}\Big(\frac{\eta_1}{2}\Big)m^2 + \operatorname{Re}\Big(\frac{\eta_1\tau+\eta_2}{2}\Big)mn + \operatorname{Re}\Big(\frac{\eta_2\tau}{2}\Big)n^2 = \operatorname{Re}(\eta_1)\Big(\frac{m^2}{2} + \rho mn + \frac{\rho^2}{2}n^2\Big)$$
$$= \frac{1}{2}\operatorname{Re}(\eta_1)(n\rho+m)^2.$$

Hence,

$$R(m,n,z_0) = \frac{1}{2}\operatorname{Re}(\eta_1)(n\rho+m)^2 + \operatorname{Re}(\eta_1 z_0)(n\rho+m)$$
$$= (n\rho+m)\Big[\frac{\operatorname{Re}(\eta_1)}{2}(n\rho+m) + \operatorname{Re}(\eta_1 z_0)\Big].$$

(Recall that by Dirichlet's theorem, there are infinitely many pairs of integers (m,n) such that either $\rho + m/n = 0$ or $|\rho + m/n| < 1/n^2$. For this reason, we need to restrict the values of m,n.) Also, since $\eta_2/\eta_1 = \tau - 2\pi i/\eta_1$ and η_2/η_1 is real, we have $\operatorname{Im}(\tau - 2\pi i/\eta_1) = 0$. So,

$$\operatorname{Im}(\tau) = 2\pi \operatorname{Re}\Big(\frac{1}{\eta_1}\Big) = 2\pi\frac{\operatorname{Re}(\eta_1)}{\operatorname{Re}(\eta_1)^2 + \operatorname{Im}(\eta_1)^2}.$$

Since $\operatorname{Im}(\tau) > 0$, we have $\operatorname{Re}(\eta_1) > 0$.

Case 1: $\rho > 0$. Suppose $m > 0, n > 0$. Then there exists a positive constant $r = r(\Omega)$ such that whenever $|z| > r$, we have

$$R(m,n,z_0) \geq c_1(n\rho+m)^2 \geq c_2 \max(|m|,|n|)^2$$

for some positive constants c_1, c_2 depending only on Ω. However,

$$|z_0 + m + n\tau| \leq c_3 \max(|m|,|n|)$$

for some constant $c_3 = c_3(\Omega) > 0$. Hence,

$$|\sigma_\Omega(z_0 + m + n\tau)| \geq |\sigma_\Omega(z_0)|e^{c_4|z_0+m+n\tau|^2} \tag{2.1}$$

for some constant $c_4 = c_4(\Omega) > 0$. If $m < 0, n < 0$, then consider the point $-z_0 - m - n\tau$. Clearly, $-z_0 \in P$. Therefore, from (2.1), we obtain

$$|\sigma_\Omega(-z_0 - m - n\tau)| \geq |\sigma_\Omega(-z_0)|e^{c_4|-z_0-m-n\tau|^2} = |\sigma_\Omega(-z_0)|e^{c_4|z_0+m+n\tau|^2}.$$

However, since $\sigma_\Omega(z)$ is an odd function,

$$|\sigma_\Omega(-z_0 - m - n\tau)| = |\sigma_\Omega(z_0 + m + n\tau)| \quad \text{and} \quad |\sigma_\Omega(-z_0) = |\sigma_\Omega(z_0)|,$$

and the required result follows.

Case 2: $\rho < 0$. The proof of this case is similar to Case 1 and therefore we omit it. □

3. Proof of Theorem 1.1

Recall that $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a lattice in $\mathbb{C}$ and $\rho = \eta_2/\eta_1$ is a nonzero real number. Throughout this section, let r and C denote the constants from Proposition 2.1. In the

https://doi.org/10.1017/S0004972722001575 Published online by Cambridge University Press

following, $c_5, \dots, c_{17}$ denote positive constants depending only on Ω (and independent of d and H). Since

$$\lim_{z \to 0} \frac{\sigma_\Omega(z)}{z} = 1,$$

there exists an ε with $0 < \varepsilon < 1/2$ such that

$$|\log|\sigma_\Omega(z)| - \log|z|| \le 1 \tag{3.1}$$

whenever $|z| < \varepsilon$. We fix such an ε.

LEMMA 3.1. *Let $z \in \mathcal{A}_\rho$ and $z_0 \in P$ be such that $z - z_0 \in \Omega$ with $|z_0| \ge \varepsilon$. Assume that $|z| \ge r$ and both z, $\sigma_\Omega(z)$ are algebraic with $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d$ and $H(z, \sigma_\Omega(z)) \le H$ for some $d \ge e$ and $H \ge e$. Then $|z| \le C_8\sqrt{d \log H}$ for some constant $C_8 = C_8(\Omega) > 0$.*

PROOF. Let $S = \{z \in P : |z| < \varepsilon\}$. Note that $P \setminus S$ is compact. Since $\sigma_\Omega(z)$ is continuous and nonzero in $P \setminus S$, for all $z \in P \setminus S$, we have $|\sigma_\Omega(z)| \ge c_5$. Since $|z_0| \ge \varepsilon$, we have $|\sigma_\Omega(z_0)| \ge c_5$. Now from Proposition 2.1,

$$|\sigma_\Omega(z)| \ge |\sigma_\Omega(z_0)| e^{C|z|^2}.$$

However, since $[\mathbb{Q}(\sigma_\Omega(z)) : \mathbb{Q}] \le d$ and $H(\sigma_\Omega(z)) \le H$, we have $|\sigma_\Omega(z)| \le H^d$. So

$$C|z|^2 \le \log|\sigma_\Omega(z)| - \log|\sigma_\Omega(z_0)| \le d \log H - \log c_5 \le c_6 d \log H,$$

and therefore,

$$|z| \le c_7\sqrt{d \log H}.$$

This completes the proof of the lemma. □

LEMMA 3.2. *Let $z \in \mathcal{A}_\rho$ and $z_0 \in P$ be such that $z - z_0 \in \Omega$ with $|z_0| < \varepsilon$. Assume that $|z| \ge r$ and both z, $\sigma_\Omega(z)$ are algebraic with $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d$ and $H(z, \sigma_\Omega(z)) \le H$ for some $d \ge e$ and $H \ge e$. For all $B > 0$ and for all $N \ge \sqrt{d \log H}$, if $|z| \ge \sqrt{(2 + B)/C}N$, then $\log|z_0| \le -BN^2$.*

PROOF. Let $z \in \mathcal{A}_\rho$ with $|z| \ge r$. Let $z_0 \in P$ be such that $z - z_0 \in \Omega$. From Proposition 2.1,

$$|\sigma_\Omega(z)| \ge |\sigma_\Omega(z_0)| e^{C|z|^2}.$$

Using $|\sigma_\Omega(z)| \le H^d$ and $N \ge \sqrt{d \log H}$, we obtain

$$C|z|^2 + \log|\sigma_\Omega(z_0)| \le \log|\sigma_\Omega(z)| \le d \log H \le N^2. \tag{3.2}$$

For any $B > 0$, put

$$A = \sqrt{\frac{2 + B}{C}}.$$

https://doi.org/10.1017/S0004972722001575 Published online by Cambridge University Press

If $|z| \geq AN$, then from (3.2), we deduce that $CA^2N^2 + \log|\sigma_\Omega(z_0)| \leq N^2$. So, $\log|\sigma_\Omega(z_0)| \leq (1 - CA^2)N^2$. Since $|z_0| \leq \varepsilon$, applying (3.1), we obtain

$$\log|z_0| \leq \log|\sigma_\Omega(z_0)| + 1 \leq (1 - CA^2)N^2 + 1 \leq (2 - CA^2)N^2 = -BN^2.$$

Thus, the result follows. □

LEMMA 3.3. *Assume that ω_1 and ω_2 are both algebraic. Let $z \in \mathcal{A}_\rho$ be such that $|z| \geq r$ and both $z, \sigma_\Omega(z)$ are algebraic with $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \leq d$ and $H(z, \sigma_\Omega(z)) \leq H$, where $d \geq e$ and $H \geq e$. Then there exists a constant $C_9 = C_9(\Omega) > 0$ such that $|z| \leq C_9 d\sqrt{\log H}$.*

PROOF. Suppose $z \in \mathcal{A}_\rho$. Choose $z_0 \in P$ such that $z - z_0 \in \Omega$. If $|z_0| \geq \varepsilon$, then by Lemma 3.1, $|z| \leq c_8\sqrt{d \log H}$. So we assume that $|z_0| < \varepsilon$. Let $\omega = z - z_0$. By [2, Lemma 3.5], if $\omega = k\omega_1 + l\omega_2$, then

$$\max(|k|, |l|) \leq c_9|\omega| \leq c_9(|z| + |z_0|) \leq c_9(|z| + |\omega_1| + |\omega_2|) \leq c_{10}|z|.$$

However, since $H(z) \leq H$ and $[\mathbb{Q}(z) : \mathbb{Q}] \leq d$, we deduce that $|z| \leq H^d$. So $H(k) = |k| \leq c_{10}|z| \leq c_{10}H^d$ and similarly $H(l) \leq c_{10}H^d$. Now, using the inequality

$$H(z_0) \leq 2H(z)H(\omega) \leq 4H(z)H(k)H(\omega_1)H(l)H(\omega_2) \leq c_{11}H^{2d+1}$$

together with the bounds

$$[\mathbb{Q}(z_0) : \mathbb{Q}] = [\mathbb{Q}(z - \omega) : \mathbb{Q}] \leq [\mathbb{Q}(\omega_1, \omega_2) : \mathbb{Q}]d \leq c_{12}d$$

and

$$M(z_0) = M(z_0^{-1}) \geq 1/|z_0|,$$

we deduce that

$$\begin{aligned}\log|z_0| &\geq \log(1/M(z_0)) = -[\mathbb{Q}(z_0) : \mathbb{Q}]\log(H(z_0)) \\ &\geq -c_{12}d((2d + 1)\log H + \log c_{11}) \geq -c_{13}d^2 \log H,\end{aligned}$$

where $M(\alpha)$ is the Mahler measure of α. Applying Lemma 3.2 with $B = c_{13}$ and $N = d\sqrt{\log H}$, we deduce that $|z| \leq c_{14}d\sqrt{\log H}$, where $c_{14} = \sqrt{(2 + c_{13})/C}$. Taking $C_9 = \max(c_8, c_{14})$, we obtain the required result. □

PROOF OF THEOREM 1.1. Define

$$\mathcal{Z}_1 = \{z \in \mathcal{A}_\rho : [\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \leq d,\ H(z, \sigma_\Omega(z)) \leq H\}.$$

Put

$$Z = 4C_9 d\sqrt{\log H}, \quad A = 2/Z.$$

From Lemma 3.3, $|z| \leq C_9 d\sqrt{\log H} \leq Z$ and $|z - z'| \leq 1/A$ for all $z, z' \in \mathcal{Z}_1$. However, from Proposition 1.6, there exists a constant $c_{15} \geq 1$ such that for all $z \in \mathcal{A}_\rho$,

$$|\sigma_\Omega(z)| \leq c_{15}^{|z|^2}.$$

https://doi.org/10.1017/S0004972722001575 Published online by Cambridge University Press

Put $M = c_{15}^{Z^2}$. Then, $|z| \le M$ and $|\sigma_\Omega(z)| \le M$ for all $|z| \le 2Z$. With these choices of A, Z and M, the conditions of Proposition 1.5 are satisfied. If we take $T = c_{16} d^3 \log H$ for a sufficiently large $c_{16} > 0$, then (1.2) is satisfied. Hence by Proposition 1.5, there exists a nonzero polynomial $P \in \mathbb{Z}[X, Y]$ of total degree at most T such that $P(z, \sigma_\Omega(z)) = 0$ for all $z \in \mathcal{Z}_1$. Finally taking $R = C_9 d\sqrt{\log H}$ and $T = c_{16} d^3 \log H$ in Proposition 1.4, we deduce that there are at most

$$c_{17} d^6 (\log d)(\log H)^2 \log\log H$$

zeros of $P(z, \sigma_\Omega(z))$ in the region $|z| \le R$. Hence, the number of elements in the set $\mathcal{Z}_1$ is at most $c_{17} d^6 (\log d)(\log H)^2 \log\log H$. This completes the proof. □

4. Proof of Theorem 1.2

Throughout this section, let Ω denote a lattice in $\mathbb{C}$ with algebraic invariants g_2, g_3. In this section, $c_{18}, \ldots, c_{27}$ denote various constants which depend only on Ω. We first state the following transcendence measure for the nonzero elements of Ω, due to David and Hirata-Kohno.

LEMMA 4.1 [3, Theorem 1.6]. *Let Ω be a lattice in $\mathbb{C}$. Let $d \ge 1$ and $H \ge 3$ be real numbers. Let α be an algebraic number with $[\mathbb{Q}(\alpha) : \mathbb{Q}] \le d$ and $H(\alpha) \le H$. Then there exists a constant $C_{10} = C_{10}(\Omega) > 0$ such that*

$$\log|\alpha - \omega| \ge -C_{10} d^4 (\log d)^2 (\log H) |\omega|^2 (1 + \max\{0, \log|\omega|\})^3$$

for all $\omega \in \Omega \setminus \{0\}$.

The following is an analogue of [2, Proposition 4.2].

LEMMA 4.2. *Assume that $\rho = \eta_2/\eta_1$ is a nonzero real number. Let $d \ge 1, H \ge 3$ be real numbers. There exist positive constants C_{11}, C_{12} depending only on Ω such that the following holds. If $z, z' \in \mathcal{A}_\rho$ with $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d, [\mathbb{Q}(z', \sigma_\Omega(z')) : \mathbb{Q}] \le d$, $H(z, \sigma_\Omega(z)) \le H$ and $H(z', \sigma_\Omega(z')) \le H$, then*

$$\min\{|z|, |z'|\} \le C_{11} \sqrt{d^9 (\log d)^2 \log H}$$

or there exists $\omega, \omega' \in \Omega$ such that

$$\max\{\log|z - \omega|, \log|z' - \omega'|\} \le -C_{12} d^9 (\log d)^2 \log H$$

with $z'/z = \omega'/\omega \in \mathbb{Q}$.

PROOF. The proof follows the same line of argument as in [2, Proposition 4.2], so we omit it here. □

LEMMA 4.3. *Assume that $\rho = \eta_2/\eta_1$ is a nonzero real number and g_2, g_3 are both algebraic. Let $d \ge 1, H \ge 3$ be real numbers. Let r be from Proposition 2.1 and C_{11} be from Lemma 4.2. Consider the set*

https://doi.org/10.1017/S0004972722001575 Published online by Cambridge University Press

$$S = \Big\{ z \in \mathcal{A}_\rho : [\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d, H(z, \sigma_\Omega(z)) \le H,$$
$$\textit{and } |z| > \max\Big\{ r, C_{11}\sqrt{d^9 (\log d)^2 \log H} \Big\} \Big\}.$$

Then there exists a positive constant $C_{13} = C_{13}(\Omega)$ such that the number of elements of S is at most

$$C_{13}\sqrt{d^5 (\log d)^2 (\log H)(1 + d \log H)^3}.$$

PROOF. We follow the strategy given in [2]. Suppose $z, z' \in S$. Then,

$$\min\{|z|, |z'|\} > C_{11}\sqrt{d^9 (\log d)^2 \log H}.$$

So by Lemma 4.2, there exist $\omega, \omega' \in \Omega$ such that

$$\max\{\log|z - \omega|, \log|z' - \omega'|\} \le -C_{12} d^9 (\log d)^2 \log H$$

with $z'/z = \omega'/\omega \in \mathbb{Q}$. This implies z, z' are not periods of Ω.

Put $\omega'/\omega = q$. Let $\omega^* \in \Omega \setminus \{0\}$ be of minimum modulus on the line joining 0 and ω. Then $\omega = m\omega^*$ for some nonzero integer m. Let $z^* = z/m$. Note that $z^* \in \mathcal{A}_\rho$. Also, since ω' and ω lie on the same line, $\omega' = m_1\omega^*$ for some $m_1 \in \mathbb{Z}$. So $qm\omega^* = m_1\omega^*$. Hence, $qm = m_1$. Now, $z' = qz = qmz^* = m_1 z^*$. So z' is an integer multiple of z^*. Thus, if we show that

$$n^2 \le c_{18} d^5 (\log d)^2 (\log H)(1 + d \log H)^3,$$

whenever $nz^* \in S$ for some $n \in \mathbb{Z}$, we are done. Indeed, just now we have seen that if $z' \in S$, then $z' = nz^*$ for some $n \in \mathbb{N}$. Accordingly, we assume $nz^* \in S$ for some $n \in \mathbb{Z}$. Put $nz^* = z''$. Thus, $z'' \in \mathcal{A}_\rho$. Let $z_0 \in P$ be such that $z'' - z_0 = \omega'' \in \Omega$. Since both z'' and z belong to S, we have $z''/z = \omega''/\omega$. Hence, $nz^*/mz^* = \omega''/m\omega^*$, or equivalently $\omega'' = n\omega^*$. Since $z'' \in \mathcal{A}_\rho$, from Proposition 2.1, we obtain

$$\log|\sigma_\Omega(z'')| \ge \log|\sigma_\Omega(z_0)| + C|z''|^2.$$

Note that $z_0 \neq 0$. Therefore, $|\sigma_\Omega(z_0)/z_0| > e^{c_{18}}$. Hence,

$$\begin{aligned} \log|\sigma_\Omega(z'')| &\ge \log|z_0| + c_{19} + C|z''|^2 \\ &= \log|z'' - \omega''| + c_{19} + C|z''|^2 \\ &\ge \log|nz^* - n\omega^*| + c_{19} + Cn^2|z^*|^2 \\ &\ge \log|z^* - \omega^*| + c_{19} + Cn^2|z^*|^2. \end{aligned}$$

Write $\omega = k\omega_1 + l\omega_2$ with integers k, l. As we have seen earlier in the proof of Lemma 3.3, $\max(|k|, |l|) \le c_{20}|z| \le c_{20}H^d$. Further, since $\omega = m\omega^*$, we see that m divides both k, l. We deduce that $|m| \le c_{20}H^d$. So,

$$\log H(z^*) = \log H(z/m) \le c_{21} d \log H.$$

https://doi.org/10.1017/S0004972722001575 Published online by Cambridge University Press

Since $z^* = z/m$ is algebraic, by Lemma 4.1, we deduce that

$$\begin{aligned}\log|z^* - \omega^*| \geq & - c_{22} d^4 (\log d)^2 d (\log H) |\omega^*|^2 (1 + \max\{0, \log|\omega^*|\})^3 \\ \geq & - c_{23} d^5 (\log d)^2 (\log H) |z^*|^2 (1 + \max\{0, \log|z^*|\})^3.\end{aligned}$$

So,

$$\begin{aligned}-c_{23} d^5 (\log d)^2 (\log H) |z^*|^2 (1 + \max\{0, \log|z^*|\})^3 + c_{19} + Cn^2 |z^*|^2 \leq & \log|\sigma_\Omega(z'')| \\ \leq & \, d \log H.\end{aligned}$$

In other words,

$$n^2 \leq c_{24} d^5 (\log d)^2 (\log H)(1 + d \log H)^3.$$

This completes the proof of the lemma. □

Proof of Theorem 1.2. To prove Theorem 1.2, by Lemma 4.3, we only need to count the number of elements in the set

$$\mathcal{Z}_2 = \Big\{ z \in \mathcal{A}_\rho : [\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \leq d,\ H(z, \sigma_\Omega(z)) \leq H, \\ \text{and } |z| \leq C_{11} \sqrt{d^9 (\log d)^2 \log H} \Big\}.$$

Put

$$Z = 4C_{11} \sqrt{d^9 (\log d)^2 \log H}, \quad A = 2/Z.$$

Then $|z| \leq Z$ and $|z - z'| \leq 1/A$ for all $z, z' \in \mathcal{Z}_2$. By Proposition 1.6, for all $z \in \mathcal{A}_\rho$,

$$|\sigma_\Omega(z)| \leq c_{25}^{|z|^2}.$$

Put $M = c_{25}^{Z^2}$. Then, $|z| \leq M$ and $|\sigma_\Omega(z)| \leq M$ for all $|z| \leq 2Z$. With these choices of A, Z and M, the conditions of Proposition 1.5 are satisfied. If we take $T = c_{26} d^{10} (\log d)^2 \log H$ for a sufficiently large $c_{26} > 0$, then (1.2) is satisfied. Thus, by Proposition 1.5, we deduce that there exists a nonzero polynomial $P \in \mathbb{Z}[X, Y]$ of total degree at most T such that $P(z, \sigma_\Omega(z)) = 0$ for all $z \in \mathcal{Z}_2$.

Finally taking $R = C_{11} \sqrt{d^9 (\log d)^2 \log H}$ and $T = c_{26} d^{10} (\log d)^2 \log H$ in Proposition 1.4, we deduce that there are at most

$$c_{27} d^{20} (\log d)^5 (\log H)^2 \log\log H$$

zeros of $P(z, \sigma_\Omega(z))$ in the region $|z| \leq R$. Hence, the number of elements in the set $\mathcal{Z}_2$ is at most $c_{27} d^{20} (\log d)^5 (\log H)^2 \log\log H$. Since

$$\sqrt{c_{24} d^5 (\log d)^2 (\log H)(1 + d \log H)^3} \leq c_{27} d^{20} (\log d)^5 (\log H)^2 \log\log H,$$

from Lemma 4.3, there are at most

$$2c_{27} d^{20} (\log d)^5 (\log H)^2 \log\log H$$

https://doi.org/10.1017/S0004972722001575 Published online by Cambridge University Press

algebraic numbers z such that $z \in \mathcal{A}_\rho$, $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d$ and $H(z, \sigma_\Omega(z)) \le H$. This completes the proof of the theorem. □

5. Proof of Theorem 1.3

Throughout this section, let δ, r denote the constants from the statements of Theorem 1.3 and Proposition 2.1.

LEMMA 5.1. *Let $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$ with $\rho = \eta_2/\eta_1$ a nonzero real number. Let $z \in \mathcal{A}_\rho$ be such that $|z| \ge r$ and $\mathrm{dist}(z, \Omega) \ge \delta$. For any $d \ge e$ and $H \ge e$, if both z and $\sigma_\Omega(z)$ are algebraic with $[\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d$ and $H(z, \sigma_\Omega(z)) \le H$, then there exists a constant $C_{14} = C_{14}(\delta, \Omega)$ such that $|z| \le C_{14}\sqrt{d \log H}$.*

PROOF. Let $z \in \mathcal{A}_\rho$ and $z_0 \in P$ be such that $z - z_0 = m_1\omega_1 + n_1\omega_2 \in \Omega$. Since $\mathrm{dist}(z, \Omega) \ge \delta$, we have $|z_0| = |z - m_1\omega_1 - n_1\omega_2| \ge \delta$. Let ε denote the constant from Lemma 3.1.

Case 1: $\delta \ge \varepsilon$. Since $|z_0| \ge \delta$, we have $|z_0| \ge \varepsilon$. So by Lemma 3.1, there exists a constant $c_{28} = c_{28}(\Omega)$ such that $|z| \le c_{28}\sqrt{d \log H}$. Hence, the lemma is proved.

Case 2: $\delta < \varepsilon$. Suppose $\delta \le \mathrm{dist}(z, \Omega) < \varepsilon$, so $\delta \le |z_0| < \varepsilon$. Hence,

$$|z_0| \ge e^{-\log 1/\delta} \ge e^{-d \log H \log 1/\delta} = e^{-BN^2},$$

where $B = \log 1/\delta$ and $N = \sqrt{d \log H}$. Since $|z_0| < \varepsilon$, applying Lemma 3.2, we obtain

$$|z| \le \sqrt{\frac{2 + \log 1/\delta}{C}}\sqrt{d \log H},$$

where C is as in Lemma 3.2.

Now suppose $\mathrm{dist}(z, \Omega) \ge \varepsilon$, so $|z_0| \ge \varepsilon$. As in Case 1, $|z| \le c_{28}\sqrt{d \log H}$. Taking $C_{14} = \max\big(c_{28}, \sqrt{(2 + \log 1/\delta)/C}\big)$, we get the required result. □

PROOF OF THEOREM 1.3. Define

$$\mathcal{Z}_3 = \{z \in \mathcal{A}_\rho : \mathrm{dist}(z, \Omega) \ge \delta, [\mathbb{Q}(z, \sigma_\Omega(z)) : \mathbb{Q}] \le d \text{ and } H(z, \sigma_\Omega(z)) \le H\}.$$

Put

$$Z = 4C_{14}\sqrt{d \log H}, \quad A = 2/Z.$$

From Lemma 5.1, $|z| \le C_{14}\sqrt{d \log H} \le Z$ and $|z - z'| \le 1/A$ for all $z, z' \in \mathcal{Z}_3$. By Proposition 1.6,

$$|\sigma_\Omega(z)| \le c_{29}^{|z|^2}.$$

Put $M = c_{29}^{Z^2}$. Then, $|z| \le M$ and $|\sigma_\Omega(z)| \le M$ for all $|z| \le 2Z$. With these choices of A, Z and M, the conditions of Proposition 1.5 are satisfied. If we take $T = c_{30}d^2 \log H$ for some sufficiently large constant $c_{30} = c_{30}(\delta, \Omega)$, then (1.2) is satisfied. Thus, by

https://doi.org/10.1017/S0004972722001575 Published online by Cambridge University Press

applying Proposition 1.5, there exists a nonzero polynomial $P \in \mathbb{Z}[X, Y]$ of total degree at most T such that $P(z, \sigma_\Omega(z)) = 0$ for all $z \in \mathcal{Z}_3$.

Finally taking $R = C_{14}\sqrt{d \log H}$ and $T = c_{30}d^2 \log H$ in Proposition 1.4, we deduce that there are at most

$$c_{31}d^4(\log d)(\log H)^2 \log\log H$$

zeros of $P(z, \sigma_\Omega(z))$ in the region $|z| \leq R$ for some constant $c_{31} = c_{31}(\delta, \Omega) > 0$. Hence, the number of elements in the set $\mathcal{Z}_3$ is at most $c_{31}d^4(\log d)(\log H)^2 \log\log H$. This completes the proof of the theorem. □

References

[1] E. Besson, 'Points algébriques de la fonction sigma de Weierstrass', Preprint, 2015. Available online at https://rivoal.perso.math.cnrs.fr/sigma.pdf.

[2] G. Boxall, T. Chalebgwa and G. Jones, 'On algebraic values of Weierstrass σ-functions', *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* **32**(4) (2021), 819–833.

[3] S. David and N. Hirata-Kohno, 'Linear forms in elliptic logarithms', *J. reine angew. Math.* **628** (2009), 37–89.

[4] M. Heins, 'On the pseudo-periods of the Weierstrass zeta functions. II', *Nagoya Math. J.* **30** (1967), 113–119.

[5] D. Masser, *Elliptic Functions and Transcendence*, Lecture Notes in Mathematics, 437 (Springer, Berlin–Heidelberg, 1975).

[6] D. Masser, 'Rational values of the Riemann zeta function', *J. Number Theory* **131**(11) (2011), 2037–2046.

[7] D. Masser, *Auxiliary Polynomials in Number Theory*, Cambridge Tracts in Mathematics, 207 (Cambridge University Press, Cambridge, 2016).

GOREKH PRASAD SENA, National Institute of Science Education and Research, Bhubaneswar, An OCC of Homi Bhabha National Institute,
Khurda 752050, Odisha, India
e-mail: gorekh.sena@niser.ac.in

K. SENTHIL KUMAR, National Institute of Science Education and Research, Bhubaneswar, An OCC of Homi Bhabha National Institute,
Khurda 752050, Odisha, India
e-mail: senthil@niser.ac.in

https://doi.org/10.1017/S0004972722001575 Published online by Cambridge University Press