CONCAVITY PROPERTIES FOR CERTAIN LINEAR COMBINATIONS OF STIRLING NUMBERS

J. C. AHUJA

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In the notation of Riordan ([5], p. 33), the Stirling numbers, $s(n,k)$ and $S(n,k)$, of the first and second kind respectively are defined by the relations

$$(x)_n = \sum_{k=1}^{n} s(n,k)x^k \tag{1}$$

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k \tag{2}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ is the factorial power function. They have been used by Jordan ([3], p. 184) to define the numbers $C(m,k)$ and $D(m,k)$, as linear combinations of $s(n,k)$ and $S(n,k)$ respectively, given by

$$C(m,k) = \sum_{j=m+1}^{2m-k+1} (-1)^{j+k} \binom{2m-k}{j} s(j,j-m) \tag{3}$$

where $C(m,k) = 0$ for $k > m-1$, $C(1,0) = -1$, $C(m,m-1) = (-1)^m m!$ and $C(m,0) = (-1)^m\ 1.3.5\cdots(2m-1)$, and

$$D(m,k) = \sum_{j=m+1}^{2m-k+1} (-1)^{k+j} \binom{2m-k}{j} S(j,j-m) \tag{4}$$

where $D(m,k) = 0$ for $k > m-1$, $D(1,0) = D(m,m-1) = 1$, and $D(m,0) = 1.3.5\cdots(2m-1)$.

As indicated by Jordan [3], the numbers $C(m,k)$ and $D(m,k)$ satisfy the following partial difference equations:

$$C(m+1,k) = -(2m-k+1)[C(m,k-1) + C(m,k)] \tag{5}$$

and

$$D(m+1,k) = (m-k+1)D(m,k-1) + (2m-k+1)D(m,k). \tag{6}$$

Harper [2] has proved the unimodality conjecture for the numbers $S(n,k)$, while Lieb [4] has shown that the Stirling numbers $s(n,k)$ and $S(n,k)$ are both strong logarithmic concave (SLC) functions of k for fixed n, that is, they satisfy the inequalities:

$$[s(n,k)]^2 > s(n,k+1)s(n,k-1) \tag{7}$$

and

$$[S(n,k)]^2 > S(n,k+1)S(n,k-1) \tag{8}$$

for $k=2,3,\cdots,n-1$. To prove the SLC property of $s(n,k)$ and $S(n,k)$, Lieb [4] used the following result of Newton's inequality given in Hardy, Littlewood, and Polya ([1], p. 52): If the polynomial $P(x)=\sum_{k=1}^{n} c_k x^k$ has only real roots, then

$$c_k^2 > c_{k+1}c_{k-1} \tag{9}$$

for $k=2,3,\cdots,n-1$.

The purpose of this paper is to show that the numbers $C(m,k)$ and $D(m,k)$ defined above are also SLC function of k for fixed m. To do this, we need the following two lemmas:

LEMMA 1. *If $P_{m-1}(x)=\sum_{k=0}^{m-1} C(m,k)x^k$, then the $(m-1)$ roots of $P_{m-1}(x)$ are real, negative, and distinct for all $m=1,2,\cdots$.*

PROOF. It can be easily verified that $P_{m-1}(x)$, using (5), may be written in the form

$$\begin{aligned} P_{m-1}(x) &= \sum_{k=0}^{m-1} C(m,k)x^k \\ &= -\sum_{k=0}^{m-1} (2m-k-1)[C(m-1,k-1)+C(m-1,k)]x^k \\ &= -[(2m-2)x+(2m-1)]P_{m-2}(x)+x(x+1)dP_{m-2}(x)/dx. \end{aligned} \tag{10}$$

By induction $P_0(x)=-1$, $P_1(x)=2x+3$, and $P_2(x)=-(6x^2+20x+15)$, so that the statement is true for $m=1,2$, and 3. For $m>3$, assume that $P_{m-2}(x)$ has $m-2$ real, negative, and distinct roots.

If we define

$$Q_m(x)=[(x+1)/x^{2m+1}]P_{m-1}(x) \tag{11}$$

then the roots of $P_{m-1}(x)$ are among those of $Q_m(x)$, and the identity (10) for $P_{m-1}(x)$ gives

$$Q_m(x)=[(x+1)/x]dQ_{m-1}(x)/dx. \tag{12}$$

Using (10), it can be easily verified that $P_{m-1}(-1) = (-1)^m$, which shows that $Q_{m-1}(x)$ has $m-1$ real, negative, and distinct roots. $Q_{m-1}(x)$ also has $-\infty$ as a root, and by Rolle's theorem between any two roots of $Q_{m-1}(x)$, $dQ_{m-1}(x)/dx$ will have a root. This proves the result by induction.

LEMMA 2. *If* $H_{m-1}(x) = \sum_{k=0}^{m-1} D(m,k)x^k$, *then the* $(m-1)$ *roots of* $H_{m-1}(x)$ *are real, negative, and distinct for all* $m = 1, 2, \cdots$.

PROOF. The proof is similar to that of Lemma 1 except that, in this case, we can write $H_{m-1}(x)$, using (6), in the form

$$H_{m-1}(x) = [(m-1)x + (2m-1)]H_{m-2}(x) - x(x+1)dH_{m-2}(x)/dx \tag{13}$$

and if we define

$$T_m(x) = [(x+1)^{m+1}/x^{2m+1}]H_{m-1}(x), \tag{14}$$

then the identity (13) for $H_{m-1}(x)$ becomes

$$T_m(x) = [-(x+1)^2/x]dT_{m-1}(x)/dx. \tag{15}$$

As before, using (13), we find that $H_{m-1}(-1) = m!$. The rest of the argument is the same as in Lemma 1.

It is now easily seen that Lemma 1 and Lemma 2 together with the inequality (9) provide us the following *SLC* property of the numbers $C(m,k)$ and $D(m,k)$:

THEOREM. *For* $m \geqq 3$, *and* $k = 1, 2, \cdots, m-2$, *the numbers* $C(m,k)$ *and* $D(m,k)$ *defined by* (3) *and* (4) *satisfy the inequalities*

$$\text{(i)} \qquad [C(m,k)]^2 > C(m,k+1)C(m,k-1)$$

and

$$\text{(ii)} \qquad [D(m,k)]^2 > D(m,k+1)D(m,k-1)$$

respectively.

References

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Portland State University
Portland, Oregon, U. S. A.

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