A CONVERGENCE THEOREM FOR CERTAIN RIEMANN SUMS

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For a Riemann integrable function f on the interval $[0, 1]$, let

$$I = \int_0^1 f$$

and consider the Riemann sums

$$R_n(f;a) = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k-a}{n}\right), \qquad 0 \leq a \leq 1 .$$

THEOREM. *If* f *is absolutely continuous on* $[0, 1]$, *then*

$$R_n(f;\tfrac{1}{2}) - I = o\left(\frac{1}{n}\right) . \tag{1}$$

This theorem gives some asymptotic information for certain finite sums. For example, taking $f(t) = t^b$, $b \geq 0$, we obtain

$$\frac{1}{n^b} \sum_{k=1}^{n} (k - \tfrac{1}{2})^b - \frac{n}{1+b} \to 0 ,$$

and if we consider $f(t) = \sin \pi t$, we get

$$\sum_{k=1}^{n} \sin \frac{(k - \frac{1}{2})\pi}{n} - \frac{2n}{\pi} \to 0 .$$

However, it is interesting to note that the theorem no longer holds if we replace $R_n(f;\frac{1}{2})$ by $R_n(f;a)$ with $a \neq \frac{1}{2}$, $0 \leq a \leq 1$. This is obvious by taking $f(t) = t$, which gives

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$$R_n(f;a) - I = (\tfrac{1}{2} - a)/n .$$

On the other hand, we have the following result.

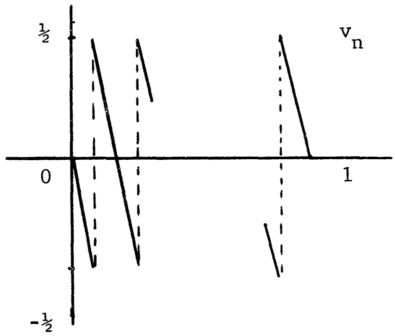
COROLLARY. *If* f *is absolutely continuous on* $[0,1]$ *such that* $f(0) = f(1)$, *then*

$$R_n(f;a) - I = o(\tfrac{1}{n}), \qquad 0 \le a \le 1 . \tag{2}$$

Proof of the above results. Consider the step functions

$$s_n(t) = \begin{cases} 0, & 0 \le t < \tfrac{1}{2}n , \\ k, & (k - \tfrac{1}{2})/n \le t < (k + \tfrac{1}{2})/n, \quad k = 1, \ldots, n-1 , \\ n, & (n - \tfrac{1}{2})/n \le t \le 1 ; \end{cases}$$

and set $v_n(t) = s_n(t) - nt$. Note that $v_n(0) = v_n(1) = 0$, v_n lies between $-\tfrac{1}{2}$ and $\tfrac{1}{2}$, and is linear with an exception of n unit jumps at $(k - \tfrac{1}{2})/n$, $k = 1, \ldots, n$.



By a proof similar to that of the Riemann-Lebesgue Theorem, it can be seen that if g is a Lebesgue integrable function on $[0,1]$, then $\int_0^1 g\, v_n \to 0$. Now, let $h = f - I$, where f is the given absolutely continuous function.

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Then

$$\int_0^1 h = \int_0^1 f - I = 0 ;$$

and being absolutely continuous, h is an indefinite integral of some Lebesgue integrable function g on $[0, 1]$. Hence,

$$\int_0^1 h \, dv_n = [h \, v_n]_0^1 - \int_0^1 g \, v_n = - \int_0^1 g \, v_n \to 0 .$$

On the other hand,

$$\int_0^1 h \, dv_n = \int_0^1 h \, ds_n - n \int_0^1 h = \sum_{k=1}^{n} h\left(\frac{k - \frac{1}{2}}{n}\right)$$

$$= \sum_{k=1}^{n} f\left(\frac{k - \frac{1}{2}}{n}\right) - nI = n \{ R_n(f ; \tfrac{1}{2}) - I \} .$$

Combining these two relations, we obtain (1). To prove the corollary, we observe that if $f(0) = f(1)$, then $h(0) = h(1)$. Extend v_n periodically to the real line and let

$$\tilde{v}_n(t) = v_n\left(t + \frac{a - \frac{1}{2}}{n}\right) .$$

Note that $\tilde{v}_n(0) = \tilde{v}_n(1)$. Hence, using $\tilde{v}_n$ in place of v_n in the above proof, we obtain (2).

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