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ISOMORPHISMS BETWEEN SPACES OF VECTOR-VALUED CONTINUOUS FUNCTIONS

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1. Introduction

A theorem due to Milutin **[12]** (see also **[13]**) asserts that for any two uncountable compact metric spaces Ω_1 and Ω_2, the spaces of continuous real-valued functions $C(\Omega_1)$ and $C(\Omega_2)$ are linearly isomorphic. It immediately follows from consideration of tensor products that if X is any Banach space then $C(\Omega_1; X)$ and $C(\Omega_2; X)$ are isomorphic.

The purpose of this paper is to show that this conclusion is false for general non-locally convex quasi-Banach spaces. In fact, it fails in a quite strong manner. We shall show that if X is a quasi-Banach space containing no copy of c_0 which is isomorphic to a closed subspace of a space with a basis and $C(I;X)\cong C(\Delta; X)$, (where I is the unit interval and Δ is the Cantor set) then we can conclude that X is locally convex.

The proof requires building some machinery concerning operators on spaces of continuous functions. The locally convex analogues of these results are to be found in the work of Batt and Berg **[1]** or Brooks and Lewis **[2]**.

Operators on spaces $C(\Omega)$ into general non-locally convex spaces have been treated in an important paper of Thomas **[19]**. Unfortunately this paper has not been published. Thomas's main result can be expressed in the language of this paper as follows

Theorem. (Thomas) *Let X be a quasi-Banach space and let Ω be a compact Hausdorff space. If $T\colon C(\Omega)\to X$ is an exhaustive linear operator then there is a regular X-valued measure μ on the Borel sets of Ω such that*

$$T\phi=\int \phi\, d\mu \qquad \phi\in C(\Omega).$$

Here "T is exhaustive" means that if ϕ_n is a sequence bounded in $C(\Omega)$ with disjoint supports then $T\phi_n\to 0$. This may be regarded as a generalisation of the Riesz Representation Theorem. We shall refer to it as the Riesz–Thomas theorem.

We derive here a similar representation for operators on spaces $C(\Omega; X)$ (Theorem 4.3). We give the proof in some detail partly because Thomas's theorem is not easily accessible. Schuchat **[18]** obtained some results in this direction.

Our other main weapon is a version of Liapunoff's theorem for non-locally convex vector-valued measures (Theorem 3.1). The absence of local convexity requires some

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tricks in the proof not usually necessary for the corresponding results in locally convex spaces (cf. [5]).

Throughout this paper a quasi-Banach space will be a *real* vector space equipped with a complete quasi-norm topology. (See [10], p. 159). The quasi-norm will always be assumed p-subadditive for some p, $0<p\leqq 1$, i.e.

$$\|x_1+x_2\|^p \leqq \|x_1\|^p + \|x_2\|^p \qquad x_1, x_2 \in X.$$

If X is a quasi-Banach space and Ω is a compact Hausdorff space then $C(\Omega; X)$ is the space of continuous X-valued functions quasi-normed by

$$\|f\| = \max_{\omega\in\Omega} \|f(\omega)\|$$

If $x \in X$ and $\phi \in C(\Omega)$ then $\phi \otimes x \in C(\Omega; X)$ is given by

$$\phi \otimes x(\omega) = \phi(\omega)x.$$

If $f \in C(\Omega; X)$ and $\phi \in C(\Omega)$ then $\phi f \in C(\Omega; X)$ is given by

$$\phi f(\omega) = \phi(\omega) f(\omega).$$

We shall have need of the concept of a *compactly determined* quasi-Banach space. We say X is compactly determined (or has a compactly determined topology) if it may be equivalent quasi-normed so that

$$\|x\| = \sup(\|Kx\| : K \in \mathscr{K})$$

where $\mathscr{K}$ is some collection of compact operators into a quasi-Banach space Z. The easiest examples are spaces embeddable in a space with a basis. However there are examples of such spaces with trivial duals. We shall assume that the quasi-norm on a compactly determined space satisfies the above criterion.

2. Submeasures

Let Ω be an abstract set and let Σ be a σ-algebra of subsets of Ω. We recall that a (continuous) *submeasure* ρ on (Ω, Σ) is a map $\rho: \Sigma \to \mathbb{R}$ satisfying

$$\rho(A) \leqq \rho(A \cup B) \leqq \rho(A) + \rho(B) \qquad A, B \in \Sigma \tag{2.0.1}$$

$$\rho(\emptyset) = 0 \tag{2.0.2}$$

$$\text{If} \quad A_n \downarrow \emptyset \quad \text{then} \quad \rho(A_n) \downarrow 0. \tag{2.0.3}$$

It is an unsolved problem posed by Maharam [10] whether every submeasure has an equivalent measure, i.e. given a submeasure ρ does there exist a positive measure

$\lambda:\Sigma\to\mathbb{R}$ so that $\rho(A)=0$ if and only if $\lambda(A)=0$. See Christensen and Herer [4], Christensen [3] and Popov [15].

A set $A\in\Sigma$ is a *ρ-atom* if $\rho(A)>0$ and whenever $B\in\Sigma$ with $B\subset A$ then $\rho(B)=0$ or $\rho(A\backslash B)=0$. The set of atoms is at most countable. We say ρ is *diffuse* if there are no ρ-atoms. The following lemma will be required later.

Lemma 2.1. *Let $\rho:\Sigma\to\mathbb{R}$ be a non-zero diffuse submeasure. Then there exists $\varepsilon>0$ and $\{A_n: n=1,2,\ldots\}\subset\Sigma$ so that $\rho(A_i\,\Delta A_j)\geqq\varepsilon$ whenever $i\neq j$.*

Proof. Let N be the σ-ideal of ρ-null sets and let Σ/N be the quotient Boolean algebra. It is easy to verify that Σ/N is a complete metric space when equipped with the metric

$$d([A],[B])=\rho(A\,\Delta B)\qquad A,B\in\Sigma.$$

If the conclusion of the Lemma fails, then $(\Sigma/N,d)$ becomes a compact abelian topological group under the group operation of symmetric difference. As Σ/N is a 2-group this implies that there is a continuous character $\chi:\Sigma/N\to\mathbb{Z}_2$ with $\chi([\Omega])=1$, where $\mathbb{Z}_2=\{0,1\}$ is the discrete group of order 2. Hence there is some $\delta>0$ so that $\rho(A)<\delta$ implies $\chi([A])=0$.

Suppose $(B_i: i\in I)$ is a maximal collection of disjoint sets in Σ so that $0<\rho(B_i)<\delta$. Then I is countable (this follows from condition (2.0.3)). Let $B=\cup(B_i: i\in I)$; then $\chi([B])=0$. Let $A=\Omega\backslash B$; then $\chi([A])=1$ and so $\rho(A)\geqq\delta$. Since ρ is diffuse we may find a descending sequence $A_n\in\Sigma$ with $A_1=A$, $\rho(A_n)>0$ and $\rho(A_n\backslash A_{n+1})>0$, for $n=1,2,\ldots$. Clearly $\rho(A_n\backslash A_{n+1})\geqq\delta$, by the maximality of $\{B_i: i\in I\}$. However if $A_\infty=\cap A_n$, then $\rho(A_n\backslash A_\infty)\downarrow 0$ and we have a contradiction.

Now suppose Ω is a compact Hausdorff space and let $\mathscr{B}$ denote the σ-algebra of Borel subsets of Ω. We shall say that a submeasure $\rho:\mathscr{B}\to\mathbb{R}$ is *regular* if given $B\in\mathscr{B}$ and $\varepsilon>0$ there exists a compact $K\subset B$ and an open $V\supset B$ so that $\rho(V\backslash K)\leqq\varepsilon$. In this setting it can be shown that if $A\in\mathscr{B}$ is a ρ-atom then there exists $\omega\in A$ with $\rho\{\omega\}=\rho(A)$ (cf. [6]).

We shall require the following lemma; a very similar result is given by Dobrakov ([6] p. 29).

Lemma 2.2. *Let Ω be a compact Hausdorff space and let $\mathscr{V}$ be the collection of open subsets of Ω. Let $\theta:\mathscr{V}\to\mathbb{R}$ be a map satisfying:*

$$\theta(\emptyset)=0 \tag{2.2.1}$$

$$\theta(V_1)\leqq\theta(V_2)\qquad V_1\subset V_2 \tag{2.2.2}$$

$$\theta(V_1\cup V_2)\leqq\theta(V_1)+\theta(V_2) \tag{2.2.3}$$

For $V\in\mathscr{V}$ and $\varepsilon>0$, there exists a compact $K\subset V$ with $\theta(V\backslash K)\leqq\varepsilon$. (2.2.4)

Then there is a regular submeasure $\rho:\mathscr{B}\to\mathbb{R}$ such that $\rho(V)=\theta(V)$ for $V\in\mathscr{V}$.

Proof. The proof is a more or less standard construction. We sketch the ideas only. Define for *every* subset $A \subset \Omega$,

$$\rho(A) = \inf(\theta(V): V \supset A). \tag{2.2.5}$$

Then $\rho(A) = \theta(A)$ if $A \in \mathscr{V}$. Then define a set $B \subset \Omega$ to be ρ-measurable if given $\varepsilon > 0$ there exist an open U and a closed K with $U \supset B \supset K$ with $\rho(U \backslash K) = \theta(U \backslash K) \leqq \varepsilon$. The class $\mathscr{M}$ of ρ-measurable sets is easily seen to be an algebra of sets containing $\mathscr{V}$, and ρ satisfies conditions (2.0.1) and (2.0.2). The proof is completed by showing that ρ satisfies (2.0.3) and that $\mathscr{M}$ is a σ-algebra; then certainly $\mathscr{M} \supset \mathscr{B}$.

For (2.0.3), if $B_n \in \mathscr{M}$ and $B_n \downarrow \emptyset$ select, for given $\varepsilon > 0$, $K_n \subset B_n$ compact with $\rho(B_n \backslash K_n) \leqq \varepsilon/2^n$. For some N, $K_1 \cap \ldots \cap K_N = \emptyset$; if $n \geqq N$ then

$$\rho(B_n) \leqq \sum_{i=1}^{N} \rho(B_n \backslash K_i) \leqq \varepsilon.$$

To show $\mathscr{M}$ is a σ-algebra, suppose now (B_n) is a descending sequence in $\mathscr{M}$ and let $B = \cap B_n$. Choose compact K_n and open V_n so that $K_n \subset B_n \subset V_n$ and $\theta(V_n \backslash K_n) \leqq \varepsilon/2^{n+1}$. Then

$$\rho(V_1 \cap \ldots \cap V_N \backslash K_1 \cap \ldots \cap K_N) \leqq \varepsilon/2$$

and so if $W_N = V_1 \cap \ldots \cap V_N$, $K = \cap K_N$,

$$\rho(W_N \backslash K) \leqq \rho(W_N \backslash K_1 \cap \ldots \cap K_N) + \rho(K_1 \cap \ldots \cap K_N \backslash K)$$

By the above argument

$$\lim_{N \to \infty} \rho(K_1 \cap \ldots \cap K_N \backslash K) = 0$$

and so

$$\limsup_{N \to \infty} \rho(W_N \backslash K) \leqq \varepsilon/2.$$

Hence for large enough N, $K \subset B \subset W_N$ and $\theta(W_N \backslash K) \leqq \varepsilon$.

Suppose now that X is a quasi-Banach space. Denote by $C(\Omega; X)$ the space of continuous X-valued functions on Ω and by $B(\Omega; X)$ the space of bounded X-valued Borel functions on Ω with separable range. On both spaces we let

$$\|f\|_\infty = \sup_{\omega \in \Omega} \|f(\omega)\|.$$

The topology of convergence in ρ-measure on $B(\Omega; X)$ is defined to be the topology with a base of neighborhoods of the form $V_\varepsilon = \{f : \rho(\omega: \|f(\omega)\| > \varepsilon) < \varepsilon\}$ for $\varepsilon > 0$. This is a pseudo-metrizable topology. We shall require the following lemma.

Lemma 2.3. *Suppose* $f \in B(\Omega; X)$ *and* $\|f\|_\infty \leqq 1$. *Then there is a sequence* $g_n \in C(\Omega; X)$ *such that* $\|g_n\|_\infty \leqq 1$ *and* $g_n \to f$ *in* ρ-*measure.*

Proof. Given $\varepsilon > 0$, we may find a countable-valued function $f_1 \in B(\Omega; X)$ with $\|f_1\|_\infty \leqq 1$ and $\|f - f_1\|_\infty < \varepsilon$. Let

$$f_1 = \sum_{n=1}^{\infty} x_n \otimes 1_{B_n}$$

where $B_n \in \mathscr{B}$ are disjoint with $\cup B_n = \Omega$ and $x_n \in X$ satisfy $\|x_n\| \leqq 1$. There exists $N < \infty$ such that

$$\rho\left(\bigcup_{N+1}^{\infty} B_i\right) < \varepsilon/2.$$

Select compact sets $K_i \subset B_i$ for $1 \leqq i \leqq N$ so that $\rho(B_i \backslash K_i) < \varepsilon/2N$.

Now let $E \subset X$ be the linear span of $\{x_1, \ldots, x_N\}$. By the Tietze extension theorem there is a continuous map $g: \Omega \to E$ so that

$$g(\omega) = x_i \qquad \omega \in K_i \qquad 1 \leqq i \leqq N.$$

Let

$$h(\omega) = g(\omega)/\max(1, \|g(\omega)\|) \qquad \omega \in \Omega.$$

Then $h \in C(\Omega, X)$, $\|h\|_\infty \leqq 1$ and

$$h(\omega) = x_i \qquad \omega \in K_i \qquad 1 \leqq i \leqq N.$$

Hence

$$\|f(\omega) - h(\omega)\| < \varepsilon \qquad \omega \in K_1 \cup \ldots \cup K_N$$

and $\rho(\Omega \backslash (K_1 \cup \ldots \cup K_N)) < \varepsilon$.

3. Vector measures

Again first let Ω be any set and Σ be a σ-algebra of sets on Ω. Let X be a quasi-Banach and let $\mu: \Sigma \to X$ be a vector measure (a countably additive set-function). Then it is well known ([16], [20]) that $\operatorname{co} \mu(\Sigma)$ is a bounded set and so if $\phi: \Omega \to \mathbb{R}$ is a bounded Σ-measurable function we may define $\int \phi \, d\mu$ by the standard procedure of approximating by simple functions.

A submeasure $\rho: \Sigma \to \mathbb{R}$ *controls* μ if $\rho(A) = 0$ implies $\mu(A) = 0$. ρ is said to be a *control submeasure* for μ if whenever $A \in \Sigma$ satisfies $\mu(B) = 0$ for every $B \in \Sigma$ with $B \subset A$ then $\rho(A)$

$=0$. Every such vector measure has a control submeasure defined by

$$\rho(A)=\sup_{B\subset A}\|\mu(B)\|^p, \tag{3.0.1}$$

where p is associated with the quasi-norm as in the introduction.

A set $A\in\Sigma$ is a *μ-atom* if $\mu(A)\neq 0$ and whenever $B\in\Sigma$ with $B\subset A$ then either $\mu(B)=0$ or $\mu(A\backslash B)=0$. It is easy to show that A is a μ-atom if and only if it is a ρ-atom for any control submeasure ρ for μ. μ is said to be *diffuse* if it has no μ-atoms; this is equivalent to the statement that some diffuse submeasure controls μ.

The measure μ is said to be *compact* if $\mu(\Sigma)$ is a relatively compact set. The following theorem is a Liapunoff-type result. The chief point here is that μ need not have a control measure, otherwise a proof like that given in [5] could be reproduced. It may be worth conjecturing that every compact vector measure has a control measure.[1]

Theorem 3.1. *Suppose $\mu:\Sigma\to X$ is a compact diffuse vector measure. Then $\overline{\mu(\Sigma)}$ is a convex set.*

Proof. It will suffice to show that $\frac{1}{2}\mu(\Omega)\in\overline{\mu(\Sigma)}$. For then it will follow that if $A\in\Sigma$ then $\frac{1}{2}\mu(A)\in(\mu(B): B\subset A,\ B\in\Sigma)$ and hence if A, $B\in\Sigma$ then $\frac{1}{2}\mu(A)\in(\mu(B): B\subset A,\ B\in\Sigma)$ and hence if A, $B\in\Sigma$ then

$$\tfrac{1}{2}(\mu(A)+\mu(B))=\mu(A\cap B)+\tfrac{1}{2}\mu(A\,\Delta B)$$

belongs to $\overline{\mu(\Sigma)}$ (simply find $C_n\subset A\Delta B$ with $\mu(C_n)\to\frac{1}{2}\mu(A\Delta B)$, and then

$$\mu((A\cap B)\cup C_n)\to\tfrac{1}{2}(\mu(A)+\mu(B)).$$

Define a control submeasure ρ for μ by (3.0.1). Then ρ is diffuse. For each $A\in\Sigma$ define $v(A)$ to be the infimum of all $\delta>0$ such that there is a finite collection of sets $B_1,\dots,B_n\in\Sigma$ with $B_i\subset A$ $(1\leqq i\leqq n)$ and $\min\rho(C\Delta B_j)<\delta$ wherever $C\subset A$.

Fix $\varepsilon>0$. Then by induction we may define a sequence $(E_n)_{n=1}^{\infty}$ of disjoint sets in Σ so that

$$\|\mu(E_{2n-1})-\mu(E_{2n})\|\leqq 2^{-n/p}\varepsilon \qquad n=1,2,\dots \tag{3.1.1}$$

$$\rho(E_{2n-1}\cup E_{2n})\geqq\tfrac{1}{2}v\Big(\Omega\Big\backslash\bigcup_{i<2n-1}E_i\Big) \qquad n=1,2,\dots \tag{3.1.2}$$

Indeed suppose $\{E_i: 1\leqq i\leqq 2n-2\}$ have been determined where $n\geqq 1$; if $n=1$ this collection is empty. Then there is an infinite collection $(B_n)_{n=1}^{\infty}$ of subsets of $(\Omega\backslash\bigcup_{i=1}^{2n-2}E_i)$ so that

$$\rho(B_j\Delta B_k)\geqq\tfrac{1}{2}v\Big(\Omega\Big\backslash\bigcup_{i=1}^{2n-2}E_i\Big) \qquad j\neq k.$$

[1]Added in proof: this conjecture is true, see N. J. Kalton and J. W. Roberts, Uniformly exhaustive submeasures and nearly additive set functions, *Trans. Amer. Math. Soc.*, to appear.

Since $\mu(\Sigma)$ is relatively compact, there exist $j \neq k$ so that

$$\|\mu(B_j)-\mu(B_k)\| \leqq 2^{-n/p}\varepsilon.$$

Let $E_{2n-1}=B_j \backslash B_k$ and $E_{2n}=B_k \backslash B_j$. Then (3.1.1) and (3.1.2) are satisfied.

Let $F=\bigcup_{n=1}^{\infty} E_{2n-1}$ and $G=\bigcup_{n=1}^{\infty} E_{2n}$. Then

$$\|\mu(F)-\mu(G)\| \leqq \varepsilon.$$

However $\rho(E_{2n-1} \cup E_{2n}) \to 0$ and so by (3.1.2) $v(\Omega \backslash \bigcup_{i=1}^{2n-2} E_i) \to 0$ and hence $v(\Omega \backslash (F \cup G)) = 0$. Now we can apply Lemma 2.1 to deduce that $\rho(\Omega \backslash (F \cup G))=0$ and hence $\mu(F \cup G) = \mu(\Omega)$. Thus

$$\|\tfrac{1}{2}\mu(\Omega)-\mu(F)\| = \tfrac{1}{2}\|\mu(F)-\mu(G)\| < \varepsilon.$$

Now suppose again that Ω is a compact Hausdorff space and $\mathscr{B}$ is its collection of Borel subsets. We say that a vector measure $\mu: \mathscr{B} \to X$ is *regular* if its associated control submeasure is regular. We shall require the following simple lemma whose proof we omit:

Lemma 3.2. *If $\mu: \mathscr{B} \to X$ is a regular vector measure and $\int \phi \, d\mu = 0$ for every $\phi \in C(\Omega)$ then $\mu = 0$.*

We shall also need to observe that an atom of a regular measure may be taken to be a single point.

4. The Riesz-Thomas theorem for $C(\Omega; X)$

Let Ω be an abstract set and Σ be a σ-algebra of subsets of Ω; let X and Y be quasi-Banach spaces. Then an additive map $\Lambda: \Sigma \to \mathscr{L}(X, Y)$ will be called a *totally σ-additive operator measure* if

$$\text{For } x \in X,\ A \mapsto \Lambda(A)x \text{ is a } Y\text{-valued vector measure.} \tag{4.0.1}$$

Whenever $\{x_n\}$ is a bounded sequence in X and $\{A_n\}$ is a disjoint sequence in Σ then $\sum \Lambda(A_n)x_n$ converges. (4.0.2)

It follows simply from (4.0.2) that $\|\Lambda(A_n)\| \to 0$ for any disjoint sequence $\{A_n\}$. Here the quasi-norm on $\mathscr{L}(X, Y)$ is defined exactly as for normed spaces. By (4.0.1) we obtain:

Lemma 4.1. *If $\Lambda: \Sigma \to \mathscr{L}(X, Y)$ is a totally σ-additive operator measure then μ is σ-additive for the topology of $\mathscr{L}(X, Y)$.*

Now let ρ be a control submeasure for $\Lambda: \Sigma \to \mathscr{L}(X, Y)$ defined by

$$\rho(A) = \sup(\|\Lambda(B)\|^p : B \subset A,\ B \in \Sigma). \tag{4.1.1}$$

Define also $\rho': \Sigma \to \mathbb{R} \cup \{\infty\}$ by

$$\rho'(A) = \sup \left\| \sum_{i=1}^{n} \Lambda(B_i) x_i \right\|^p \qquad A \in \Sigma \tag{4.1.2}$$

where the supremum is taken over all disjoint $B_1, \ldots, B_n \in \Sigma$ with $B_i \subset A$ $(1 \leqq i \leqq n)$, all $x_i \in X$ with $\|x_i\| \leqq 1$ $(1 \leqq i \leqq n)$ and all $n \in \mathbb{N}$.

Lemma 4.2 *ρ' is a submeasure on Σ equivalent to ρ (i.e. $\rho'(A) = 0$ if and only if $\rho(A) = 0$). In particular $\rho'(\Omega) < \infty$.*

Proof. Conditions (2.0.1) and (2.0.2) are immediate. Suppose $A_n \downarrow \emptyset$. Then we claim that for each n there exists $m \geqq n$, a finite disjoint collection $B_1, \ldots, B_k$ of subsets of $A_n \backslash A_m$ in Σ and $x_1, \ldots, x_k \in X$ with $\|x_i\| \leqq 1$ $(1 \leqq i \leqq k)$ so that

$$\left\| \sum_{i=1}^{k} \Lambda(B_i) x_i \right\|^p \geqq \tfrac{1}{2} \min(1, \rho'(A_n)). \tag{4.2.1}$$

Indeed given n we may choose disjoint $C_1, \ldots, C_k$ subsets of A_n in Σ and $x_1, \ldots, x_k \in X$ with $\|x_i\| \leqq 1$ $(1 \leqq i \leqq n)$ so that

$$\left\| \sum_{i=1}^{k} \Lambda(C_i) x_i \right\|^p > \tfrac{1}{2} \min(1, \rho'(A_n)).$$

Now for each $m \geqq n$

$$\left\| \sum_{i=1}^{k} \Lambda(C_i \cap A_m) x_i \right\|^p \leqq k \rho(A_m)$$

Hence for large enough m, (4.2.1) holds with $B_i = C_i \backslash (C_i \cap A_m)$.

Now (4.2.1) may be used to determine an increasing sequence $m(k)$, a disjoint sequence $\{B_n\}$, $x_n \in X$ with $\|x_n\| \leqq 1$ so that for some increasing sequence $q(k)$

$$\left\| \sum_{q(k-1)+1}^{q(k)} \Lambda(B_i) x_i \right\|^p \geqq \tfrac{1}{2} \min(1, \rho'(A_{m(k)}))$$

Hence by (4.0.2) $\rho'(A_{m(k)}) \to 0$ and so $\rho'(A_n) \to 0$.

Next we show $\rho'(\Omega) < \infty$, so that ρ' is indeed a submeasure. This will complete the proof as $\rho' \geqq \rho$ but $\rho(A) = 0$ implies $\rho'(A) = 0$.

If $\rho'(\Omega) = \infty$ we construct a decreasing sequence A_n so that $\rho'(A_n) = \infty$ for all n, but

$$\rho(A_n \backslash A_{n+1}) \geqq \tfrac{1}{2} \sup_{B \subset A_n} (\rho(A_n \backslash B) : \rho'(B) = \infty).$$

Let $A = \cap A_n$. We note $\rho'(A) = \infty$; indeed $\rho'(A_n \backslash A) \to 0$ and hence as $\rho'(A_n) \leqq \rho'(A)$

$+\rho'(A_n\backslash A)$, $\rho'(A)=\infty$. Next if $B\subset A$ then either $\rho'(B)=\infty$ or $\rho'(A\backslash B)=\infty$. If $\rho'(B)=\infty$, then

$$\rho(A_n\backslash B)\leqq 2\rho(A_n\backslash A_{n+1})\to 0 \qquad \text{as } n\to\infty.$$

Hence $\rho(A\backslash B)=0$. Similarly if $\rho'(A\backslash B)=\infty$, $\rho(B)=0$. Thus A is a ρ-atom; but then $\rho(A)=\rho'(A)<\infty$ as can be immediately seen.

Now if $f:\Omega\to X$ is a countably simple bounded Σ-measurable function say

$$f=\sum_{n=1}^{\infty} 1_{A_n}\otimes x_n$$

where $\|x_n\|\leqq 1$ and $A_n\in\Sigma$ are disjoint, we can define

$$\int f\,d\Lambda=\sum_{n=1}^{\infty}\Lambda(A_n)x_n$$

and

$$\|\int f\,d\Lambda\|\leqq\rho'(\Omega)^{1/p}\|f\|_\infty.$$

By continuity we can then extend the definition of $\int f\,d\Lambda$ to all of $B(\Omega;X)$, and if $B\in\Sigma$,

$$\left\|\int_B f d\Lambda\right\|=\|\int f.1_B\,d\Lambda\|\leqq\rho'(B)^{1/p}\|f\|_\infty.$$

Now suppose Ω is a compact Hausdorff space and $T: C(\Omega;X)\to Y$ is a linear operator. We say T is *exhaustive* if $Tf_n\to 0$ whenever f_n is a uniformly bounded sequence with $\operatorname{supp} f_i\cap\operatorname{supp} f_j=\varnothing$ for $i\neq j$ ($\operatorname{supp} f=\{\omega:\|f(\omega)\|>0\}$).

If $\Lambda:\mathscr{B}\to\mathscr{L}(X,Y)$ is a totally σ-additive operator measure then

$$Tf=\int f\,d\Lambda \qquad f\in C(\Omega;X)$$

defines an exhaustive operator. This follows easily from (4.2.1).

Theorem 4.3. *Suppose $T: C(\Omega;X)\to Y$ is an exhaustive linear operator. Then there is a unique regular totally σ-additive operator measure $\Lambda:\mathscr{B}\to\mathscr{L}(X,Y)$ so that*

$$Tf=\int f\,d\Lambda \qquad f\in C(\Omega;X).$$

Proof. Define for each open $V\subset\Omega$

$$\theta(V)=\sup(\|Tf\|^p: f\in U,\ \operatorname{supp} f\subset V) \text{ where } U \text{ is the unit ball of } C(\Omega:X) \qquad (4.3.1)$$

We claim θ satisfies the conditions of Lemma 2.2. Condition (2.2.1) and (2.2.2) are immediate.

For (2.2.3) suppose V_1, V_2 are open and that $F=\operatorname{supp} f\subset V_1\cup V_2$ where $f\in U$. Then there are continuous real functions ϕ, ψ with $\phi\geqq 0$, $\psi\geqq 0$ $\operatorname{supp}\phi\subset V_1$, $\operatorname{supp}\psi\subset V_2$ and $\psi(\omega)+\phi(\omega)=1$ for $\omega\in F$. Then $\|T(\phi f)\|^p\leqq\theta(V_1)$ and $\|T(\psi f)\|^p\leqq\theta(V_2)$, so that $\|Tf\|^p\leqq\theta(V_1)+\theta(V_2)$.

If (2.2.4) fails to be true for some $\varepsilon>0$ then by induction we can find a sequence $f_n\in U$ with $(\operatorname{supp} f_n)_{n=1}^{\infty}$ disjoint and contained in V so that $\|Tf_n\|\geqq\frac{1}{2}\varepsilon$. This contradicts the fact that T is exhaustive.

Thus there is a regular submeasure $\rho:\mathscr{B}\to\mathbb{R}$ so that $\rho(V)=\theta(V)$ for $V\in\mathscr{V}$. If $\{f_n\}$ is a uniformly bounded sequence in $C(\Omega;X)$ such that $f_n\to 0$ in ρ-measure then for every $\varepsilon>0$, $\theta(A_n)\to 0$ and $\theta(B_n)\to 0$ where $A_n=\{\omega:\|f_n(\omega)\|>\varepsilon\}$ and $B_n=\{\omega:\|f_n(\omega)\|>\frac{1}{2}\varepsilon\}$. Let $\psi_n\in C(\Omega)$ satisfy $1_{A_n}\leqq\psi_n\leqq 1_{B_n}$. Then

$$\|Tf_n\|^p\leqq\|T(\psi_n f_n)\|^p+T\|(1-\psi_n)f_n\|^p\leqq M^p\theta(B_n)+\varepsilon^p\|T\|^p$$

where $M=\sup\|f_n\|$. Hence

$$\limsup_{n\to\infty}\|Tf_n\|^p\leqq\|T\|^p\varepsilon^p.$$

i.e. $Tf_n\to 0$.

Now if $\{f_n\}$ is a Cauchy sequence in ρ-measure which is uniformly bounded, then $\lim_{n\to\infty} Tf_n$ exists. By Lemma 2.3 we can define for $f\in B(\Omega;X)$

$$T_1 f=\lim_{n\to\infty} Tg_n$$

where $g_n\in C(\Omega;X)$ is a uniformly bounded sequence with $g_n\to f$ in ρ-measure. By an interlacing argument this limit is unique for every choice of such a sequence.

Now $T_1: B(\Omega;X)\to Y$ satisfies $T_1 f=Tf$ for $f\in C(\Omega;X)$ and

$$\|T_1 f\|\leqq\|T\|\,\|f\|\qquad f\,\varepsilon\, B(\Omega;X).$$

Suppose $f_n\in B(\Omega;X)$, $\|f_n\|_\infty\leqq 1$ and $f_n\to 0$ in ρ-measure. Then we may choose $g_n\in C(\Omega;X)$ so that $\|g_n\|_\infty\leqq 1$, $g_n-f_n\to 0$ in ρ-measure and $Tg_n\to 0$; hence $T_1 f_n\to 0$. Thus T_1 is continuous in ρ-measure on bounded sets.

Now for $A\in\mathscr{B}$, define $\Lambda(A)\in\mathscr{L}(X,Y)$ by

$$\Lambda(A)(x)=T_1(x\otimes 1_A).$$

Clearly for each $x\in X$ $A\mapsto\Lambda(A)(x)$ is a Y-valued measure. If $\{x_n\}$ is a bounded sequence in X and $\{A_n\}$ is a disjoint sequence in $\mathscr{B}$ then

$$\sum_{n=1}^{\infty}\Lambda(A_n)(x_n)=T_1\left(\sum_{n=1}^{\infty}x_n\otimes 1_{A_n}\right).$$

Thus Λ is a totally σ-additive operator measure.

Note that if V is open

$$\sup_{B \subset V} \|\Lambda(B)\|^p \leqq \theta(V)$$

and so Λ is controlled by ρ and is thus regular.

Clearly

$$Tf = \int f \, d\Lambda \qquad f \in C(\Omega; X).$$

in the sense that the integral is defined following Lemma 4.2.

We shall say that an exhaustive operator $T: C(\Omega; X) \to Y$ is *diffuse* if its representing measure Λ is diffuse. We then have:

Corollary 4.4 *Suppose $T: C(\Omega; X) \to Y$ is an exhaustive operator. Then there is a sequence $\{L_n\}$ in $\mathscr{L}(X, Y)$, and a sequence $\{\omega_n\}$ in Ω so that*

$$Tf = \sum_{n=1}^{\infty} L_n(f(\omega_n)) + Sf, \qquad f \in C(\Omega; X). \tag{4.4.1}$$

where S is a diffuse exhaustive operator. Here $\|L_n\| \leqq \|T\|$ and $\sum L_n x_n$ converges for every bounded sequence x_n in X. If U is the unit ball of $C(\Omega; X)$ then $S(U)$ and $(T-S)(U)$ are contained in $\overline{T(U)}$.

Proof. Let (ω_n) be the atoms of Λ and let $L_n = \Lambda\{\omega_n\}$. Let $B = \Omega \backslash \{\omega_n : n = 1, 2, \ldots\}$. Then

$$Tf = T_1(f \, . \, 1_{\Omega \backslash B}) + T_1(f \, . \, 1_B) = \sum_{n=1}^{\infty} L_n(f(\omega_n)) + Sf$$

where

$$Sf = T_1(f \, . \, 1_B).$$

As S is controlled by the submeasure

$$\rho_1(A) = \rho(A \cap B) \qquad A \in \mathscr{B}$$

which is diffuse, S is also diffuse.

Finally we note the following:

Theorem 4.5. *Suppose Y is a quasi-Banach space containing no copy of c_0. Then every bounded linear operator $T: C(\Omega; X) \to Y$ is exhaustive.*

Proof. If $\{f_n\}$ is a sequence in $C(\Omega; X)$ with disjoint supports and $\|f_n\| = 1$, then $\{f_n\}$ is a basic sequence equivalent to the usual basis of c_0. As shown in **[8]**, if $T: C(\Omega; X) \to Y$ is a bounded linear operator then $Tf_n \to 0$.

Note also that every compact operator on $C(\Omega; X)$ is exhaustive.

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5. Diffuse operators

Theorem 5.1. *Suppose $T: C(\Omega; X) \to Y$ is a compact diffuse operator. Then $\overline{T(U)}$ is a convex set, where U is the unit ball of $C(\Omega; X)$.*

Proof. Suppose $f_1, f_2 \in U$, suppose T is represented by the diffuse regular totally σ-additive measure $\Lambda: \mathscr{B} \to \mathscr{L}(X, Y)$. Then define measures $\lambda_1: \mathscr{B} \to Y$, $\lambda_2: \mathscr{B} \to Y$ by

$$\lambda_1(B) = \int_B f_1 \, d\Lambda \qquad B \in \mathscr{B}$$

$$\lambda_2(B) = \int_B f_2 \, d\Lambda \qquad B \in \mathscr{B}$$

Then the measure $\lambda: \mathscr{B} \to Y \oplus Y$ given by $\lambda(B) = (\lambda_1(B), \lambda_2(B))$ is a regular diffuse measure. By Theorem 3.1, given $\varepsilon > 0$, there exists $B \in \mathscr{B}$ so that

$$\|\lambda_1(B) - \tfrac{1}{2}\lambda_1(\Omega)\|^p < \varepsilon^p/2$$

$$\|\lambda_2(B) - \tfrac{1}{2}\lambda_2(\Omega)\|^p < \varepsilon^p/2$$

Thus if $g = f_1 . 1_B + f_2 . 1_{\Omega \setminus B'}$

$$\|\int g \, d\mu - \tfrac{1}{2}(\lambda_1(\Omega) + \lambda_2(\Omega))\| < \varepsilon.$$

Now $g \in B(\Omega; X)$ and $\|g\|_\infty \leqq 1$. Hence $\int g \, d\mu$ belongs to the closure of $T(U)$. Thus

$$\tfrac{1}{2}(\lambda_1(\Omega) + \lambda_2(\Omega)) = \tfrac{1}{2}(Tf_1 + Tf_2)$$

is also in the closure of $T(U)$.

Corollary 5.2. *Suppose $X^* = \{0\}$. Then every compact operator $T: C(\Omega; X) \to Y$ is of the form*

$$Tf = \sum_{n=1}^{\infty} L_n(f(\omega_n)) \qquad f \in C(\Omega; X) \tag{5.2.1}$$

where (ω_n) is a sequence of points in Ω, and $L_n \in \mathscr{L}(X, Y)$ are compact operators.

Proof. We use Corollary 4.4. Write T in the form (4.4.1) and observe first that as $L_n = \Lambda\{\omega_n\}$, each L_n is compact.

Now $S: C(\Omega; X) \to Y$ is compact and diffuse. By Theorem 5.1, $S(\text{co}\, U)$ is bounded. If $S \neq 0$, we conclude $\text{co}\, U \neq C(\Omega; X)$, i.e., there exists $u \in C(\Omega; X)^*$ with $u \neq 0$. Now by Theorem 4.3,

$$u(f) = \int_\Omega f \, d\sigma \qquad f \in C(\Omega; X)$$

where $\sigma: \mathscr{B} \to \mathscr{L}(X, \mathbb{R})$ is a vector measure. However $\mathscr{L}(X, \mathbb{R}) = X^* = \{0\}$, so $u = 0$, contrary to assumption.

Theorem 5.3. *Suppose Y has a compactly determined topology and $T: C(\Omega; X) \to Y$ is a diffuse exhaustive operator. Then*

$$\|Tf\| \leqq \|T\| \qquad f \in \operatorname{co} U.$$

Proof. We may suppose that there is a quasi-Banach space Z so that if $y \in Y$

$$\|y\| = \sup \|Ky\|$$

where K runs through all compact operators $K: Y \to Z$ with $\|K\| \leqq 1$.

If $K: Y \to Z$ is compact with $\|K\| \leqq 1$ then $KT: C(\Omega; X) \to Z$ is diffuse and compact. Hence if $f \in \operatorname{co} U$

$$\|KTf\| \leqq \|KT\| \leqq \|T\|$$

by Theorem 5.1, and the result follows.

6. A converse to Milutin's theorem

In general it is an unsolved problem whether, for any compact Hausdorff space and any quasi-Banach space X, $C(\Omega) \otimes X$ is dense in $C(\Omega; X)$ (see **[9]**, **[17]**, **[21]**, **[22]**). However Schuchat has shown that this conclusion is true under certain hypotheses on either Ω or X. We shall use the fact that it is true if Ω is zero-dimensional, e.g. $\Omega = \Delta$, the Cantor set (**[17]**).

Proposition 6.1. *Suppose Ω is a compact metric space and that X is a separable quasi-Banach space. Then $C(\Omega; X)$ is isomorphic to a subspace of $C(\Delta; X)$ and thus is separable.*

This is a simple deduction from the fact that there is a continuous surjection of Δ onto Ω.

Before proceeding to our main result we shall need some preparatory lemmas. Let I denote the unit interval. We say that a quasi-Banach space X is a factor of a space Y if X is isomorphic to a complemented subspace of Y.

Lemma 6.2. *Suppose Ω is a compact metric space and there is a non-constant continuous map $\pi: I \to \Omega$. Then for any quasi-Banach space X, $C(I; X)$ is a factor of $C(\Omega; X)$.*

Proof. There is a homeomorphic embedding of I into Ω (cf. **[23]** p. 19). Since I is an absolute retract there are thus continuous maps $\eta: I \to \Omega$, $\xi: \Omega \to I$ so that $\xi \circ \eta$ is the identity on I. Thus if we define $S: C(I; X) \to C(\Omega; X)$ and $T: C(\Omega; X) \to C(I; X)$ by $Sf(\omega) = f(\xi(\omega))$ and $Tf(t) = f(\eta(t))$ then TS is the identity on $C(I; X)$ and the result is proved.

Lemma 6.3. *Let E be a closed subset of I and let $R_E: C(I;X)\to C(E;X)$ be defined by*

$$R_E f(s) = f(s) \qquad s\in E.$$

Then there exists an operator $J_E: C(E;X)\to C(I;X)$ with $\|J_E\|\leqq 2^{1/p-1}$ and $R_E J_E$ is the identity on $C\,(E;X)$.

Proof. Define $J_E f$ for $f\in C(E;X)$ by linear interpolation on $I\backslash E$ (which is a countable union of open intervals).

For any quasi-Banach space X and compact metric space Ω, let $\mathcal{M}(\Omega;X)$ be the space of regular X-valued Borel measures on Ω equipped with the topology induced by the maps

$$\mu \mapsto \int_\Omega \phi\, d\mu$$

for $\phi\in C(\Omega)$.

For any open subset V of Ω we define

$$\alpha_V(\mu) = \sup\left(\left\|\int \phi\, d\mu\right\|^p + \left\|\int \psi\, d\mu\right\|^p\right)$$

over all ϕ, $\psi\in C(\Omega)$ whose supports are disjoint and contained in V, and such that $0\leqq \phi$, $\psi\leqq 1$.

Also let

$$\beta_V(\mu) = \sup_{\omega\in V} \|\mu(\omega)\|^p \left(= \max_{\omega\in V} \|\mu\{\omega\}\|^p\right).$$

Lemma 6.3. *α_V is lower-semi-continuous and β_V is of Baire class one.*

Proof. The first statement is clear. For the second let F_n be an increasing sequence of closed subsets of Ω with $\bigcup F_n = V$. For each n, let $\phi_{n,1},\dots,\phi_{n,k(n)}$ be a finite collection of functions in $C(\Omega)$ satisfying $0\leqq \phi_{n,j}\leqq 1$, $\operatorname{supp}\phi_{n,j}\subset V$, $\operatorname{diam}(\operatorname{supp}\phi_{n,j})\leqq n^{-1}$ and

$$\max_{1\leqq j\leqq k(n)} \phi_{n,j}(\omega) = 1 \qquad \omega\in F_n.$$

Let

$$\beta_V^{(n)}(\mu) = \max_{1\leqq j\leqq k(n)} \left\|\int_\Omega \phi_{n,j}\, d\mu\right\|^p$$

Then each $\beta_V^{(n)}$ is continuous and we shall see that $\beta_V^{(n)}(\mu)\to\beta_V(\mu)$ for every $\mu\in\mathcal{M}(\Omega;X)$. For each $\omega\in V$, there exists $N\in\mathbb{N}$ with $\omega\in F_n$ for $n\geqq N$. For $n\geqq N$, let $j(n)$ be such that

$$\phi_{n,j(n)}(\omega) = 1.$$

Then

$$\left\|\int \phi_{n,j(n)}\,d\mu\right\|^p \to \|\mu\{\omega\}\|^p$$

so that

$$\beta_V(\mu) \leqq \liminf_{n\to\infty} \beta_V^{(n)}(\mu).$$

On the other hand if

$$\beta_V^{(m(n))}(\mu) \geqq r$$

for $m(1) < m(2) < m(3)\ldots$, there exist $\phi_{m(n),j(n)}$ so that

$$\left\|\int \phi_{m(n),j(n)}\,d\mu\right\|^p \geqq r.$$

By passing to a subsequence we may suppose

$$\operatorname{supp} \phi_{m(n),j(n)} \subset \{\omega' : d(\omega', \omega) < \varepsilon_n\}$$

where $\omega \in \bar{V}$, and $\varepsilon_n \to 0$.

Now it follows from the regularity of μ that

$$\left\|\int \phi_{m(n),j(n)}\,d\mu - \phi_{m(n),j(n)}(\omega)\mu\{\omega\}\right\| \to 0.$$

Hence $\omega \in V$ and $\|\mu\{\omega\}\|^p \geqq r$.

Theorem 6.4. *Suppose:*

X is a separable quasi-Banach space with a compactly determined topology (e.g. suppose X has a basis), which does not contain a copy of c_0. (6.4.1)

Ω is an uncountable compact metric space containing no homeomorphic image of I (e.g. Ω is the Cantor set Δ). (6.4.2)

Then the following conditions are equivalent:

$C(I; X)$ is isomorphic to $C(\Omega; X)$ (6.4.3)

$C(I; X)$ is a factor of $C(\Omega; X)$ (6.4.4)

X is locally convex. (6.4.5)

Remark. For example if $0 < p < 1$, $C(\Delta, l_p)$ is *not* isomorphic to $C(I; l_p)$.

Proof. That (6.4.5) implies (6.4.3) follows from Milutin's theorem according to which

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$C(I) \cong C(\Omega)$ and the fact that $C(\Omega; X)$ is simply the ε-tensor of $C(\Omega)$ and X. Clearly (6.4.3) implies (6.4.4).

Suppose (6.4.4) holds. Thus there are bounded linear operators $T: C(\Omega; X) \to C(I; X)$ and $S: C(I; X) \to C(\Omega; X)$ so that

$$TSf = f \qquad f \in C(I; X).$$

For each $t \in I$, the map

$$T_t: C(\Omega; X) \to X$$

given by $T_t f = Tf(t)$ satisfies $\|T_t\| \leqq \|T\|$ and is exhaustive.

Thus we can write T_t in the form (4.4.1) or

$$T_t f = \sum_{\omega \in \Omega} L(\omega, t)(f(\omega)) + D_t f \tag{6.4.6}$$

Here $L(\omega; t) \in \mathscr{L}(X)$ for (ω, t), and the set $F_t = \{\omega: L(\omega; t) \neq 0\}$ is countable for each t. The operator D_t is diffuse, $D_t \in \mathscr{L}(X)$ and $D_t(U) \subset \overline{T(U)}$. By Theorem 5.3

$$\|D_t(f_t + \cdots + f_n)\| \leqq \|T\|(\|f_1\| + \cdots + \|f_n\|) \tag{6.4.7}$$

for $f_1, \ldots, f_n \in C(\Omega; X)$.

Fix an $f \in C(\Omega; X)$ and consider the operators $M_t: C(\Omega) \to X$ defined by

$$M_t(\phi) = T_t(\phi f).$$

M_t is exhaustive and by using Thomas's theorem,

$$M_t \phi = \int \phi \, d\mu_t$$

where μ_t is a regular X-valued measure. Clearly

$$M_t \phi = \sum_{\omega \in \Omega} L(\omega, t)(f(\omega))\phi(\omega) + D_t(\phi f)$$

and $\phi \mapsto D_t(\phi . f)$ is controlled by a diffuse submeasure and is thus diffuse. Hence the atoms of μ_t are those $\omega \in F_t$ such that $L(\omega, t)(f(\omega)) \neq 0$.

Let $\{V_k\}$ be a base for the open sets of Ω. Noting that the map $t \mapsto \mu_t$ is necessarily continuous from I into $\mathscr{M}(\Omega; X)$, we have that the maps α_k, β_k are Baire class one, where

$$\alpha_k(t) = \alpha_{V_k}(\mu_t) \qquad k = 1, 2, \ldots$$

$$\beta_k(t) = \beta_{V_k}(\mu_t) \qquad k = 1, 2, \ldots$$

Hence there is a dense G_δ-subset of I, $G = G(f)$, say, so that each α_k, β_k is continuous at every point $s \in G$. (See Kuratowski [11] p. 394).

Suppose $s \in G$ and that ω is an atom of μ_s. We shall show that there is a neighborhood V of s such that ω is also an atom of μ_t for $t \in V$. Suppose

$$\|\mu_s\{\omega\}\| = \delta > 0.$$

Fix k so that $\omega \in V_k$ and if $\phi \in C(\Omega)$ with $|\phi| \leqq 1$ then

$$\left\| \int_{V_k \setminus \{\omega\}} \phi \, d\mu_t \right\|^p < \frac{\delta^p}{10}.$$

Then α_k, β_k are continuous at s. Clearly

$$\beta_k(s) = \delta^p$$

while it is easy to calculate that

$$\alpha_k(s) \leqq \tfrac{6}{5} \delta^p.$$

Pick a neighborhood I_0 of s in I, which is a closed interval so that

$$\beta_k(t) > \tfrac{9}{10} \delta^p \qquad t \in I_0$$

$$\alpha_k(t) < \tfrac{13}{10} \delta^p \qquad t \in I_0.$$

For each $t \in I_0$ there is exactly one $\omega = \omega(t)$ so that $\omega(t) \in V_k$ and

$$\|\mu_t\{\omega(t)\}\|^p > \tfrac{9}{10} \delta^p.$$

We claim $t \mapsto \omega(t)$ is a continuous map. It suffices to consider the case $t_n \to t_0$, $\omega(t_n) \to \omega_0 \neq \omega(t_0)$. Let ϕ be a continuous function with $\operatorname{supp} \phi \subset V_k$ and suppose $0 \leqq \phi \leqq 1$. Let $\phi = 1$ on a neighborhood of $\omega(t_0)$ and let $\phi = 0$ on a neighborhood of ω_0. For large enough $n, \phi = 0$ on a neighborhood of $\omega(t_n)$. We may choose a continuous function ψ with $0 \leqq \psi \leqq 1$ so that $\operatorname{supp} \psi \subset V \setminus \operatorname{supp} \phi$ and

$$\|\textstyle\int \psi \, d\mu_{t_n}\|^p \geqq \tfrac{4}{5} \delta^p$$

Thus

$$\|\textstyle\int \phi \, d\mu_{t_n}\|^p \leqq \tfrac{13}{10} \delta^p - \tfrac{4}{5} \delta^p = \tfrac{1}{2} \delta^p$$

Let $n \to \infty$; thus

$$\|\textstyle\int \phi \, d\mu_{t_0}\|^p \leqq \tfrac{1}{2} \delta^p$$

and letting $\operatorname{supp} \phi$ contract to $\{\omega(t_0)\}$ we obtain

$$\|\mu_{t_0}\{\omega(t_0)\}\|^p \leqq \tfrac{1}{2} \delta^p$$

contrary to assumption. Hence $\omega(t)$ is constant on I_0, i.e. ω is an atom of μ_t for $t \in I_0$.

If we repeat this for a dense countable subset (f_n) of $C(\Omega; X)$ we can find one single dense G_δ-set H in I so that the sets $\{t\in H: L(\omega,t)\neq 0\}$ are open relative to H for every $\omega\in\Omega$. Since H has a countable base of open sets it follows that $\bigcup(F_t: t\in H)$ is again countable. Denote this set by $\{\omega_1,\omega_2,\omega_3,\ldots\}$. Then we may rewrite (6.4.6), for $t\in H$, as

$$T_t f=\sum_{n=1}^{\infty} L(\omega_n,t)(f(\omega_n))+D_t f. \tag{6.4.8}$$

Now for any $f\in C(\Omega; X)$ the maps

$$t\mapsto \sum_{n=k+1}^{\infty} L(\omega_n,t)f(\omega_n)$$

are Borel maps on I. This is most easily seen by noting that

$$\sum_{n=k+1}^{\infty} L(\omega_n,t)f(\omega_n)=\int_{A_k} f\,d\Lambda_t$$

where $A_k=\{\omega_{k+1},\ldots\}$ and Λ_t is the totally σ-additive operator measure corresponding to T_t.

Hence by the separability of $C(\Omega; X)$, the maps

$$\gamma_k(t)=\sup_{\|f\|\leqq 1}\left\|\sum_{n=k+1}^{\infty} L(\omega_n,t)f(\omega_n)\right\|$$

are also Borel on I. From Lemma 4.2 it follows that $\gamma_k(t)\to 0$ pointwise on H and hence there is an uncountable compact subset E of H such that

$$\lim_{k\to\infty}\sup_{t\in E}\gamma_k(t)=0.$$

(e.g. apply Egoroff's theorem for some diffuse measure on H).

Now suppose $\{\phi_n\}$ is a sequence in $C(E)$ with $0\leqq\phi_n\leqq 1$ and $\operatorname{supp}\phi_n\cap\operatorname{supp}\phi_m=\emptyset$ for $m\neq n$. Let $x_1,\ldots,x_N\in X$ be such that $\|x_i\|\leqq 1$. Define $(f_{n,k}: 1\leqq n<\infty,\ 1\leqq k\leqq N)$ in $C(\Omega; X)$ by

$$f_{n,k}=SJ_E(\phi_n\otimes x_k).$$

If P is any finite subset of the positive integers

$$\left\|\sum_{n\in P} f_{n,k}\right\|\leqq\|SJ_E\|$$

and hence, as X contains no copy of c_0, for any fixed $\omega\in\Omega$

$$\lim_{n\to\infty}\|f_{n,k}(\omega)\|=0 \qquad k=1,2,\ldots,N.$$

If $t \in E$

$$\left\| \sum_{j=1}^{\infty} L(\omega_j, t) f_{n,k}(\omega_j) \right\|^p \leqq \left\| \sum_{j=1}^{r} L(\omega_j, t) f_{n,k}(\omega_j) \right\|^p + \gamma_r(t)^p$$

for any r. As $\|L(\omega_j, t)\| \leqq \|T\|$ for all j, we conclude

$$\lim_{n\to\infty} \sup_{t\in E} \left\| \sum_{j=1}^{\infty} L(\omega_j, t) f_{n,k}(\omega_j) \right\|^p = 0.$$

Now by (6.4.7)

$$\left\| D_t \sum_{k=1}^{N} f_{n,k} \right\| \leqq N \|SJ_E\| \, \|T\|$$

and so we have

$$\limsup_{n\to\infty} \left\| R_E T \sum_{k=1}^{N} f_{n,k} \right\| \leqq N \|SJ_E\| \, \|T\|$$

i.e.

$$\limsup_{n\to\infty} \|\phi_n \otimes (x_1 + \cdots + x_N)\| \leqq N \|SJ_E\| \, \|T\|$$

or

$$\|x_1 + \cdots + x_N\| \leqq N \|SJ_E\| \, \|T\|.$$

As this is true for any $x_1, \ldots, x_N$ in the unit ball of X we conclude that X is locally convex.

7. Problems

Here we list three problems that appear to be of interest.

Problem 7.1. (Klee [9]) Is $C(\Omega) \otimes X$ dense in $C(\Omega; X)$ in general?

Problem 7.2. Is $C(\Delta; L_p) \cong C(I; L_p)$ for $0 < p < 1$?

Problem 7.3. Is $C(I; l_p) \cong C(I^2, l_p)$?

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