A NOTE ON HYPERNILPOTENT RADICAL PROPERTIES FOR ASSOCIATIVE RINGS

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Introduction. We work entirely in the category of associative rings. We show that if $\mathbf{P}_1$ is a homomorphically closed class which contains the zero rings, then the lower Kurosh radical $\mathbf{P}$ of $\mathbf{P}_1$ is the class $\mathbf{P}_2$ of all rings R such that every non-zero homomorphic image of R has non-zero ideals in $\mathbf{P}_1$, provided that $\mathbf{P}_1$ is closed under extensions by zero rings (i.e., if I is a $\mathbf{P}_1$-ideal of R and $(R/I)^2 = 0$, then $R \in \mathbf{P}_1$). The latter assumption replaces the hypothesis that $\mathbf{P}_1$ be hereditary for ideals in a similar result of Anderson–Divinsky–Sulinsky in **(2)**. This leads to a brief proof that the lower radical construction of Kurosh terminates at $\mathbf{P}_{\omega_0}$ (where ω_0 is the first infinite ordinal) when $\mathbf{P}_1$ is a homomorphically closed class of associative rings containing the zero rings. This was proved for arbitrary homomorphically closed classes $\mathbf{P}_1$ of associative rings in **(2)**.

1. Radical classes. Let $\mathbf{P}_1$ be a homomorphically closed class of rings. For any ordinal $\alpha > 1$, $\mathbf{P}_\alpha$ is defined as the class of rings R, each of whose non-zero homomorphic images contains non-zero ideals from $\mathbf{P}_\beta$ for some $\beta < \alpha$. The union of the classes $\mathbf{P}_\alpha$ is the lower radical $\mathbf{P}$ of Kurosh, which is the smallest radical class $\mathbf{P}$ containing the class $\mathbf{P}_1$. Anderson–Divinsky–Sulinski **(2)** have shown that in the category of associative rings this construction stops with $\mathbf{P} = \mathbf{P}_{\omega_0}$, where ω_0 is the first infinite ordinal. For alternative rings the construction stops at $\mathbf{P} = \mathbf{P}_{\omega_0 2}$. Also, it is shown in **(2)** that if $\mathbf{P}_1$ is a hereditary, homomorphically closed class of rings containing the zero rings, then $\mathbf{P} = \mathbf{P}_2$. We show that the hypothesis that $\mathbf{P}_1$ be hereditary can be replaced by a different condition in the following theorem.

THEOREM. *Let $\mathbf{P}_1$ be a homomorphically closed class of associative rings containing the zero rings and having the additional property that if $I \in \mathbf{P}_1$ is an ideal of R with R/I a zero ring then $R \in \mathbf{P}_1$ (i.e., $\mathbf{P}_1$ is closed under extensions by zero rings); then $\mathbf{P}_2$ is the lower radical containing $\mathbf{P}_1$.*

Proof. We verify that the class $\mathbf{SP}_1$ of rings having only the zero ideal in $\mathbf{P}_1$ is hereditary. Then by the result of Amitsur **(1**, Theorem 8.1, p. 118**)** it follows that $\mathbf{P}_2$ is the lower radical. Let $R \in \mathbf{SP}_1$ and let I be an ideal of R. Suppose I has an ideal K with $K \in \mathbf{P}_1$. Then, if K_1 is the ideal of R generated by K, we have $(K_1/K)^3 = 0$, so that $(K_1^2 + K)/K$ is a zero ring and hence $K_1^2 + K$ is a member of $\mathbf{P}_1$. Also, K_1 is an extension of the ring $K_1^2 + K$ in

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$\mathbf{P}_1$ by the zero ring $K_1/(K_1^2 + K)$, so it is in $\mathbf{P}_1$, which shows that $K = K_1 = 0$; hence $I \in \mathbf{SP}_1$ as desired.

COROLLARY (Anderson–Divinsky–Sulinsky). *Let $\mathbf{P}_1$ be a homomorphically closed class of associative rings containing the zero rings. Then the lower radical construction terminates at the class $\mathbf{P}_{\omega_0}$, where ω_0 is the first infinite ordinal.*

Proof. Let $\mathbf{N}_1$ be the class of all rings R such that R is in some class $\mathbf{P}_n$ for $n \geqslant 1$. Then since each class $\mathbf{P}_n$ is homomorphically closed, it follows that $\mathbf{N}_1$ is homomorphically closed. Moreover, if I is an ideal of a ring R with $I \in \mathbf{P}_n$ and $(R/I)^2 = 0$, it follows that $R \in \mathbf{P}_{n+1}$. For let K be an ideal of R with $K \neq R$. If $I \subseteq K$, then $R/K \in \mathbf{P}_1$, as a homomorphic image of R/I. On the other hand, if $I \subseteq K$, $(I + K)/K$ is a non-zero ideal of R/K and a member of $\mathbf{P}_n$. Hence $R \in \mathbf{P}_{n+1}$. Thus by the theorem the lower radical of $\mathbf{N}_1$ is $\mathbf{N}_2$, which clearly is just the class $\mathbf{P}_{\omega_0}$.

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