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# CHROMATIC SUMS FOR ROOTED PLANAR TRIANGULATIONS, III: THE CASE $\lambda = 3$

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**Summary.** In this paper we are chiefly concerned with the chromatic sums we have called $l$ and $h$, with colour-number 3. In this case $h$ can be interpreted as enumerating the rooted Eulerian triangulations with a given number of faces, and $l$ as enumerating such triangulations with a given number of faces and a given valency for the root-vertex. The series $h$ has been determined already, by summation from the formula enumerating even slicings [**3**]. However our formula for $l$ does not seem to have been published before, though it could presumably be derived in a similar way. The object of the present paper is not only to obtain formulae for $l$ and $h$, but to derive them by a method analogous to that used in Paper II of this series for the case $\lambda = \tau + 1$. It is thought that such analogies may help eventually in the construction of a theory valid for all $\lambda$. (See [**4**; **5**].)

**1. The number** 3. It is well-known that a planar triangulation $T$ can be 3-coloured if and only if it is Eulerian, that is the valency of each vertex is even. If $T$ is Eulerian there is essentially only one 3-colouring. But to allow for the six permutations of the three colours we write $P(T, 3) = 6$.

By definition $h_{2n}$, the coefficient of $z^{2n}$ in $h$ is the sum of $P(M, 3)$ over all rooted planar near-triangulations $M$ with a digon as root-face and with $2n$ triangular faces. It has been remarked in I and II that such a near-triangulation $M$, if non-degenerate, can be converted into a true triangulation by erasing the non-root edge of the digon. So for $\lambda = 3$ and $n > 0$ the coefficient $h_{2n}$ is six times the number of rooted Eulerian planar triangulations with $2n$ faces. Dually, it is six times the number of rooted planar bicubic maps with $2n$ vertices. Such bicubic maps are counted in [**2**]. From the formula given in that paper we have

$$h_{2n} = \frac{9 \cdot 2^n \cdot (2n)!}{n!(n+2)!} \qquad (n \geqq 1). \tag{1}$$

In what follows we derive this formula by a new method, analogous to that used in II for the case $\lambda = \tau + 1$.

In II we gave some special theorems for the case $\lambda = \tau + 1$. We now give one special theorem for the case $\lambda = 3$. It has analogies with Theorem 1.3 of II, but it relates to a pentagon $a_1a_2a_3a_4a_5$ instead of to a quadrilateral $VWXY$.

Consider a rooted near-triangulation $N$ in which the root-face is a pentagon

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$P = a_1a_2a_3a_4a_5$. We take the root-vertex to be $a_1$ and the root-edge $E$ to be the edge $a_1a_2$ of $P$. Let $K$ denote the face other than $P$ incident with the root-edge. We now define $Z_i$ as the triangulation obtained from $N$ by subdividing $P$ by means of two diagonals $a_ia_{i+2}$ and $a_ia_{i+3}$. ($1 \leqq i \leqq 5$ and addition and subtraction in the sufficies are modulo 5.) In $Z_i$ we retain the root-vertex and root-edge of $N$, and we take the new root-face to be inside $P$. If $a_i$ is not divalent in $N$ we define $Y_i$ as the triangulation obtained from $N$ by identifying $a_{i-1}$ with $a_{i+1}$, and correspondingly identifying the edge $a_ia_{i-1}$ with $a_ia_{i+1}$. The face $P$ reduces to a triangle $a_{i+1}a_{i+2}a_{i+3}$. In $Y_i$ we retain the same root-vertex and root-edge as in $N$, and we define the new root-face as the face other than $K$ incident with the root-edge (see Figure 1).

If $a_{i-1}$ is joined directly to $a_{i+1}$ in $N$ the above construction for a triangulation $Y_i$ fails. The identifications can be carried out and they yield a planar map $Y_i$. But $Y_i$ has a loop and so does not satisfy our definition of a triangulation (I, § 1). Because of this loop the chromial $P(Y_i, \lambda)$ is identically zero. We acknowledge $Y_i$ in this case as a degenerate kind of triangulation whose chromial sometimes appears formally in equations. But as the chromial is zero this makes no real difference.

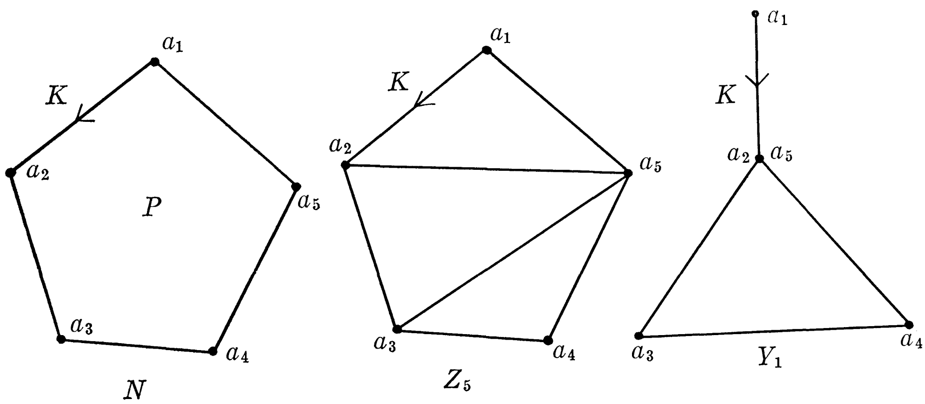


FIGURE 1

We can now state our theorem.

1.1. *For each suffix $i$,*

$$P(Y_i, 3) = P(Z_{i+2}, 3) + P(Z_{i+3}, 3).$$

*Proof.* Let $J_i = 1$ if $a_{i+2}$ and $a_{i+3}$ are the only vertices of $N$ of odd valency, and let $J_i = 0$ otherwise. Then evidently

$$P(Z_i, 3) = 6J_i,$$
$$P(Y_i, 3) = 6(J_{i+2} + J_{i+3}),$$

for each suffix $i$. The theorem follows.

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Professor D. W. Hall has pointed out to the author that the theorem can also be deduced from the Birkhoff-Lewis equations for the 5-ring [**1**].

**2. An equation for $l$.** We introduce the generating series

$$f(y, z) = \sum_T y^{n(T)} z^{t(T)+1} P(T, 3), \tag{2}$$

where the sum is over all rooted triangulations $T$. We write $l = l(y, z)$ for $l(y, z, 3)$. As in II there is a simple relation between $f(y, z)$ and $l(y, z)$. It is now

$$l(y, z) = 6y + yf(y, z). \tag{3}$$

In our next definitions we use the notation of Section 1.

Let $S(Z_i)$ denote the contribution to $f(y, z)$ of all rooted planar triangulations $T$ such that $T = Z_i$ for some choice of $N$.

We proceed to determine $S(Z_4)$ in terms of $f(y, z)$.

Consider a rooted planar triangulation $T$ with root-vertex $a_1$, a root-edge $E = a_1a_2$, and a root-face $F = a_1a_2a_4$. Let the other face incident with $a_1a_4$ be $F_1$, and let its third vertex be $a_5$.

It may happen that $a_5 = a_2$. In this case $T$ can be represented by the diagram of Figure 2. The shaded regions correspond to rooted near-triangulations $M_1$ and $M_2$ as indicated. $M_1$ has the same root-edge as $T$. Each shaded region is shown bounded by a digon, but each of them may degenerate into a single edge. The chromials of $T$, $M_1$ and $M_2$ are related, for general $\lambda$, by Equation (6) of II.

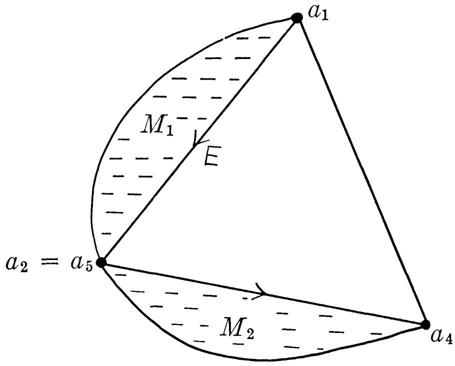


FIGURE 2

The contribution to $f(y, z)$ of triangulations $T$ of this kind is

$$\sum_{(M_1, M_2)} y^{n(M_1)+1} z^{t(M_1)+t(M_2)+2} P(M_1, 3) P(M_2, 3)/6,$$

where $M_1$ and $M_2$ are arbitrary rooted near-triangulations each with a digon

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as root-face. This expression can be abbreviated as

$$yz^2lh/6.$$

In the remaining case we are dealing with triangulations $T$ in which there is a quadrilateral $a_1a_2a_4a_5$ subdivided into two faces $F$ and $F_1$ by a diagonal $a_1a_4$. The contribution to $f(y, z)$ of triangulations of this kind is

$$f(y, z) - (yz^2lh/6).$$

For such a triangulation $T$ let $F_2$ be the non-root face incident with the edge $a_2a_4$ of $F$, and let its third vertex be $a_3$.

It may happen that $a_3 = a_1$. In that case $T$ can be represented by the diagram of Figure 3. Again we have two shaded regions, bounded by digons,

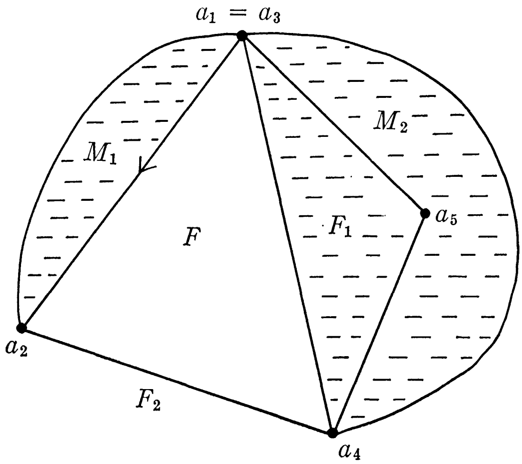


FIGURE 3

corresponding to rooted near-triangulations $M_1$ and $M_2$. Now $M_1$ may reduce to a single edge, but $M_2$ cannot. Otherwise there is no restriction on the structures of $M_1$ and $M_2$. The contribution to $f(y, z)$ of triangulations $T$ of this kind is

$$\sum_{(M_1, M_2)} y^{n(M_1)+n(M_2)} z^{t(M_1)+t(M_2)+2} P(M_1, 3)P(M_2, 3)/6,$$

that is

$$z^2l(l - 6y)/6.$$

Another possibility is $a_3 = a_5$. In this case $T$ has a subgraph that is a complete 4-graph, and therefore $T$ has no 3-colouring. The contribution to $f(y, z)$ of such triangulations is zero.

In the remaining case we have only the triangulations such as $Z_4$. We deduce from the foregoing results that

$$S(Z_4) = f(y, z) - (yz^2lh/6) - (z^2l(l - 6y)/6). \tag{4}$$

We go on to determine $S(Z_5)$. We can start with a general rooted planar triangulation $T$ with root-vertex $a_1$, root-edge $E = a_1a_2$ and root-face $F = a_1a_2a_5$. Let $F_1$ be the second face incident with the edge $a_2a_5$ of $F$, and let its third vertex be $a_3$.

It may happen that $a_3 = a_1$. Then $T$ is represented by the diagram of Figure 2, with each $a_i$ replaced by $a_{i+1}$, and with the common root of $T$ and $M_1$ reversed. We deduce that the contribution to $f(y, z)$ of triangulations $T$ of this kind is

$$z^2l^2/6.$$

In the remaining case we are dealing with triangulations $T$ in which there is a quadrilateral $a_1a_2a_3a_5$ subdivided into two faces $F = a_1a_2a_5$ and $F_1 = a_2a_3a_5$ by a diagonal $a_2a_5$. The contribution to $f(y, z)$ of such triangulations is

$$f(y, z) - (z^2l^2/6).$$

Let $F_2$ be the second face incident with the edge $a_3a_5$ of $F_1$, and let its third vertex be $a_4$.

It may happen that $a_4 = a_2$. Then $T$ is represented by the diagram of Figure 4. The shaded region $M_1$, shown bounded by a digon, can be interpreted as a rooted near-triangulation. It may degenerate into a single edge.

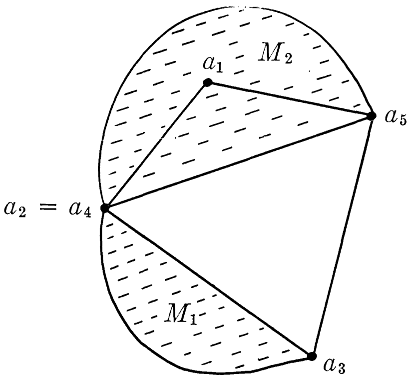


FIGURE 4

The shaded region $M_2$ can be interpreted as a rooted planar triangulation in which one edge, not incident with the root-vertex, is expanded into a digon. The rooting of this triangulation is the same as for $T$. We deduce that the contribution to $f(y, z)$ of triangulations $T$ of this kind is

$$fhz^2/6 = y^{-1}z^2h(l - 6y)/6$$

Another possibility is $a_4 = a_1$. But then $T$ has a complete 4-graph as a subgraph, and its contribution to $f(y, z)$ is zero.

In the remaining case we have only the triangulations such as $Z_5$. We deduce that

$$(5) \qquad S(Z_5) = f(y, z) - (z^2l^2/6) - (y^{-1}z^2h(l - 6y)/6).$$

Now let $S$ denote the class of all rooted planar triangulations $T$ contributing to $S(Z_4)$. With $T$ as $Z_4$ we write $\theta(T)$ for the corresponding $Z_5$, and $\phi(T)$ for the corresponding $Y_2$. We put $f_1 = S(Z_4)$, and we define $f_2$ and $f_3$ as follows.

(6) $$f_2 = \sum_{T\in S} y^{n(T)}z^{t(T)+1}P(\theta(T), 3),$$

(7) $$f_3 = \sum_{T\in S} y^{n(T)}z^{t(T)+1}P(\phi(T), 3).$$

By Theorem 1.1 we have

(8) $$f_3 = f_1 + f_2.$$

We can rewrite (6) as

$$f_2 = \sum_{T\in S} y^{n(\theta(T))+1}z^{t(\theta(T))+1}P(\theta(T), 3).$$

Thus

(9) $$f_2 = yS(Z_5)$$

by the 1-1 correspondence between $Z_4$ and $Z_5$.

We can rewrite (7) as

(10) $$f_3 = \sum_{T\in S'} y^{n(T)}z^{t(\phi(T))+3}P(\phi(T), 3),$$

where $S'$ is the set of all members $T$ of $S$ such that $\phi(T)$ has no loop.

Any rooted planar triangulation $K$ can be considered as a possible $\phi(T)$. Let the root-vertex of $K$ be $a_1$ and let the other end of the root-edge $E$ be $a_2$. Suppose there is a face $F_1 = a_1a_4a_5$ of $K$ incident with $a_1$ but not with $a_2$ (see Figure 5). Let us cut out the face $F_1$, cut along the edge $E$, and open out $E$

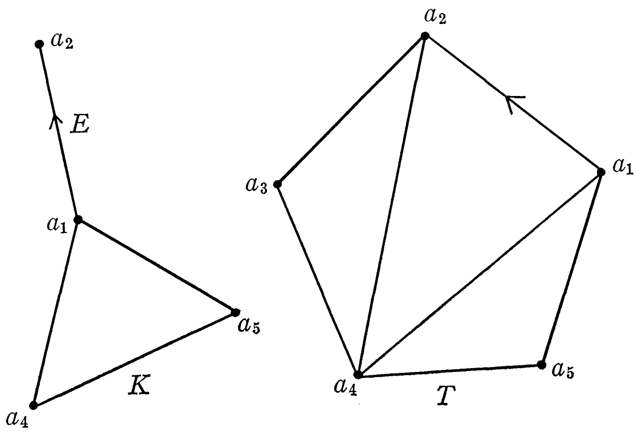


FIGURE 5

so as to form with the original triangular hole $a_1a_4a_5$ a pentagonal hole $a_1a_2a_3a_4a_5$. Here the vertices $a_1$ and $a_3$ both arise from the original $a_1$ of $K$.

Next let us fill up the pentagonal hole with three new triangular faces $a_1a_4a_5$, $a_1a_2a_4$ and $a_2a_3a_4$. We thus obtain a planar triangulation $T$. We take its root-vertex to be $a_1$, its root-edge to be the edge $a_1a_2$ of the pentagonal hole, and its root-face to be the new triangle $a_1a_2a_4$. Evidently $T$ is a member of $S'$. Let us agree to adjust the notation so that the non-root face incident with the root-edge is the same in $T$ as in $K$. Then evidently $K = \phi(T)$.

For a given $K$ there are at most $n(K) - 2$ faces that can be taken as $F_1$. These are the faces incident with $a_1$ but not with $E$. Some of them may be incident with $a_2$, and therefore inadmissible. If we neglect this possibility and consider the $n(K) - 2$ faces in their order at $a_1$ we must conclude that they give rise to $n(K) - 2$ distinct rooted triangulations $T$ such that $K = \phi(T)$, with $n(T)$ taking all values from 3 to $n(K)$. We are thus led to the following first approximation $\Phi_3$ to $f_3$.

$$\Phi_3 = \sum_K \left\{ \sum_{j=2}^{n(K)-1} y^{j+1} \right\} z^{t(K)+3} P(K, 3). \tag{11}$$

Here each value of $j$ corresponds to a face $F_1$ of $K$ incident with $a_1$ but not with $E$. To obtain a correct formula for $f_3$ we must subtract the contributions of all pairs $(K, j)$ corresponding to triangles $F_1$ incident with $a_2$.

Consider first the case in which $a_4 = a_2$. Then $K$ is represented by the diagram of Figure 6. In this case $K$ can be decomposed into two non-de-

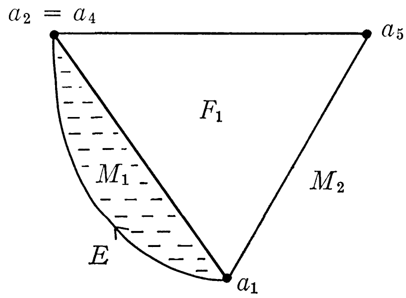


FIGURE 6

generate rooted near-triangulations $M_1$ and $M_2$ with root-faces both bounded by the 2-circuit made up of $E$ and the edge $a_1a_4$ of $F_1$. We take $M_2$ to be the one not having $F_1$ as a face, and we assign to each of $M_1$ and $M_2$ the same root-vertex and root-edge as for $K$. The contribution of $K$ to $\Phi_3$ is found to be

$$y^{n(M_2)} z^{t(M_1)+t(M_2)+2} P(M_1, 3) P(M_2, 3)/6.$$

We deduce that the total contribution to $\Phi_3$ of all triangulations $K$ of the kind being considered is

$$z^2(l - 6y)(h - 6)/6.$$

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Consider next the case $a_5 = a_2$. Then $K$ is represented by Figure 6 with the suffixes 4 and 5 interchanged. The analysis is similar to that of the preceding case, but the roles of $M_1$ and $M_2$ are interchanged. The contribution of $K$ to $\Phi_3$ is found to be

$$y^{n(M_1)+1}z^{t(M_1)+t(M_2)+2}P(M_1, 3)P(M_2, 3)/6.$$

The extra 1 in the index of $y$ is due to the fact that the special face $F_1$ of $K$ is a face of $M_2$ but not a face of $M_1$. We deduce that the total contribution to $\Phi_3$ of all rooted triangulations $K$ of the kind now being considered is

$$yz^2(l - 6y)(h - 6)/6.$$

Subtracting from $\Phi_3$ the contributions of all cases in which $F_1$ is incident with $a_2$ we find that

$$\begin{aligned} f_3 = \sum_K \left\{\sum_{j=1}^{n(K)} y^{j+1}\right\} z^{t(K)+3}P(K, 3) \\ - y^2\sum_K z^{t(K)+3}P(K, 3) - \sum_K y^{n(K)+1}z^{t(K)+3}P(K, 3) \\ - z^2(1 + y)(lh - 6l - 6yh + 36y)/6 \\ = y^2z^2 \triangle (f) - y^2z^2(h - 6) - z^2(l - 6y) \\ - (1 + y)z^2(lh/6) + (1 + y)z^2l + (y + y^2)z^2h \\ - 6(1 + y)yz^2, \end{aligned}$$

where $f = f(y, z)$. Simplifying, we find

$$f_3 = y^2z^2 \triangle (f) - (1 + y)z^2(lh/6) + yz^2l + yz^2h. \tag{12}$$

But $f_3 = f_1 + f_2 = S(Z_4) + yS(Z_5)$, by (8) and (9), and the definition of $f_1$. So by (4) and (5),

$$f_3 = (1 + y)f - (1 + y)z^2(lh/6) - (1 + y)z^2(l^2/6) + yz^2l + yz^2h.$$

Comparing this with (12) we deduce that

$$(1 + y)f = (1 + y)z^2(l^2/6) + y^2z^2 \triangle (f). \tag{13}$$

We deduce from (3) that

$$\triangle(l) = 6 + f + \triangle(f) = y^{-1}l + \triangle(f).$$

We can therefore write (13) entirely in terms of $l$, as follows.

$$6(1 + y)(l - 6y) = (1 + y)yz^2l^2 - 6y^2z^2l + 6y^3z^2 \triangle (l). \tag{14}$$

It is of some interest to compare this equation with Equation (18) of II.

**3. Solution of the difference equation.** Since $\triangle(l) = (h - l)/(1 - y)$ we can multiply (14) by $(1 - y)$ and then rewrite it as

$$(1 - y^2)yz^2l^2 - 6(1 - y^2 + y^2z^2)l + 36y(1 - y^2) + 6y^3z^2h = 0. \tag{15}$$

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This is equivalent to

$$\text{(16)}\quad \{(1-y^2)yz^2l-3(1-y^2+y^2z^2)\}^2 = 9(1-y^2+y^2z^2)^2-36y^2(1-y^2)^2z^2-6y^4z^4(1-y^2)h.$$

Let us write

$$\text{(17)}\qquad 9H = 6z^4h$$

and denote the expression on the right of (16) by $9D$. Then

$$\text{(18)}\quad D = 1-2(1+z^2)y^2+(1+6z^2+z^4-H)y^4+(H-4z^2)y^6.$$

We solve for $l$ and $h$ by the same method as in II. First we introduce a power series $\xi$ in $z$ such that

$$\text{(19)}\qquad (1-\xi^2)\xi z^2 l(\xi, z) - 3(1-\xi^2+\xi^2z^2) = 0.$$

Now $l(y, z)$ involves only odd powers of $y$ and even powers of $z$, with non-zero coefficients. Hence (19) uniquely determines $\xi^2$ as a power series in $z^2$. As in II there are two solutions for $\xi$, but now one of them is merely the negative of the other.

We deduce from (16) that $D$ and its derivative with respect to $y^2$ both vanish when $y^2$ is set equal to $\xi^2$. We thus have the following equations.

$$\text{(20)}\quad 1-2(1+z^2)\xi^2+(1+6z^2+z^4)\xi^4-4z^2\xi^6-H\xi^4(1-\xi^2) = 0,$$

$$\text{(21)}\quad -2(1+z^2)+2(1+6z^2+z^4)\xi^2-12z^2\xi^4-H\xi^2(2-3\xi^2) = 0.$$

Eliminating $H$ between these two equations we find that

$$-2+(5+2z^2)\xi^2+(-4-4z^2)\xi^4+(1+2z^2+z^4)\xi^6 = 0.$$

Other forms of this equation are

$$\xi^6z^4+2\xi^2(1-\xi^2)^2z^2-(1-\xi^2)^2(2-\xi^2) = 0,$$

$$\text{(22)}\qquad \xi^2\{\xi^2z^2-(1-\xi^2)\}\{\xi^4z^2+(1-\xi^2)(2-\xi^2)\} = 0.$$

Hence the relation between $z^2$ and $\xi^2$ is

$$\text{(23)}\qquad z^2 = \xi^{-2}(1-\xi^2)$$

or

$$\text{(24)}\qquad z^2 = -\xi^{-4}(1-\xi^2)(2-\xi^2).$$

From (20) and (21) we have

$$\text{(25)}\qquad \xi^6H = 2-2(1+z^2)\xi^2+4\xi^6z^2.$$

If we assume (23) it follows that

$$\begin{aligned} \xi^6H &= 2-2\xi^2-2(1-\xi^2)+4\xi^4(1-\xi^2),\\ H &= 4z^2,\\ hz^4 &= 6z^2, && \text{by (17).}\end{aligned}$$

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But this is impossible since $h$ is a power series in $z$ with no negative indices. We conclude that (24) holds. Accordingly

$$\xi^6 H = 2 - \xi^2(2 - 2\xi^{-4}(1 - \xi^2)(2 - \xi^2)) - 4\xi^2(1 - \xi^2)(2 - \xi^2)$$
$$= 2(1 - \xi^2)(1 + \xi^{-2}(2 - \xi^2) - 2\xi^2(2 - \xi^2)),$$
$$\xi^8 H = 4(1 - \xi^2)(1 - 2\xi^4 + \xi^6),$$

(26) $$H = 4\xi^{-8}(1 - \xi^2)^2(1 + \xi^2 - \xi^4),$$

(27) $$z^4 h = 6\xi^{-8}(1 - \xi^2)^2(1 + \xi^2 - \xi^4).$$

The above equation (22) corresponds to Equation (29) of II. There also we derive two alternative expressions for $z^2$. But in II both these are legitimate, in the sense that they can be used to determine $h$. Here only one of them is.

If we write $\theta = \xi^{-2}$ we have

(28) $$z^2 = -(1 - \theta)(1 - 2\theta),$$

(29) $$z^4 h = -6(1 - \theta)^2(1 - \theta - \theta^2).$$

We can find $h$ as a power series in $z^2$ by elimination of $\theta$ between these two equations.

Substituting for $z^2$ and $H$ in (18) we find that

$$D = 1 + y^2(-2 + 2(1 - \theta)(1 - 2\theta)) + y^4(1 - 6(1 - \theta)(1 - 2\theta)$$
$$+ (1 - \theta)^2(1 - 2\theta)^2 + 4(1 - \theta)^2(1 - \theta - \theta^2))$$
$$+ y^6(4(1 - \theta)(1 - 2\theta) - 4(1 - \theta)^2(1 - \theta - \theta^2))$$
$$= 1 - 2y^2(3\theta - 2\theta^2) + y^4(9\theta^2 - 8\theta^3) - 4y^6(\theta^3 - \theta^4),$$

(30) $$D = (1 - \theta y^2)^2(1 - 4\theta(1 - \theta)y^2).$$

We can now use (16) to determine $l$ in terms of $\theta$.

$$(1 - y^2)yz^2 l - 3(1 - y^2 - y^2(1 - \theta)(1 - 2\theta)) = \pm 3\sqrt{D},$$
$$(1 - y^2)yz^2 l - 3(1 - \theta y^2 - 2y^2(1 - \theta)^2) = \pm 3(1 - \theta y^2)(1 - 4\theta(1 - \theta)y^2)^{\frac{1}{2}}.$$

Since $l = 0$ when $y = 0$ we must resolve the ambiguity by taking the negative sign. We interpret the square root as a power series in $y$ and $\theta$ with constant term 1. Our equation for $l$ is thus

(31) $$(1 - y^2)yz^2 l + 6y^2(1 - \theta)^2 = 3(1 - \theta y^2)(1 - (1 - 4\theta(1 - \theta)y^2)^{\frac{1}{2}}).$$

**4. The series $h$.** Let us write

(32) $$u = -(1 - 2\theta)^{-1}.$$

It can be verified, using (28) and (29), that

$$u = 1 + 2u^2 z^2,$$
$$2h = 3(1 + 4u - u^2).$$
$$(dh/du) = 3(2 - u).$$

When $z = 0$ we have $u = 1$ and $h = 6$.

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Applying Lagrange's theorem we find that

$$h = 6 + 3\sum_{n=1}^{\infty}\left(\frac{(2z^2)^n}{n!}\right)\left[\left(\frac{d}{du}\right)^{n-1}\{u^{2n}(2-u)\}\right]_{u=1}$$

$$= 6 + 3\sum_{n=1}^{\infty}\left(\frac{(2z^2)^n}{n!}\right)\left\{\frac{2\cdot(2n)!}{(n+1)!} - \frac{(2n+1)!}{(n+2)!}\right\},$$

(32) $$h = 6 + 9\sum_{n=1}^{\infty}\left\{\frac{2^n\cdot(2n)!z^{2n}}{n!(n+2)!}\right\}.$$

We thus recover Equation (1).

For large $n$ we can apply Stirling's formula to obtain the following asymptotic approximation:

(33) $$h_{2n} \sim 9\pi^{-(1/2)}n^{-(5/2)}8^n.$$

**5. The series $q$.** We do not attempt to find an explicit formula for $q(x, z, 3)$. However for the convenience of anyone who may wish to extend the theory we note that there is in principle a method for determining $q$ when $l$ is known, a method valid for every $\lambda$.

The chromatic equation (I, (13)) can be written as

(34) $$g\cdot(x - \lambda^{-1}yzq - yz + x^2y^2z(y-1)^{-1}) = x^2(xy\lambda(\lambda-1) - yzl + y^2z(y-1)^{-1}q).$$

We now introduce $v$, regarded as a power series in $z$ whose coefficients are functions of $y$. It is defined by

(35) $$v - \lambda^{-1}yzq(v, z, \lambda) - yz + v^2y^2z(y-1)^{-1} = 0.$$

From this equation we can determine, in terms of the coefficients in $q$, the coefficients of successive powers of $z$ in $v$ as far as we please. Thus $v$ is well-defined.

Substituting $v$ for $x$ in (34) we find that $v$ must also satisfy

(36) $$vy\lambda(\lambda-1) - yzl(y, z, \lambda) + y^2z(y-1)^{-1}q(v, z, \lambda) = 0.$$

If $l$ is given by an equation such as (15) or (31) we can eliminate it between this equation and (36). We can then eliminate $y$ between the resulting equation and (35). We will then have an equation giving $q(v, z, \lambda)$ directly in terms of $v$, $z$ and $\lambda$.

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