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ON HIGHER TORSION CLASSES

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Abstract. Building on the embedding of an n-abelian category $\mathscr{M}$ into an abelian category $\mathcal{A}$ as an n-cluster-tilting subcategory of $\mathcal{A}$, in this paper, we relate the n-torsion classes of $\mathscr{M}$ with the torsion classes of $\mathcal{A}$. Indeed, we show that every n-torsion class in $\mathscr{M}$ is given by the intersection of a torsion class in $\mathcal{A}$ with $\mathscr{M}$. Moreover, we show that every chain of n-torsion classes in the n-abelian category $\mathscr{M}$ induces a Harder–Narasimhan filtration for every object of $\mathscr{M}$. We use the relation between $\mathscr{M}$ and $\mathcal{A}$ to show that every Harder–Narasimhan filtration induced by a chain of n-torsion classes in $\mathscr{M}$ can be induced by a chain of torsion classes in $\mathcal{A}$. Furthermore, we show that n-torsion classes are preserved by Galois covering functors, thus we provide a way to systematically construct new (chains of) n-torsion classes.

§1. Introduction

Higher homological algebra has its origin in the study of n-cluster-tilting subcategories of abelian and triangulated categories in [16], [17]. The subject has greatly developed since its introduction with more and more of the classical notions emerging in the higher setting. The key idea of higher homological algebra is the study of categories where the shortest nonsplit exact sequences are composed of $n+2$ objects, for a fixed positive integer n. In particular, 1-homological algebra corresponds to the classical theory of abelian, exact and triangulated categories and their classical generalizations such as quasi-abelian and extriangulated categories.

In recent years, the importance of higher homological algebra is starting to emerge through articles showing connections between this subject and other branches of the mathematical sciences, such as combinatorics and homological mirror symmetry [10], [15], [19], [27], [35].

Since its inception, it has been shown that many of the fundamental homological concepts in the classical theory have an analogue in higher homological algebra. Classical homological algebra is a by now well-developed subject, and many of the fundamental concepts have several equivalent definitions characterizing different properties and aspects of the various concepts. However, this poses a difficulty in generalizing these ideas into the setting of higher homological algebra, since the classically equivalent definitions might

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lift to nonequivalent concepts in the higher setting. Therefore, even if n-exact sequences are easy to identify, the search for the best definition for higher analogues of classical notions is not an easy task. A breakthrough in this direction was achieved in [20], where the definitions of n-abelian and n-exact categories were introduced and in [11], [24] where it is shown that any n-abelian category arises as an n-cluster tilting subcategory of an abelian category.

A key notion in representation theory and homological algebra is the concept of torsion theories, introduced by Dickson in [9]. Their natural relevance, for example, in relation to derived categories and tilting theory, have led to the use of homological algebra in many branches of mathematics, including algebraic geometry and mathematical physics. Torsion theory is built on the notion of torsion pairs, where a torsion pair is a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ with no nonzero morphism from the torsion class $\mathcal{T}$ to the torsion-free class $\mathcal{F}$. A definition of torsion classes in higher homological algebra has recently been given by Jørgensen in [22] based on the classical characterization of the existence of a unique torsion subobject and a unique torsion-free quotient for every object in the category.

Part of our motivation for writing this paper is to show that when considering an n-abelian category in the context of its ambient abelian category, that is viewed as an n-cluster tilting subcategory of an abelian category, then the definition of higher torsion classes in [22] seems to encode all the relevant properties one would expect. More precisely, one of the main ideas of the paper is built on the comparison of n-torsion classes in an n-abelian category and the corresponding minimal torsion classes generated by these n-torsion classes in an abelian category which contains the n-abelian category as an n-cluster-tilting subcategory. In particular, we do this in the context of Harder–Narasimhan filtrations and n-Harder–Narasimhan filtrations which we define and which we also study in the context of Galois coverings.

Our first result characterizes the minimal torsion classes in an abelian category containing an n-torsion class. Note that for every $n \geq 1$, the set of all n-torsion classes in an n-abelian category $\mathscr{M}$ forms a poset under the natural order given by the inclusion (see Corollary 3.3 and Theorem 3.6).

THEOREM 1.1. *Let $\mathscr{M}$ be an n-cluster-tilting subcategory of a skeletally small abelian length category $\mathcal{A}$. Then there is an injective morphism of posets*

$$T : \{n\text{-torsion classes in } \mathscr{M}\} \to \{\text{torsion classes in } \mathcal{A}\}$$

given by sending an n-torsion class $\mathscr{U}$ in $\mathscr{M}$ to the minimal torsion class in $\mathcal{A}$ containing $\mathscr{U}$. Moreover, a torsion class $\mathcal{T}$ in $\mathcal{A}$ is of the form $T(\mathscr{U})$, for $\mathscr{U}$ an n-torsion class in $\mathscr{M}$, if and only if the following hold:

1. *$tM \in \mathscr{U}$ for all $M \in \mathscr{M}$; where $t = t_{\mathcal{T}}$ is the torsion functor associated to the torsion class $\mathcal{T}$;*
2. *$\mathcal{T}$ is the minimal torsion class in $\mathcal{A}$ containing $\{tM : M \in \mathscr{M}\}$; and*
3. *$Ext^{n-1}_{\mathcal{A}}(X,Y) = 0$, for all $X \in \{tM : M \in \mathscr{M}\}$ and $Y \in \{coker(tM \hookrightarrow M) \mid M \in \mathscr{M}\}$.*

In this case, $\mathscr{U} = \mathcal{T} \cap \mathscr{M} = \{tM : M \in \mathscr{M}\}$.

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Stability conditions were introduced in [26] to attack algebro-geometric problems. Given their simplicity and effectiveness, their definition was later adapted to other contexts, such as quiver representations [23], [31], abelian categories [30], and triangulated categories [5].

One of the features of stability conditions relies on the fact that every stability condition determines for each nonzero object in a category a stratification by more well-behaved objects. This stratification, usually known as the *Harder–Narasimhan filtration*, has been used to make possible calculations that otherwise would be highly complicated or even impossible. Applications of this can be found, for example, in the study of Donaldson–Thomas invariants and in the mirror symmetry program [6], [28].

In a recent paper [34], Treffinger has introduced an axiomatic approach to Harder–Narasimhan filtrations for abelian categories, by showing that the existence of such a filtration for every object in an abelian category is equivalent to the existence of a chain of torsion classes in the category. Since this construction of Harder–Narasimhan filtrations does not depend on the existence of a stability condition, it allows the introduction of Harder–Narasimhan filtrations to nonabelian settings such as quasi-abelian categories [32]. In the second main result of this paper, we push this idea further by showing that chains of n-torsion classes induce Harder–Narasimhan filtrations in n-abelian categories. Moreover, we show that the Harder–Narasimhan filtrations obtained in this way coincide with the Harder–Narasimhan filtrations in the ambient abelian category which contains the n-abelian category as an n-cluster tilting subcategory (see Theorems 4.6 and 5.2).

THEOREM 1.2. *Let $\mathscr{M}$ be an n-abelian category, and let δ be a chain of n-torsion classes in $\mathscr{M}$. Then, for M, a nonzero object in $\mathscr{M}$, the following hold:*

1. *δ induces an n-Harder–Narasimhan filtration of M which is unique up to isomorphism.*
2. *If $\mathscr{M}$ is the n-cluster-tilting subcategory of a skeletally small abelian length category $\mathcal{A}$, then the n-Harder–Narasimhan filtration of M in $\mathscr{M}$ induced by δ is equal to the Harder–Narasimhan filtration of M in $\mathcal{A}$ induced by $T(\delta)$, where $T(\delta) = \{T(\mathscr{U}) \mid \mathscr{U} \in \delta\}$.*

Note that we need the *skeletally small* assumption in Theorems 1.1 and 1.2(2) because for these results we rely on Theorem 7.3 of [24], also Theorem 4.3 of [11], showing that under this assumption, any n-abelian category is embedded an abelian category as an n-cluster-tilting subcategory of $\mathcal{A}$ (see Theorem 2.4).

Another important concept in representation theory is the notion of Galois coverings, introduced by Gabriel in [12], [13] and studied further by many authors since. The initial aim was to reduce a problem for modules over an algebra A to that of a category $\mathcal{C}$ with an action of a group G such that A is equivalent to the orbit category $\mathcal{C}/G$. The theory has much evolved since its inception leading to a vast body of literature on the subject [1], [3], [4], [7], [14], [25], [29]. In particular, it has been shown that several nice properties, such as local finiteness or Cohen–Maculay finiteness, are preserved by Galois coverings [2], [12]. Recently, Darpö and Iyama show in [7] that n-cluster-tilting subcategories are, under certain conditions, preserved by Galois coverings. Their construction is based on the fact that under certain technical conditions, which are described in Theorem 1.3, given a Galois covering functor $P : \mathcal{C} \longrightarrow \mathcal{C}/G$, there exists a Galois precovering functor $P_\bullet : \text{mod-}\mathcal{C} \longrightarrow \text{mod-}(\mathcal{C}/G)$, called pushdown functor, between the categories of finitely presented functors over $\mathcal{C}$ and $\mathcal{C}/G$ such that $P_\bullet(\mathscr{M})$ is an n-cluster-tilting subcategory in mod-$\mathcal{C}/G$, where $\mathscr{M}$ is a certain n-cluster tilting subcategory of mod-$\mathcal{C}$. We add to this by showing that under similar

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conditions as in [7], n-torsion classes, chains of n-torsion classes, and n-Harder–Narasimhan filtrations are preserved by Galois coverings. Note that some authors use *G-covering* instead of *Galois covering* for this generalized version of the classical Galois covering theory (see, e.g., [1]). More precisely, we show the following. Recall that the action of a group G on $\mathcal{C}$ is called admissible if $gX \ncong X$ for each indecomposable object X in $\mathcal{C}$ and each $g \neq 1$ in G (see Definition 2.13; see also Theorem 6.1 and Proposition 6.3).

THEOREM 1.3. *Let $\mathcal{C}$ be a small locally bounded Krull–Schmidt $\Bbbk$-category with an admissible action of a group G on $\mathcal{C}$ inducing an admissible action on* mod-$\mathcal{C}$. *Suppose that $\mathscr{M}$ is an n-cluster-tilting G-equivariant full subcategory of* mod-$\mathcal{C}$ *such that $P_\bullet(\mathscr{M})$ is functorially finite in* mod-$(\mathcal{C}/G)$. *If $\mathscr{U}$ is a G-equivariant n-torsion class of $\mathscr{M}$, then $P_\bullet(\mathscr{U})$ is an n-torsion class of $P_\bullet(\mathscr{M})$.*

Moreover, if $\mathscr{U}$ is a G-equivariant n-torsion class in $\mathscr{M}$, then the following statements hold for $M \in \mathscr{M}$:

1. *An object U^M in $\mathscr{U}$ is the torsion object of M with respect to $\mathscr{U}$ if and only if $P_\bullet(U^M)$ is the torsion object of $P_\bullet(M)$ with respect to $P_\bullet(\mathscr{U})$.*
2. *If $\delta = \{\mathscr{U}_s : s \in [0,1]\}$ is a chain of G-equivariant n-torsion classes in $\mathscr{M}$, then*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{r-1} \subsetneq M_r = M$$

is the n-Harder–Narasimhan filtration of M with respect to δ in $\mathscr{M}$ if and only if

$$0 = P_\bullet(M_0) \subsetneq P_\bullet(M_1) \subsetneq \cdots \subsetneq P_\bullet(M_{r-1}) \subsetneq P_\bullet(M_r) = P_\bullet(M)$$

is the n-Harder–Narasimhan filtration of $P_\bullet(M)$ with respect to the chain of n-torsion classes $P_\bullet(\delta)$ in $P_\bullet(\mathscr{M})$.

3. *If $T(\mathscr{U})$ is G-equivariant, then $T(P_\bullet(\mathscr{U})) = P_\bullet(T(\mathscr{U}))$, that is, the following diagram is commutative:*

$$\begin{array}{ccc} \{G\text{-equivariant } n\text{-torsion classes in } \mathscr{M}\} & \xrightarrow{T(-)} & \{G\text{-equivariant torsion classes in mod-}\mathcal{C}\} \\ \downarrow{\scriptstyle P_\bullet(-)} & & \downarrow{\scriptstyle P_\bullet(-)} \\ \{n\text{-torsion classes in } P_\bullet(\mathscr{M})\} & \xrightarrow{T(-)} & \{\text{torsion classes in mod-}\mathcal{C}/G\}. \end{array}$$

§2. Background

An abelian category $\mathcal{A}$ is said to be a *length category* if every object of $\mathcal{A}$ is of finite length, that is, every object has a finite filtration starting with the zero object such that at each step the quotient is a simple object. Such a filtration usually is called a composition series or a Jordan–Hölder sequence of the object. A category $\mathcal{A}$ is called *skeletally small* if the class of all isomorphism classes of objects in $\mathcal{A}$ is a set. In this paper, whenever we say that $\mathcal{A}$ is an abelian category, we assume that $\mathcal{A}$ is a skeletally small abelian length category.

Given a full subcategory $\mathcal{X}$ of $\mathcal{A}$ which is closed under direct sums, we define the subcategory $\mathrm{Fac}(\mathcal{X})$ of $\mathcal{A}$ to be the full subcategory of $\mathcal{A}$ whose objects are quotients

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of objects in $\mathcal{X}$,

$$\mathrm{Fac}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists \text{ exact sequence } X \to Y \to 0, \text{ for some } X \in \mathcal{X}\}.$$

Similarly, the category $\mathrm{Sub}(\mathcal{X})$ is the full subcategory of $\mathcal{A}$ whose objects are subobjects of objects in $\mathcal{X}$,

$$\mathrm{Sub}(\mathcal{X}) = \{Y \in \mathcal{A} : \exists \text{ exact sequence } 0 \to Y \to X, \text{ for some } X \in \mathcal{X}\}.$$

We say that $\mathcal{X}$ is a *generating* subcategory of $\mathcal{A}$ if $\mathrm{Fac}(\mathcal{X}) = \mathcal{A}$. Dually, we say that $\mathcal{X}$ is *cogenerating* if $\mathrm{Sub}(\mathcal{X}) = \mathcal{A}$.

We say that an object M of $\mathcal{A}$ is filtered by $\mathcal{X}$ if there exists a finite sequence of subobjects

$$M_0 \subset M_1 \subset \cdots \subset M_n$$

such that $M_0 = 0$, $M_n = M$, and $M_i/M_{i-1} \in \mathcal{X}$ for all $1 \leq i \leq n$. We denote by $\mathrm{Filt}\mathcal{X}$ the full subcategory of all objects filtered by $\mathcal{X}$. Note that $\mathcal{X}$ is a full subcategory of $\mathrm{Fac}(\mathcal{X})$, $\mathrm{Sub}(\mathcal{X})$, and $\mathrm{Filt}(\mathcal{X})$.

2.1 n-cluster-tilting subcategories and n-abelian categories

Let n be an integer greater than or equal to 1. The theory of higher homological algebra started in [16], [17] with the study of the so-called *n-cluster-tilting subcategories* of module categories. Their definition for arbitrary abelian categories is the following.

Let us preface the definition by recalling some notions. Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$. We say that $\mathcal{X}$ is a contravariantly finite subcategory of $\mathcal{A}$ if every object $A \in \mathcal{A}$ admits a right $\mathcal{X}$-approximation, that is, for every $A \in \mathcal{A}$, there exists a morphism $\pi : M \to A$ with $M \in \mathcal{X}$ such that any other morphism $\pi' : M' \to A$, with $M' \in \mathcal{X}$, factors through π. Dually, the notion of covariantly finite subcategories is defined. A functorially finite subcategory of $\mathcal{A}$ is a subcategory which is both contravariantly and covariantly finite.

Definition 2.1 [20, Def. 3.14]. Let $\mathcal{A}$ be an abelian category. A functorially finite generating-cogenerating subcategory $\mathscr{M}$ of $\mathcal{A}$ is *n-cluster-tilting* if

$$\begin{aligned}\mathscr{M} &= \{X \in \mathcal{A} : \mathrm{Ext}^i_{\mathcal{A}}(X, M) = 0 \text{ for all } M \in \mathscr{M} \text{ and all } 1 \leq i \leq n-1\} \\ &= \{Y \in \mathcal{A} : \mathrm{Ext}^i_{\mathcal{A}}(M, Y) = 0 \text{ for all } M \in \mathscr{M} \text{ and all } 1 \leq i \leq n-1\}.\end{aligned}$$

The concept of n-abelian category was introduced in [20] as a generalization of the classical concept of abelian categories, to formalize the homological structure of n-cluster-tilting subcategories. The formal definition uses the notions of n-kernel and n-cokernel of a morphism, that we now recall. Let $f^0 : X^0 \to X^1$ be a morphism in an additive category $\mathscr{M}$. A sequence of morphisms

$$X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}$$

is called an *n-cokernel of f^0* if, for every $M \in \mathscr{M}$, the following sequence

$$0 \to \mathscr{M}(X^{n+1}, M) \to \mathscr{M}(X^n, M) \to \cdots \to \mathscr{M}(X^1, M) \to \mathscr{M}(X^0, M)$$

of abelian groups is exact. An n-cokernel of f^0 is denoted by $(f^1, \ldots, f^n)$. The notion of *n-kernel of a morphism* is defined dually. The sequence

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}$$

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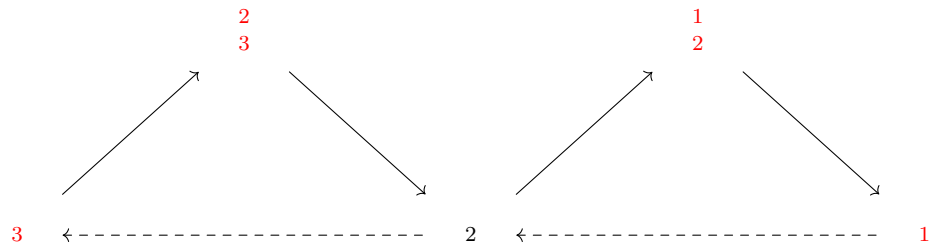

Figure 1.
The Auslander–Reiten quiver of A.

is called *n-exact* if $(f^1,\dots,f^n)$ is an n-cokernel of f^0 and $(f^0,\dots,f^{n-1})$ is an n-kernel of f^n.

DEFINITION 2.2 [20, Def. 3.1]. Let n be a positive integer. An additive category $\mathscr{M}$ is *n-abelian* if the following axioms hold:

(A0) $\mathscr{M}$ has split idempotents.
(A1) Every morphism in $\mathscr{M}$ has an n-kernel and an n-cokernel.
(A2) For every monomorphism $f^0 : X^0 \to X^1$ and any n-cokernel $(f^1,\dots,f^n)$ of f^0, the following sequence is n-exact:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

(A2^{op}) For every epimorphism $f^n : X^n \to X^{n+1}$ and any n-kernel $(f^0,\dots,f^{n-1})$ of f^n, the following sequence is n-exact:

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{n-1}} X^n \xrightarrow{f^n} X^{n+1}.$$

The motivating example for n-abelian categories are n-cluster-tilting subcategories, and indeed, as stated below, it is now known that these are the only small n-abelian categories.

THEOREM 2.3 [20, Th. 3.16]. *Let $\mathscr{M}$ be an n-cluster-tilting subcategory of an abelian category $\mathcal{A}$. Then $\mathscr{M}$ is n-abelian.*

It is worth noticing that all n-exact sequences in $\mathscr{M}$ are the n-extensions of $\mathcal{A}$ where all terms of the extensions are in $\mathscr{M}$. The converse of the previous result also holds.

THEOREM 2.4 [11, Th. 4.3], [24, Th. 7.3]. *Let $\mathscr{M}$ be an n-abelian category. Then there exists an abelian category $\mathcal{A}$ and a fully faithful functor $F : \mathscr{M} \to \mathcal{A}$ such that $\mathcal{F}(\mathscr{M})$ is an n-cluster-tilting subcategory of $\mathcal{A}$.*

Before going any further, we introduce a running example that helps us to illustrate most of the results of this paper.

EXAMPLE 2.5. Let A be the path algebra of the quiver tikz2 modulo the ideal generated by the relation $\alpha\beta$. The Auslander–Reiten quiver of A can be seen in Figure 1, where the dashed arrows correspond to the Auslander–Reiten translations in mod-A. It is well known that the subcategory

$$\mathscr{M} = \operatorname{add}\left\{ {}_3 \oplus {}^{2}_{3} \oplus {}^{1}_{2} \oplus {}_1 \right\}$$

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is a 2-cluster-tilting subcategory of mod-A. In Figure 1, you can find in red the indecomposable objects of mod-A that belong to $\mathscr{M}$.

2.2 Torsion and n-torsion classes

Generalizing the classical properties of abelian groups, Dickson introduced in [9] the notion of *torsion pair* as follows.

DEFINITION 2.6. Let $\mathcal{A}$ be an abelian category. Then the pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ is a *torsion pair* if the following conditions are satisfied:

- $\mathrm{Hom}_{\mathcal{A}}(X,Y) = 0$ for all $X \in \mathcal{T}$ and $Y \in \mathcal{F}$.
- For every module M in $\mathcal{A}$, there exists a short exact sequence

$$0 \to tM \xrightarrow{\iota_M} M \xrightarrow{\pi_M} fM \to 0,$$

where $tM \in \mathcal{T}$ and $fM \in \mathcal{F}$.

This short exact sequence is unique up to isomorphisms and is known as the *canonical short exact sequence* of M with respect to $(\mathcal{T}, \mathcal{F})$. Moreover, we say that $\mathcal{T}$ is a *torsion class* and $\mathcal{F}$ is a *torsion free class*.

In the same paper [9] where he introduced the concept of torsion pair, Dickson gave an useful characterization of torsion and torsion-free classes.

THEOREM 2.7 [9, Th. 2.3]. *A full subcategory $\mathcal{T}$ of an abelian category $\mathcal{A}$ is a torsion class if and only if $\mathcal{T}$ is closed under factors and extensions. Dually, a full subcategory $\mathcal{F}$ of an abelian category $\mathcal{A}$ is a torsion-free class if and only if $\mathcal{F}$ is closed under subobjects and extensions.*

We denote by $\mathrm{tors}(\mathcal{A})$ the set of all torsion classes in $\mathcal{A}$. It is clear that the natural inclusion of sets induces a natural partial order in $\mathrm{tors}(\mathcal{A})$.

Given a subcategory $\mathcal{X}$ of $\mathcal{A}$, we denote by $T(\mathcal{X})$ the minimal torsion class of $\mathcal{A}$ containing $\mathcal{X}$. It is well known that $T(\mathcal{X})$ coincides with all the objects of $\mathcal{A}$ filtered by elements in the category $\mathrm{Fac}(\mathcal{X})$, that is, $T(\mathcal{X}) = \mathrm{Filt}(\mathrm{Fac}(\mathcal{X}))$. For a proof, see [33, Prop. 2.1].

With the development of higher homological algebra, it is natural to consider higher analogues of torsion classes in this framework. The first such notion is introduced in [22]. The formal definition is as follows.

DEFINITION 2.8 [22, Def. 1.1]. Let $\mathscr{M}$ be an n-abelian category. A full subcategory $\mathscr{U}$ of $\mathscr{M}$ is an *n-torsion class* if for every $M \in \mathscr{M}$ there exists an n-exact sequence

$$0 \to U^M \to M \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0, \tag{1}$$

where U^M is an object of $\mathscr{U}$, and the sequence

$$0 \to \mathrm{Hom}_{\mathscr{M}}(U,V^1) \to \mathrm{Hom}_{\mathscr{M}}(U,V^2) \to \cdots \to \mathrm{Hom}_{\mathscr{M}}(U,V^n) \to 0 \tag{2}$$

is exact, for all objects U in $\mathscr{U}$. U^M is called the n-torsion subobject of M with respect to $\mathscr{U}$.

2.3 Harder–Narasimhan filtrations in abelian categories

Inspired by the relation between stability conditions and torsion classes, in [34], the relation between Harder–Narasimhan filtrations and torsion classes was studied. This was done through the introduction of chains of torsion classes as follows.

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DEFINITION 2.9 [34, Def. 2.1]. A chain of torsion classes η in an abelian category $\mathcal{A}$ is a set of torsion classes

$$\eta := \{\mathcal{T}_s : s \in [0,1],\ \mathcal{T}_0 = \mathcal{A}, \mathcal{T}_1 = \{0\} \text{ and } \mathcal{T}_s \subseteq \mathcal{T}_r \text{ if } r \leq s\}.$$

We denote by $\mathcal{T}(\mathcal{A})$ the set of all chains of torsion classes of $\mathcal{A}$.

Associated to every chain of torsion classes $\eta \in \mathcal{T}(\mathcal{A})$, there is a set

$$\mathcal{P}_\eta = \{\mathcal{P}_t : t \in [0,1]\}$$

of full subcategories of $\mathcal{A}$ where each $\mathcal{P}_t$ is defined as follows. Note that in from now on we assume $\bigcap_{s<0} \mathcal{T}_s = \mathcal{A}$ and $\bigcup_{s>1} \mathcal{T}_s = \{0\}$.

DEFINITION 2.10. Consider a chain of torsion classes $\eta \in \mathcal{T}(\mathcal{A})$. For every $t \in [0,1]$, we have the subcategory $S_t = \bigcap_{s<t} \mathcal{T}_s \setminus \bigcup_{r>t} \mathcal{T}_r$. We define $\mathcal{P}_t$ to be

$$\mathcal{P}_t := \{\operatorname{coker}(\alpha) : \text{where } \alpha : tM_{>t} \to M \text{ and } M \in S_t\},$$

where $t(-)_{>t}$ is the torsion functor associated to the torsion class $\bigcup_{s>t} \mathcal{T}_s$ applied to M. We then define the *slicing* $\mathcal{P}_\eta$ of η to be the set $\mathcal{P}_\eta := \{\mathcal{P}_t : t \in [0,1]\}$.

We note that this definition is different of the original definition introduced in [34, Def. 2.8]. In the following proposition, we show that both definitions are equivalent.

PROPOSITION 2.11. *Let η be a chain of torsion classes in $\mathcal{T}(\mathcal{A})$. Then, for every $t \in [0,1]$, we have that*

$$\mathcal{P}_t = \left(\bigcap_{s<t} \mathcal{T}_s\right) \cap \left(\bigcap_{s>t} \mathcal{F}_s\right),$$

where $\mathcal{F}_s$ is the torsion-free class such that $(\mathcal{T}_s, \mathcal{F}_s)$ is a torsion pair in $\mathcal{A}$.

Proof. First recall that, for every $t \in [0,1]$, by [34, Prop. 2.6], the tuple

$$\left(\bigcup_{s>t} \mathcal{T}_s, \bigcap_{s>t} \mathcal{F}_s\right)$$

is a torsion pair. We now prove the statement by double inclusion.

Fix $t \in [0,1]$ and $X \in \left(\bigcap_{s<t} \mathcal{T}_s\right) \cap \left(\bigcap_{s>t} \mathcal{F}_s\right)$. Then, in particular, $X \in \bigcap_{s<t} \mathcal{T}_s$. Moreover, the short exact sequence of M associated to the torsion pair $\left(\bigcup_{s>t} \mathcal{T}_s, \bigcap_{s>t} \mathcal{F}_s\right)$ is isomorphic to

$$0 \to 0 \to X \to X \to 0,$$

because $X \in \bigcup_{s>t} \mathcal{T}_s$. Hence $X \in \{\operatorname{coker}(\alpha) : \text{where } \alpha : tM_{>t} \to M \text{ and } M \in S_t\}$.

In the other direction, take $X \in \{\operatorname{coker}(\alpha) : \text{where } \alpha : tM_{>t} \to M \text{ and } M \in S_t\}$. Then X is the torsion-free quotient of an object $X' \in \bigcap_{s<t} \mathcal{T}_s \setminus \bigcup_{r>t} \mathcal{T}_r$ with respect to the torsion pair $\left(\bigcup_{s>t} \mathcal{T}_s, \bigcap_{s>t} \mathcal{F}_s\right)$. This implies that $X \in \bigcap_{s>t} \mathcal{F}_s$. Moreover, since $\bigcap_{s<t} \mathcal{T}_s$ is a torsion class, we know that is close under quotients. So, $X \in \bigcap_{s<t} \mathcal{T}_s$. Hence, $X \in \left(\bigcap_{s<t} \mathcal{T}_s\right) \cap \left(\bigcap_{s>t} \mathcal{F}_s\right)$. ∎

One of the main results in [34] shows that every chain of torsion classes $\eta \in \mathcal{T}(\mathcal{A})$ induces a Harder–Narasimhan filtration for every nonzero object $M \in \mathcal{A}$. The formal statement is the following.

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THEOREM 2.12 [34, Th. 2.9]. *Let $\mathcal{A}$ be an abelian category and $\eta \in \mathcal{T}(\mathcal{A})$. Then every object $M \in \mathcal{A}$ admits a Harder–Narasimhan filtration with respect to η. That is, a filtration*

$$M_0 \subset M_1 \subset \cdots \subset M_n$$

such that

1. $0 = M_0$ *and* $M_n = M$;
2. *there exists* $r_k \in [0,1]$ *such that* $M_k/M_{k-1} \in \mathcal{P}_{r_k}$ *for all* $1 \leq k \leq n$; *and*
3. $r_1 > r_2 > \cdots > r_n$.

Moreover, this filtration is unique up to isomorphism.

2.4 Galois coverings

Let $\mathcal{C}$ be a skeletally small Krull–Schmidt $\Bbbk$-category, where $\Bbbk$ is a field. Given a group G, we say that there is a G-action over $\mathcal{C}$ or simply $\mathcal{C}$ is a *G-category* if there is a group homomorphism $A : G \longrightarrow \mathrm{Aut}(\mathcal{C})$, where $\mathrm{Aut}(\mathcal{C})$ denotes the group of $\Bbbk$-linear automorphisms of $\mathcal{C}$. We usually write A_g instead of $A(g)$ and gX for $A_g(X)$, for each $g \in G$ and $X \in \mathcal{C}$.

DEFINITION 2.13. With the above notations, the action of G on $\mathcal{C}$ is called *admissible* if $gX \ncong X$ for each indecomposable object X in $\mathcal{C}$ and each $g \neq 1$ in G.

Let $\mathcal{C}$ be a G-category. The *orbit category* $\mathcal{C}/G$ of $\mathcal{C}$ by G is a category whose objects are the objects of $\mathcal{C}$ and for every $X, Y \in \mathcal{C}/G$, the morphism set $\mathcal{C}/G(X,Y)$ is given by

$$\left\{ (f_{h,g})_{(g,h)} \in \prod_{(g,h)\in G\times G} \mathcal{C}(gX, hY) \mid (f_{h,g})_{(g,h)} \text{ is } \mathsf{rcf} \text{ and } f_{g'h,g'g} = g'(f_{h,g}), \forall g' \in G \right\},$$

where rcf denotes $(f_{h,g})_{(g,h)}$ being row and column finite, that is, for every $g \in G$, there are finitely many $h \in G$ such that $f_{h,g} \neq 0$ and, dually, for every $h' \in G$, there are finitely many $g' \in G$ such that $f_{h',g'} \neq 0$. For two composable morphisms $X \xrightarrow{f} Y \xrightarrow{f'} Z$ in $\mathcal{C}/G$, we define

$$f'f := \left(\sum_{g'\in G} f'_{h,g'} f_{g',g} \right)_{(g,h)\in G\times G}.$$

There is a canonical functor $P : \mathcal{C} \longrightarrow \mathcal{C}/G$ which is given by $P(X) = X$ and $P(f) = (\delta_{g,h} g f)_{(g,h)}$, for every $X, Y \in \mathcal{C}$ and for every $f \in \mathcal{C}(X,Y)$.

Recall that a pair (F, φ) of a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ and a family $\varphi := (\varphi_g)_{g\in G}$ of natural isomorphisms $\varphi_g : F \longrightarrow FA_g$ is called a G-invariant functor if, for every $g, h \in G$, the following diagram is commutative:

$$\begin{array}{ccc} F & \xrightarrow{\varphi_g} & FA_g \\ & \searrow^{\varphi_{hg}} & \downarrow{\scriptstyle \varphi_h A_g} \\ & & FA_{hg} = FA_hA_g. \end{array}$$

The family $\varphi := (\varphi_g)_{g\in G}$ is called an *invariant adjuster of F*.

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DEFINITION 2.14 [1, Def. 1.7]. Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ be a G-invariant functor. Then F is called a *G-precovering functor*, if for every $X, Y \in \mathcal{C}$ the $\Bbbk$-homomorphisms

$$F^{(1)}_{X,Y} : \bigoplus_{g\in G} \mathcal{C}(gX, Y) \longrightarrow \mathcal{C}'(FX, FY), \quad (f_g)_{g\in G} \mapsto \sum_{g\in G} F(f_g)\varphi_{g,X},$$

$$F^{(2)}_{X,Y} : \bigoplus_{g\in G} \mathcal{C}(X, gY) \longrightarrow \mathcal{C}'(FX, FY), \quad (f_g)_{g\in G} \mapsto \sum_{g\in G} \varphi_{g^{-1},gY} F(f_g),$$

are isomorphisms. If F is also dense, then it is called a *G-covering functor*.

It is shown in [1, Prop. 2.6] that $P : \mathcal{C} \longrightarrow \mathcal{C}/G$ is a G-covering functor which is universal among G-invariant functors starting from $\mathcal{C}$.

We say that the $\Bbbk$-category $\mathcal{C}$ is locally bounded if, for each indecomposable $X \in \mathcal{C}$, we have that

$$\sum_{Y\in \text{ind-}\mathcal{C}} (\dim_{\Bbbk}(\text{Hom}_{\mathcal{C}}(X, Y) + \dim_{\Bbbk}(\text{Hom}_{\mathcal{C}}(Y, X)) < \infty.$$

From now on, assume moreover that $\mathcal{C}$ is a locally bounded $\Bbbk$-category.

Let $\mathcal{C}$ be a G-category. The G-action on $\mathcal{C}$ induces a G-action on Mod-$\mathcal{C}$, where Mod-$\mathcal{C}$ denotes the category of contravariant functors from $\mathcal{C}$ to Mod-$\Bbbk$. In fact, for each $g \in G$, we can define an automorphism $\overline{A}_g : \text{Mod-}\mathcal{C} \longrightarrow \text{Mod-}\mathcal{C}$ by

$$\overline{A}_g(M) = {}^gM := M \circ A_g^{-1},$$

for all $M \in \text{Mod-}\mathcal{C}$. It follows from the definitions that for every $M \in \mathcal{C}$, ${}^g\mathcal{C}(-, M) = \mathcal{C}(g^{-1}(-), M) \cong \mathcal{C}(-, gM)$.

The canonical functor $P : \mathcal{C} \longrightarrow \mathcal{C}/G$ induces a functor $P^\bullet : \text{Mod-}(\mathcal{C}/G) \longrightarrow \text{Mod-}\mathcal{C}$, given by $P^\bullet(M) = M \circ P$ for every $M \in \text{Mod-}(\mathcal{C}/G)$. This functor is called the *pull-up of P*. It is well known that $P^\bullet$ possesses a left adjoint $P_\bullet : \text{Mod-}\mathcal{C} \longrightarrow \text{Mod-}(\mathcal{C}/G)$, which is called the *pushdown functor*. For an explicit description of this functor, see the proof of Theorem 4.3 of [1]. It follows that the pushdown functor $P_\bullet$ is exact.

A functor $M \in \text{Mod-}\mathcal{C}$ is called *finitely presented* if there exists an exact sequence $(-, Y) \longrightarrow (-, X) \longrightarrow M \longrightarrow 0$ in Mod-$\mathcal{C}$. Let mod-$\mathcal{C}$ be the full subcategory of Mod-$\mathcal{C}$ consisting of all finitely presented modules. It is known [1, Th. 4.3] that the restriction of the pushdown functor to mod-$\mathcal{C}$ induces a functor

$$P_\bullet : \text{mod-}\mathcal{C} \longrightarrow \text{mod-}(\mathcal{C}/G)$$

again denoted by $P_\bullet$, which is a G-precovering functor.

The central result relating higher homological algebra and Galois coverings was proved by Darpö and Iyama, and it reads as follows. Recall that a subcategory $\mathcal{X}$ of mod-$\mathcal{C}$ is said to be *G-equivariant* if ${}^g\mathcal{X} = \mathcal{X}$, for all $g \in G$.

THEOREM 2.15 [7, Th. 2.14]. *Let $\mathcal{C}$ be a locally bounded Krull–Schmidt G-category with an admissible action of G on $\mathcal{C}$ inducing an admissible action on* mod-$\mathcal{C}$. *If $\mathscr{M}$ is a G-equivariant full subcategory of* mod-$\mathcal{C}$ *such that $P_\bullet(\mathscr{M})$ is functorially finite in* mod-$(\mathcal{C}/G)$, *then $\mathscr{M}$ is an n-cluster-tilting subcategory of* mod-$\mathcal{C}$ *if and only if $P_\bullet(\mathscr{M})$ is an n-cluster-tilting subcategory of* mod-$(\mathcal{C}/G)$.

For more details on the covering theory, we refer the reader to [1], [3], [7].

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§3. Minimal torsion classes containing n-torsion classes

Let $\mathscr{M}$ be an n-abelian category. By [24], there exist an abelian category $\mathcal{A}$ and a fully faithful functor $F:\mathscr{M}\to\mathcal{A}$ such that $F(\mathscr{M})$ is an n-cluster-tilting subcategory of $\mathcal{A}$. Throughout the section, we fix an n-abelian category $\mathscr{M}$ and consider it as the n-cluster tilting subcategory of the abelian category $\mathcal{A}$.

For an n-torsion class $\mathscr{U}\subseteq\mathscr{M}$, let $T(\mathscr{U})$ denote the smallest torsion class of $\mathcal{A}$ containing $\mathscr{U}$ and for any $M\in\mathscr{M}$, denote by U^M the n-torsion object of M with respect to the n-torsion class $\mathscr{U}$. For a torsion class $\mathcal{T}$ in $\mathcal{A}$ and for $M\in\mathcal{A}$, denote by tM the torsion object of M with respect to $\mathcal{T}$.

LEMMA 3.1. *Let $\mathscr{U}$ be an n-torsion class in $\mathscr{M}$, and let $M\in\mathscr{M}$. Then,*

$$tM\cong U^M,$$

where tM is the torsion object of M with respect to $T(\mathscr{U})$. In other words, for all $M\in\mathscr{M}$, the fundamental n-exact sequence

$$0\to U^M\xrightarrow{u}M\to V^1\xrightarrow{v^1}\cdots\xrightarrow{v^{n-1}}V^n\to 0 \tag{3}$$

of M with respect to $\mathscr{U}$ is isomorphic to

$$0\to tM\xrightarrow{\iota_M}M\to V^1\xrightarrow{v^1}\cdots\xrightarrow{v^{n-1}}V^n\to 0. \tag{4}$$

Proof. Let $M\in\mathscr{M}$, and take the canonical n-exact sequence of M with respect to $\mathscr{U}$

$$0\to U^M\xrightarrow{u}M\to V^1\xrightarrow{v_1}V^2\to\cdots\to V^n\to 0. \tag{5}$$

This induces the following exact sequence in $\mathcal{A}$:

$$0\to\operatorname{coker}u\to V^1\xrightarrow{v^1}V^2\xrightarrow{v^2}\cdots\xrightarrow{v^{n-1}}V^n\to 0. \tag{6}$$

Applying the functor $\mathscr{M}(U,-)$ with $U\in\mathscr{U}$, we obtain the following exact sequence:

$$0\to\mathscr{M}(U,\operatorname{coker}u)\to\mathscr{M}(U,V^1)\xrightarrow{v^1_*}\mathscr{M}(U,V^2). \tag{7}$$

However, from the definition of n-torsion class, we have that

$$0\to\mathscr{M}(U,V^1)\xrightarrow{v^1_*}\cdots\xrightarrow{v^{n-1}_*}\mathscr{M}(U,V^n)\to 0 \tag{8}$$

is exact for every $U\in\mathscr{U}$. In particular, the exactness of the sequence (8) implies that $v_1^*:\mathscr{M}(U,V^1)\to\mathscr{M}(U,V^2)$ is injective. Thus, $\mathscr{M}(U,\operatorname{coker}u)=0$ for every $U\in\mathscr{U}$. This implies that

$$0\to U^M\xrightarrow{u}M\longrightarrow\operatorname{coker}u\to 0$$

is such that $U^M\in T(\mathscr{U})$ and $\operatorname{coker}u\in(T(\mathscr{U}))^{\perp}$, where for a class $\mathcal{T}$ of objects of $\mathcal{A}$,

$$\mathcal{T}^{\perp}=\{Y\in\mathcal{A}\mid\operatorname{Hom}_{\mathcal{A}}(X,Y)=0,\ \text{for all}\ \ X\in\mathcal{T}\}.$$

Since the canonical short exact sequence of any object with respect to a torsion pair is unique up-to-isomorphism, we conclude that $tM\cong U^M$. □

As a direct consequence of Lemma 3.1, we obtain the following result.

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PROPOSITION 3.2. *Let $\mathscr{U}_1, \mathscr{U}_2$ be two n-torsion classes in $\mathscr{M}$. Then $T(\mathscr{U}_1) = T(\mathscr{U}_2)$ if and only if $\mathscr{U}_1 = \mathscr{U}_2$.*

Proof. The sufficiency is clear, so we show the necessity. Let $M \in \mathscr{U}_2 \setminus \mathscr{U}_1$. Then the torsion object U_1^M of M with respect to $\mathscr{U}_1$ is not isomorphic to $U_2^M = M$. By Lemma 3.1, we have that the torsion object $t_1 M$ of M with respect to $T(\mathscr{U}_1)$ is not isomorphic to the torsion object $t_2 M = M$ of M with respect to $T(\mathscr{U}_2)$. Hence, $T(\mathscr{U}_1)$ is different from $T(\mathscr{U}_2)$. □

In recent years, there has been a great deal of interest regarding the poset of torsion classes in the module category of an algebra ordered by inclusion. As a consequence of our previous result, we have the following.

COROLLARY 3.3. *The map $T(-) : n\text{-}tors(\mathscr{M}) \to tors(\mathcal{A})$ from the set of n-torsion classes $n\text{-}tors(\mathscr{M})$ of the n-cluster-tilting subcategory $\mathscr{M}$ to the set of torsion classes $tors(\mathcal{A})$ of the abelian category $\mathcal{A}$ is injective and respects the order given by the inclusion.*

Proof. It follows from Proposition 3.2 that if $\mathscr{U}_1$ and $\mathscr{U}_2$ are two distinct n-torsion classes in $\mathscr{M}$, then $T(\mathscr{U}_1)$ and $T(\mathscr{U}_2)$ are two distinct torsion classes in $\mathcal{A}$. The fact that $T(\mathscr{U}_1) \subset T(\mathscr{U}_2)$ if $\mathscr{U}_1 \subset \mathscr{U}_2$ is immediate. □

LEMMA 3.4. *Let $\mathscr{U}$ be an n-torsion class of $\mathscr{M} \subset \mathcal{A}$. If $\mathcal{T} = T(\mathscr{U})$ is the minimal torsion class of $\mathcal{A}$ containing $\mathscr{U}$, then $\mathscr{U} = t\mathscr{M}$, where $t\mathscr{M} = \{tM : M \in \mathscr{M}\}$. In particular, we have $\mathcal{T} = T(t\mathscr{M})$.*

Proof. The fact that $\mathscr{U} \supseteq \{tM : M \in \mathscr{M}\}$ follows directly from Lemma 3.1. Let $U \in \mathscr{U} \subset \mathscr{M}$. Then $U \in \mathcal{T} = T(\mathscr{U})$. Hence, $tU = U$. Thus, $U \in \{tM : M \in \mathscr{M}\}$. □

LEMMA 3.5. *Let $\mathscr{M} \subseteq \mathcal{A}$ be an n-cluster-tilting subcategory, and let $\mathcal{T} \subseteq \mathcal{A}$ be a torsion class satisfying $t\mathscr{M} \subseteq \mathscr{M}$. Then $t\mathscr{M}$ is an n-torsion class of $\mathscr{M}$ if and only if $Ext_{\mathcal{A}}^{n-1}(X, Y) = 0$ for all $X \in t\mathscr{M}$ and all $Y \in f\mathscr{M}$, where $f\mathscr{M} = \{coker(tM \hookrightarrow M) \mid M \in \mathscr{M}\}$.*

Proof. Assume first that $t\mathscr{M} \subseteq \mathscr{M}$ is an n-torsion class. Let $X \in t\mathscr{M}$ and $Y \in f\mathscr{M}$ be arbitrary elements. So, $Y = fM$, for some $M \in \mathscr{M}$, and there exists a short exact sequence

$$0 \to tM \xrightarrow{\iota_M} M \to fM \to 0, \tag{9}$$

where the monomorphism $tM \overset{\iota_M}{\hookrightarrow} M$ sits in the canonical n-exact sequence

$$0 \to tM \xrightarrow{\iota_M} M \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0, \tag{10}$$

in which the sequence

$$0 \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0 \tag{11}$$

is $t\mathscr{M}$-exact.

The sequence (9) induces a long exact sequence which contains

$$\mathrm{Ext}_{\mathcal{A}}^{n-1}(X, M) \to \mathrm{Ext}_{\mathcal{A}}^{n-1}(X, fM) \to \mathrm{Ext}_{\mathcal{A}}^{n}(X, tM) \xrightarrow{\mathrm{Ext}_{\mathcal{A}}^{n}(X, \iota_M)} \mathrm{Ext}_{\mathcal{A}}^{n}(X, M). \tag{12}$$

Since $X \in t\mathscr{M} \subseteq \mathscr{M}$ while $\mathscr{M}$ is n-cluster tilting, the first term is zero and so the sequence reads as follows:

$$0 \to \mathrm{Ext}_{\mathcal{A}}^{n-1}(X, fM) \to \mathrm{Ext}_{\mathcal{A}}^{n}(X, tM) \xrightarrow{\mathrm{Ext}_{\mathcal{A}}^{n}(X, \iota_M)} \mathrm{Ext}_{\mathcal{A}}^{n}(X, M). \tag{13}$$

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Moreover, $X \in \mathscr{M}$ implies that the sequence (10) induces a long exact Hom- Ext^n-sequence which contains the following (see [21, Prop. 2.2]):

$$\mathcal{A}(X, V^{n-1}) \xrightarrow{v_*^{n-1}} \mathcal{A}(X, V^n) \to \mathrm{Ext}^n_{\mathcal{A}}(X, tM) \xrightarrow{\mathrm{Ext}^n_{\mathcal{A}}(X, \iota_M)} \mathrm{Ext}^n_{\mathcal{A}}(X, M). \tag{14}$$

Since the sequence (11) is $t\mathscr{M}$-exact, we deduce that v_*^{n-1} is surjective and so the morphism $\mathrm{Ext}^n_{\mathcal{A}}(X, \iota_M)$ is injective. This in view of the sequence (13) implies that $\mathrm{Ext}^{n-1}_{\mathcal{A}}(X, Y) = 0$.

For the converse, assume that $\mathrm{Ext}^{n-1}_{\mathcal{A}}(X, Y) = 0$, for all $X \in t\mathscr{M}$ and all $Y \in f\mathscr{M}$. Consider $M \in \mathscr{M}$. Since $t\mathscr{M} \subseteq \mathscr{M}$, we have an n-exact sequence

$$0 \to tM \xrightarrow{\iota_M} M \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0 \tag{15}$$

in $\mathscr{M}$, where $\mathrm{coker}(tM \overset{\iota_M}{\hookrightarrow} M) = fM \in f\mathscr{M}$. To conclude the result, we should show that the sequence

$$0 \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0$$

is $t\mathscr{M}$-exact. Since (15) is an n-exact sequence, we just need to show that for every $X \in t\mathscr{M}$, the induced morphism $\mathcal{A}(X, v^1)$ is an injection and the induced morphism $\mathcal{A}(X, v^{n-1})$ is a surjection. However, $\mathcal{A}(X, v^1)$ is injective, because $\mathcal{A}(t\mathscr{M}, fM) = 0$. To see that $\mathcal{A}(X, v^{n-1})$ is surjective, it is enough to show that the morphism $\mathrm{Ext}^n_{\mathcal{A}}(X, \iota_M)$ in the sequence (14) is injective. However, this follows from the sequence (12) in view of the fact that by assumption $\mathrm{Ext}^{n-1}_{\mathcal{A}}(X, fM) = 0$. □

We are now in place to give a characterization of the torsion classes $\mathcal{T}$ in $\mathcal{A}$ which are of the form $\mathcal{T} = T(\mathscr{U})$ for some n-torsion class $\mathscr{U}$ of $\mathscr{M}$.

THEOREM 3.6. *Let $\mathscr{M} \subseteq \mathcal{A}$ be an n-cluster-tilting subcategory, and let $\mathcal{T}$ be a torsion class of $\mathcal{A}$. Then $\mathcal{T}$ is of the form $\mathcal{T} = T(\mathscr{U})$ for some n-torsion class $\mathscr{U}$ of $\mathscr{M}$ if and only if the following holds:*

1. *$t\mathscr{M} \subseteq \mathscr{M}$;*
2. *$\mathcal{T} = T(t\mathscr{M})$; and*
3. *$Ext^{n-1}_{\mathcal{A}}(X, Y) = 0$ for all $X \in t\mathscr{M}$ and $Y \in f\mathscr{M}$.*

Moreover, in this case, $\mathscr{U} = \mathcal{T} \cap \mathscr{M} = \{tM : M \in \mathscr{M}\}$.

Proof. *Necessity.* Suppose that $\mathcal{T} = T(\mathscr{U})$ for an n-torsion class $\mathscr{U} \subseteq \mathscr{M}$. We must show that $\mathcal{T}$ has all three characteristics as in the statement. Parts 1. and 2. follow from Lemma 3.4. In particular, we have $\mathscr{U} = t\mathscr{M} \subseteq \mathscr{M}$, so Lemma 3.5 applies to complete the proof of this part.

Sufficiency. Suppose that $\mathcal{T}$ is a torsion class in $\mathcal{A}$ satisfying 1.–3. We must show $\mathcal{T} = \mathcal{T}(\mathscr{U})$ for an n-torsion class $\mathscr{U} \subseteq \mathscr{M}$, and in view of 2., this holds if $t\mathscr{M}$ is an n-torsion class in $\mathscr{M}$. However, $t\mathscr{M} \subseteq \mathscr{M}$ holds by 1., and so by 3., in view of Lemma 3.5, the sequence (10) is a fundamental n-exact sequence for M with respect to $t\mathscr{M}$.

Now, we show the moreover part of the statement. It is already proved in Lemma 3.4 that $\mathscr{U} = \{tM : M \in \mathscr{M}\}$. So, we only need to prove that $\mathscr{U} = \mathcal{T} \cap \mathscr{M}$. We do it by double inclusion. The fact that $\mathscr{U} \subset \mathscr{M} \cap \mathcal{T}$ follows immediately from the fact that $\mathcal{T} = T(\mathscr{U})$. Now, if $X \in \mathscr{M} \cap \mathcal{T}$, we have that $X = tX$. Then $X \in \mathscr{U}$ by Lemma 3.1. □

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EXAMPLE 3.7. Let A as in Example 2.5, and consider the torsion classes $\mathcal{T}_1 = \operatorname{add}\{\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\}$ and $\mathcal{T}_2 = \operatorname{add}\{\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\}$ and $\mathcal{T}_3 = \operatorname{add}\{1 \oplus 3\}$ in mod-A.

First, we have that $\mathcal{T}_1$ verifies 1.–3. in the previous theorem and hence is the minimal torsion class containing the 2-torsion class $\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\}$. On the other hand, neither $\mathcal{T}_2$ nor $\mathcal{T}_3$ are minimal classes containing a 2-torsion class.

In the case of $\mathcal{T}_2$, one can see that 2. fails, that is, $\mathcal{T}_2 \neq t\mathscr{M}$. Indeed,

$$\mathcal{T}_2 = \operatorname{add}\left\{\begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\} \neq \operatorname{add}\left\{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\} = t\mathscr{M}.$$

Nevertheless, we point out that $\mathcal{T}_2 \cap \mathscr{M} = \operatorname{add}\left\{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\right\}$ is a 2-torsion class in $\mathscr{M}$.

Finally, in the case of $\mathcal{T}_3$, the problem is that 3. is not satisfied since the canonical exact sequence of $\begin{smallmatrix}2\\3\end{smallmatrix}$ with respect to the torsion pair $(\mathcal{T}_3, \mathcal{F}_3)$ is

$$0 \longrightarrow 3 \longrightarrow \begin{smallmatrix}2\\3\end{smallmatrix} \longrightarrow 2 \longrightarrow 0$$

and $\operatorname{Ext}^1_A(1,2) \ncong 0$. In fact, the minimal 2-torsion class in $\mathscr{M}$ containing $\operatorname{add}\{1 \oplus 3\}$ is the whole of $\mathscr{M}$.

§4. Harder–Narasimhan filtrations in n-abelian categories

We start this section by introducing chains of n-torsion classes.

DEFINITION 4.1. A chain of n-torsion classes δ in an n-abelian category $\mathscr{M}$ is a set of n-torsion classes

$$\delta := \{\mathscr{U}_s : s \in [0,1], \mathscr{U}_0 = \mathscr{M}, \mathscr{U}_1 = \{0\} \text{ and } \mathscr{U}_s \subseteq \mathscr{U}_r \text{ if } r \leq s\}.$$

We denote by $\mathcal{T}(\mathscr{M})$ the set of all chains of n-torsion classes in $\mathscr{M}$.

In this section, we show that every chain of n-torsion classes δ in $\mathcal{T}(\mathscr{M})$ induces an n-Harder–Narasimhan filtration for every nonzero object $M \in \mathscr{M}$. We first need some preliminary results.

LEMMA 4.2. *Let $\mathscr{U}_1 \subset \mathscr{U}_2$ be two n-torsion classes in an n-abelian category $\mathscr{M}$, and let M be an object of $\mathscr{M}$. Take the n-torsion subobjects U_1^M and U_2^M of M with respect to $\mathscr{U}_1$ and $\mathscr{U}_2$, respectively. Then, the following hold:*

1. *U_1^M is a subobject of U_2^M.*
2. *The torsion object $U_1^{U_2^M}$ of U_2^M with respect to $\mathscr{U}_1$ is isomorphic to U_1^M.*

Proof. 1. This follows directly from the definition of n-torsion classes and the fact that U_1^M is an object of $\mathscr{U}_2$.

2. Since U_2^M is a subobject of M, we have that $U_1^{U_2^M}$ is a subobject of M which belongs to $\mathscr{U}_1$. Then, by the universal properties of n-torsion objects, we have that $U_1^{U_2^M}$ is a subobject of U_1^M. On the other hand, we have by 1. that U_1^M is a subobject of U_2^M. Hence, the universal properties of torsion objects imply that U_1^M is a subobject of $U_1^{U_2^M}$. We can then conclude that U_1^M is isomorphic to $U_1^{U_2^M}$. □

PROPOSITION 4.3. *Let $\mathscr{M}$ be an n-abelian category, and let δ be a chain of n-torsion classes in $\mathscr{M}$. Then $\bigcup_{r>s} \mathscr{U}_r$ is an n-torsion class in $\mathscr{M}$ for all $s \in [0,1)$ and $\bigcap_{t<s} \mathscr{U}_t$ is an n-torsion class in $\mathscr{M}$ for all $s \in (0,1]$.*

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Proof. Let $\delta \in \mathcal{T}(\mathscr{M})$, and let $M \in \mathscr{M}$. We first show that $\bigcup_{r>s} \mathscr{U}_r$ is an n-torsion class for all $s \in [0,1)$. Take for every $t > s$ the n-torsion subobject U_t^M of M with respect to $\mathscr{U}_t$. Then Lemma 4.2 implies that we have an ascending chain of subobjects of M as follows:

$$0 = U_1^M \subset \cdots \subset U_t^M \subset \cdots \subset M.$$

Recall that $\mathcal{A}$ is a length category, in particular, $\mathcal{A}$ is an Artinian category. This implies that the above ascending chain of subobjects of M stabilizes. In other words, this implies the existence of a $t_M > s$ such that $U_{t_M}^M = U_t^M$ for all $s < t < t_M$.

Given that $U_{t_M}^M$ is a subobject of M, there is a monomorphism $\alpha : U_{t_M}^M \to M$. Then we obtain an n-exact sequence in $\mathscr{M}$

$$0 \to U_{t_M}^M \xrightarrow{\alpha} M \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0 \tag{16}$$

by taking the n-cokernel of α. We claim that

$$0 \to \mathscr{M}(X, V^1) \to \mathscr{M}(X, V^2) \to \cdots \to \mathscr{M}(X, V^n) \to 0 \tag{17}$$

is exact for all $X \in \bigcup_{r>s} \mathscr{U}_r$. Indeed, for each $X \in \bigcup_{r>s} \mathscr{U}_r$, there exists a real number $t_1 \in (s,1]$ such that $X \in \mathscr{U}_{t_1}$. If $t_M \leq t_1$, we have that $X \in \mathscr{U}_{t_M}^M$. Then (17) is exact because $U_{t_M}^M$ is the torsion object of M in $\mathscr{U}_{t_M}$. Otherwise, let $s < t_1 < t_M$. Then we have that $U_{t_1}^M \cong U_{t_M}^M$ by construction and (17) is exact for all $X \in \bigcup_{r>s} \mathscr{U}_r$. Hence, we have shown that, for each $M \in \mathscr{M}$, there exists a subobject $U_{>s}^M := U_{t_M}^M$ in $\bigcup_{r>s} \mathscr{U}_r$ such that (17) is exact for all $X \in \bigcup_{r>s} \mathscr{U}_r$. Thus, $\bigcup_{r>s} \mathscr{U}_r$ is an n-torsion class.

To show that $\bigcap_{t<s} \mathscr{U}_t$ is an n-torsion class in $\mathscr{M}$ for all $s \in (0,1]$, we start by noting that, for all $M \in \mathscr{M}$, there exists a descending chain of subobjects

$$0 \subset \cdots \subset U_t^M \subset \cdots \subset U_0^M = M,$$

where $t \in [0,s)$. Since $\mathcal{A}$ is a length category, we have that $\mathcal{A}$ is Noetherian, which implies the existence of $t_M \in [0,s)$ such that $U_{t_M}^M = U_t^M$ for all $t_M < t < s$. Let $\alpha : U_{t_M}^M \to M$ be a monomorphism and consider the exact sequence

$$0 \to U_{t_M}^M \xrightarrow{\alpha} M \to V^1 \xrightarrow{v^1} \cdots \xrightarrow{v^{n-1}} V^n \to 0 \tag{18}$$

in $\mathscr{M}$ that comes from taking the n-cokernel of α. Then the sequence

$$0 \to \mathscr{M}(X, V^1) \to \mathscr{M}(X, V^2) \to \cdots \to \mathscr{M}(X, V^n) \to 0 \tag{19}$$

is exact for all $X \in \bigcap_{t<s} \mathscr{U}_t \subset \mathscr{U}_{t_M}$ because $U_{t_M}^M$ is the torsion object of M with respect to $\bigcap_{t<s} \mathscr{U}_t$. Hence, for every $M \in \mathscr{M}$, there exists a subobject $U_{<s}^M := U_{t_M}^M$ of M such that $U_{<s}^M \in \bigcap_{t<s} \mathscr{U}_t$ and (19) is exact for every $X \in \bigcap_{t<s} \mathscr{U}_t$. In other words, $\bigcap_{t<s} \mathscr{U}_t$ is an n-torsion class. ∎

Given an object M in $\mathscr{M}$, δ a chain of n-torsion classes in $\mathscr{M}$, and $s \in (0,1)$, we denote by $U_{>s}^M$ and $U_{<s}^M$ the torsion object of M with respect to $\bigcup_{r>s} \mathscr{U}_r$ and $\bigcap_{t<s} \mathscr{U}_t$, respectively.

The above results allow us to define the notion of slicing for chains of n-torsion classes which enable us to show the existence of Harder–Narasimhan filtrations from chains of n-torsion classes.

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DEFINITION 4.4. Let δ be a chain of n-torsion classes. For every $t \in [0,1]$, we have the subcategory $S_t = \bigcap_{s<t} \mathscr{U}_s \setminus \bigcup_{r>t} \mathscr{U}_r$. We define $\mathcal{Q}_t$ to be

$$\mathcal{Q}_t := \{n\text{-coker}(\alpha) : \text{where } \alpha : U^M_{>t} \to M \text{ and } M \in S_t\}.$$

Moreover, the *slicing* $\mathcal{Q}_\delta$ of δ is the set $\mathcal{Q}_\delta := \{\mathcal{Q}_t : t \in [0,1]\}$.

REMARK 4.5. Note that for a given chain of n-torsion classes δ, $\mathcal{Q}_t$ might be empty for some $t \in [0,1]$.

THEOREM 4.6. *Let $\mathscr{M}$ be an n-abelian category, and let δ be a chain of n-torsion classes in $\mathscr{M}$. Then δ induces an n-Harder–Narasimhan filtration for every $M \in \mathscr{M}$. That is, a filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{r-1} \subsetneq M_r = M$$

such that there exists a finite ordered set $s_1 > s_2 > \cdots > s_r$ such that $s_i \in [0,1]$ and the n-cokernel of the inclusion $M_{k-1} \to M_k$ is in $\mathcal{Q}_{s_k}$ for every $1 \leq k \leq r$. Moreover, this filtration is unique up-to-isomorphism.

In particular, we have that M_{i-1} is the torsion subobject of M_i with respect to $\bigcup_{r>s_i} \mathscr{U}_r$ for all $1 \leq i \leq r$.

Proof. Given $M \in \mathscr{M}$, we begin by showing the existence of a filtration with the desired properties. For this, we need to show the existence of $s_r \in [0,1]$ such that $M \in \bigcap_{s<s_r} \mathscr{U}_s \setminus \bigcup_{t>s_r} \mathscr{U}_t$. Clearly, $M \in \mathcal{T}_0 = \mathscr{M}$ and $M \notin \mathcal{T}_1 = \{0\}$. Moreover, either $M \in \mathscr{U}_s$ or $M \notin \mathscr{U}_s$ for all $s \in [0,1]$. Hence,

$$s_r = \inf\{t \in [0,1] : M \notin \mathscr{U}_t\} = \sup\{s \in [0,1] : M \in \mathscr{U}_s\}$$

is well-defined and uniquely determined by M. Now, consider the n-torsion subobject $U^M_{>s_r}$ of M with respect to the n-torsion class $\bigcup_{t>s_r} \mathscr{U}_t$. Note that $U^M_{>s_r}$ is a proper subobject of M since $M \notin \bigcup_{t>s_r} \mathscr{U}_t$. Hence, the n-cokernel of the natural inclusion $U^M_{>s_t} \to M$ is in $\mathcal{Q}_{s_r}$.

Set $M_r := M$ and $M_{r-1} := U^M_{>s_r}$. Applying the above argument to M_{r-1}, there is a unique $s_{r-1} \in [0,1]$ such that $M_{r-1} \in \bigcap_{s<s_{r-1}} \mathscr{U}_s \setminus \bigcup_{t>s_{r-1}} \mathscr{U}_t$. Moreover,

$$s_{r-1} = \inf\{t \in [0,1] : M_{r-1} \notin \mathscr{U}_t\} = \sup\{s \in [0,1] : M_{r-1} \in \mathscr{U}_s\}.$$

Note that $M_{r-1} \in \bigcup_{t>s_r} \mathscr{U}_t \subset \mathscr{U}_{s_r}$. In particular, this implies that $s_{r-1} > s_r$.

Applying this process inductively, we get an ascending sequence $s_r < s_{r-1} < \cdots$ corresponding to a descending chain of subobjects of M

$$\cdots \subset M_{r-i} \subset M_{r-i+1} \subset \cdots \subset M_r = M.$$

Recall that, by Theorem 2.4, $\mathscr{M}$ is a full subcategory of an abelian category $\mathcal{A}$, which is assumed to be a length category. Hence, M is of finite length in $\mathcal{A}$. Thus, there is a positive integer k that $M_{r-k} = 0$. Without loss of generality, we can suppose that $k = r$. This shows the existence of a filtration with the desired properties.

We now show the uniqueness of this filtration. Suppose that there exists a second filtration

$$0 = M'_0 \subsetneq M'_1 \subsetneq \cdots \subsetneq M'_{t-1} \subsetneq M'_t = M$$

as in the statement of the theorem. Then M'_{t-1} is the torsion object $U^M_{>s_t}$ of M with respect to the n-torsion class $\bigcup_{x>s_t} \mathscr{U}_x$ for some s_t in $[0,1]$ such that $M \in$

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$\bigcap_{s<s_t} \mathscr{U}_s \setminus \bigcup_{t>s_t} \mathscr{U}_t$. However, we have shown that there exists a unique $s_t \in [0,1]$ such that $M \in \bigcap_{s<s_t} \mathscr{U}_s \setminus \bigcup_{t>s_t} \mathscr{U}_t$. This implies that $s_t = s_r$. Moreover, we have that $M_{r-1} \cong M'_{t-1}$ since the torsion objects are unique up to isomorphism.

Repeating this process, we show that $s_{t-i} = s_{r-i}$ and that $M_{r-i} \cong M'_{t-i}$ for all positive integer i. In particular, we have that $0 \neq M_1 \cong M'_{r-t+1}$ and $0 = M_0 \cong M'_{r-t}$, implying that $r = t$ and the proof is complete. □

EXAMPLE 4.7. Consider once again the algebra A with 2-cluster tilting subcategory $\mathscr{M}$ as in Example 2.5. Then,

$$\delta = \begin{cases} \mathscr{U}_t = \mathscr{M}, & \text{if } t \in [0,1/3) \\ \mathscr{U}_t = \operatorname{add}\{3\}, & \text{if } t \in [1/3,2/3) \\ \mathscr{U}_t = \operatorname{add}\{0\}, & \text{if } t \in [2/3,1] \end{cases}$$

is a chain of 2-torsion classes in $\mathscr{M}$. The following is a complete list of the 2-Harder–Narasimhan filtrations induced by δ in the indecomposable objects of $\mathscr{M}$.

Object	2-Harder–Narasimhan filtration	$\{s_1,\dots,s_m\}$
1	$0 \subset 1$	$s_1 = 1/3$
$\begin{smallmatrix}1\\2\end{smallmatrix}$	$0 \subset \begin{smallmatrix}1\\2\end{smallmatrix}$	$s_1 = 1/3$
$\begin{smallmatrix}2\\3\end{smallmatrix}$	$0 \subset 3 \subset \begin{smallmatrix}2\\3\end{smallmatrix}$	$s_1 = 2/3,\ s_2 = 1/3$
3	$0 \subset 3$	$s_1 = 2/3$

§5. Embedding of n-Harder–Narasimhan filtrations

In §3, we have shown that the map $T(-) : n\text{-tors}(\mathscr{M}) \to \operatorname{tors}(\mathcal{A})$ embeds the poset of n-torsion classes $n\text{-tors}(\mathscr{M})$ in an n-cluster-tilting subcategory $\mathscr{M}$ of an abelian category $\mathcal{A}$ into the poset $\operatorname{tors}(\mathcal{A})$ of torsion classes in $\mathcal{A}$. This implies, in particular, that every chain of n-torsion classes δ in $\mathscr{M}$ induces naturally a chain of torsion classes $T(\delta)$ in $\mathcal{A}$ by setting

$$T(\delta) := \{T(\mathscr{U}_s) : \mathscr{U}_s \in \delta \text{ for all } s \in [0,1]\}.$$

In order to construct (n-)Harder–Narasimhan filtrations and show that they are unique, we use infinite unions and intersections of (n-)torsion classes. We now show that the map $T(-) : n\text{-tors}(\mathscr{M}) \to \operatorname{tors}(\mathcal{A})$ commutes with infinite unions and intersections.

PROPOSITION 5.1. *Let $\delta = \{\mathscr{U}_s : s \in [0,1]\}$ be a chain of n-torsion classes in $\mathscr{M}$. Then,*

$$T\left(\bigcup_{r>s} \mathscr{U}_r\right) = \bigcup_{r>s} T(\mathscr{U}_r) \quad \text{and} \quad T\left(\bigcap_{r<s} \mathscr{U}_r\right) = \bigcap_{r<s} T(\mathscr{U}_r)$$

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for all $s \in [0,1]$. Moreover, if $F(\mathscr{U})$ is the torsion-free class in $\mathcal{A}$ such that $(T(\mathscr{U}), F(\mathscr{U}))$ is a torsion pair in $\mathcal{A}$, then

$$\left(\bigcup_{r>s} T(\mathscr{U}_r), \bigcap_{r>s} F(\mathscr{U}_r)\right) \quad \text{and} \quad \left(\bigcap_{r<s} T(\mathscr{U}_r), \bigcup_{r<s} F(\mathscr{U}_r)\right)$$

are torsion pairs in $\mathcal{A}$ for all $s \in [0,1]$.

Proof. We prove that $T\left(\bigcup_{r>s} \mathscr{U}_r\right) = \bigcup_{r>s} T(\mathscr{U}_r)$.

Clearly, $\mathscr{U}_r \subset T(\mathscr{U}_r)$ for all $r > s$. Then $\bigcup_{r>s} \mathscr{U}_r \subset \bigcup_{r>s} T(\mathscr{U}_r)$. We can then apply $T(-)$ to both sets to obtain $T\left(\bigcup_{r>s} \mathscr{U}_r\right) \subset T\left(\bigcup_{r>s} T(\mathscr{U}_r)\right)$. Now, we have that $\bigcup_{r>s} T(\mathscr{U}_r)$ is a torsion class by [34, Prop. 2.3]. So, $T\left(\bigcup_{r>s} \mathscr{U}_r\right) \subset T\left(\bigcup_{r>s} T(\mathscr{U}_r)\right) = \bigcup_{r>s} T(\mathscr{U}_r)$.

In the other direction, recall that $T(X) = \mathrm{Filt}(\mathrm{Fac}(X))$ (cf. [8]). Then we have the following inclusions:

$$\mathscr{U}_r \subset \bigcup_{r>s} \mathscr{U}_r, \qquad \text{for all } r > s,$$

$$\mathrm{Filt}(\mathrm{Fac}(\mathscr{U}_r)) \subset \mathrm{Filt}\left(\mathrm{Fac}\left(\bigcup_{r>s} \mathscr{U}_r\right)\right), \qquad \text{for all } r > s,$$

$$T(\mathscr{U}_r) \subset T\left(\bigcup_{r>s} \mathscr{U}_r\right), \qquad \text{for all } r > s,$$

$$\bigcup_{r>s} T(\mathscr{U}_r) \subset T\left(\bigcup_{r>s} \mathscr{U}_r\right).$$

The moreover part of the statement follows directly from [34, Prop. 2.7]. □

By now, we have seen that, for every nonzero object $M \in \mathscr{M} \subset \mathcal{A}$, we have the n-Harder–Narasimhan filtration induced by δ given by Theorem 4.6 and the Harder–Narasimhan filtration induced by $T(\delta)$ given by Theorem 2.12. We show that both filtrations coincide.

THEOREM 5.2. *Let δ be a chain of n-torsion classes in $\mathscr{M}$, let $T(\delta)$ be the chain of torsion classes in $\mathcal{A}$ induced by δ, and let M be an object in $\mathscr{M}$. Consider the n-Harder–Narasimhan filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots M_{t-1} \subsetneq M_t = M$$

of M induced by δ and the Harder–Narasimhan filtration

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots M_{t'-1} \subsetneq M_{t'} = M$$

of M induced by $T(\delta)$. Then we have that $t = t'$ and $M_i \cong N_i$ for all $1 \leq i \leq t$.

Proof. Let M be an object of $\mathscr{M}$, and let δ be a chain of n-torsion classes. Consider the n-Harder–Narasimhan filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots M_{t-1} \subsetneq M_t = M$$

of M induced by δ. By Theorem 4.6, we have that M_{i-1} is the n-torsion subobject of M_i with respect to the n-torsion class $\bigcup_{r>s_i} \mathscr{U}_r$. Applying Lemma 3.1 and Proposition 5.1, we

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obtain that M_{i-1} is the torsion object of M_i with respect of $\bigcup_{r>s_i} T(\mathscr{U}_r)$. In other words,

$$0 \longrightarrow M_{i-1} \xrightarrow{\alpha} M_i \longrightarrow \operatorname{coker}(\alpha) \longrightarrow 0$$

is the canonical short exact sequence of M_i with respect to the torsion pair

$$\left(\bigcup_{r>s_i} T(\mathscr{U}_r), \bigcap_{r>s_i} F(\mathscr{U}_r) \right),$$

where $F(\mathscr{U}_r)$ is the torsion-free class in $\mathcal{A}$ such that $(T(\mathscr{U}_r), F(\mathscr{U}_r))$ is a torsion pair (see [34, Prop. 2.7]). Thus, $\operatorname{coker}(\alpha)$ belongs to $\bigcap_{r>s_i} F(\mathscr{U}_r)$.

On the other hand, it follows from Theorem 4.6 that $M_i \in \mathscr{U}_s$ for all $s < s_i$. Then $M_i \in T(\mathscr{U}_s)$ for all $s < s_i$. Hence, $M_i \in \bigcap_{s<s_i} T(\mathscr{U}_s)$. So, $\operatorname{coker}(\alpha)$ belongs to $\bigcap_{s<s_i} T(\mathscr{U}_s)$, since $\bigcap_{s<s_i} T(\mathscr{U}_s)$ is a torsion class by [34, Prop. 2.7].

We can then conclude that $\operatorname{coker}(\alpha) \in \mathcal{P}_{s_i} = \bigcap_{s<s_i} \mathcal{T}(\mathscr{U}_s) \cap \bigcap_{s>s_i} F(\mathscr{U}_s)$ for all $1 \leq i \leq t$.

This means that

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots M_{t-1} \subsetneq M_t = M$$

is a filtration of M such that

1. $0 = M_0$ and $M_n = M$;
2. there exists $s_k \in [0,1]$ such that $M_k/M_{k-1} \in \mathcal{P}_{s_k}$, for all $1 \leq k \leq t$; and
3. $s_1 > s_2 > \cdots > s_t$.

Then this is the Harder–Narasimhan filtration of M with respect to the chain of torsion classes $T(\delta)$. □

As a consequence of the last theorem, we have the following.

COROLLARY 5.3. *Let δ be a chain of n-torsion classes of $\mathscr{M} \subset \mathcal{A}$, and consider the family of subcategories $\{\mathcal{P}_s : s \in [0,1]\}$ be the slicing associated to $T(\delta)$ as defined in Definition 2.10. If there exists a nonzero object in $\mathscr{M}$, then there exists an $s_1 \in [0,1]$ such that $\mathcal{P}_{s_1} \cap \mathscr{M}$ contains a nonzero object.*

Proof. Let δ be a chain of n-torsion classes in $\mathscr{M}$, and let M be a nonzero object of $\mathscr{M}$. Then, Theorem 4.6 gives us the n-Harder–Narasimhan filtration of M

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots M_{t-1} \subsetneq M_t = M.$$

Moreover, Theorem 5.2 implies that $M_i/M_{i-1} \in \mathcal{P}_{s_i}$ for all $1 \leq i \leq t$. In particular, $M_1/M_0 \cong M_1 \in \mathcal{P}_{s_1}$. □

EXAMPLE 5.4. To finish this section, let A and M be as in Example 2.5, and let

$$\delta = \begin{cases} \mathscr{U}_t = \mathscr{M}, & \text{if } t \in [0,1/3) \\ \mathscr{U}_t = \operatorname{add}\left\{ {}_1 \oplus {}^1_2 \oplus {}^2_3 \right\}, & \text{if } t \in [1/3,2/3) \\ \mathscr{U}_t = \operatorname{add}\{0\}, & \text{if } t \in [2/3,1] \end{cases}$$

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be a chain of 2-torsion classes in $\mathscr{M}$. Is easy to see that

$$T(\delta) = \begin{cases} T(\mathscr{U}_t) = \text{mod-}A, & \text{if } t \in [0,1/3) \\ T(\mathscr{U}_t) = \text{add}\left\{ {}_1 \oplus {}^1_2 \oplus {}^2_3 \oplus {}_2 \right\}, & \text{if } t \in [1/3,2/3) \\ T(\mathscr{U}_t) = \text{add}\{0\}, & \text{if } t \in [2/3,1] \end{cases}$$

is the chain of torsion classes $T(\delta)$ induced by δ in mod-A.

§6. Galois coverings of n-torsion classes

Assume that $\mathcal{C}$ is a locally bounded Krull–Schmidt $\Bbbk$-category, where $\Bbbk$ is a field, with an admissible G-action on $\mathcal{C}$ inducing an admissible action on mod-$\mathcal{C}$. Then, as in §2.4, the functor $P : \mathcal{C} \to \mathcal{C}/G$ induces a functor $P_\bullet : \text{mod-}\mathcal{C} \longrightarrow \text{mod-}(\mathcal{C}/G)$, which is a G-precovering map. Moreover, we know by Theorem 2.15 that, under these assumptions, a G-equivariant subcategory $\mathscr{M}$ of mod-$\mathcal{C}$ such that $P_\bullet(\mathscr{M})$ is functorially finite in mod-$(\mathcal{C}/G)$ is an n-cluster-tilting subcategory of mod-$\mathcal{C}$ if and only if $P_\bullet(\mathscr{M})$ is an n-cluster-tilting subcategory of mod-$\mathcal{C}/G$. For more details, see §2.4.

We start this section by showing that G-equivariant n-torsion classes behave well under the pushdown functor $P_\bullet : \text{mod-}\mathcal{C} \longrightarrow \text{mod-}(\mathcal{C}/G)$.

THEOREM 6.1. *Let $\mathcal{C}$ be a locally bounded Krull–Schmidt $\Bbbk$-category with an admissible action of a group G on $\mathcal{C}$ inducing an admissible action on* mod-$\mathcal{C}$. *Suppose that $\mathscr{M}$ is an n-cluster-tilting G-equivariant full subcategory of* mod-$\mathcal{C}$ *such that $P_\bullet(\mathscr{M})$ is functorially finite in* mod-$\mathcal{C}/G$. *Let $\mathscr{U}$ be a G-equivariant full subcategory of $\mathscr{M}$. If $\mathscr{U}$ is an n-torsion class of $\mathscr{M}$, then $P_\bullet(\mathscr{U})$ is an n-torsion class of $P_\bullet(\mathscr{M})$.*

Proof. First, note that, by Theorem 2.15, $P_\bullet(\mathscr{M})$ is an n-cluster-tilting subcategory of mod-$\mathcal{C}/G$. Let $\mathscr{U}$ be an n-torsion class of $\mathscr{M}$, and let $P_\bullet(M)$ be an object in $P_\bullet(\mathscr{M})$. By definition, there is an n-exact sequence

$$0 \longrightarrow U^M \xrightarrow{\theta} M \xrightarrow{\varphi^1} V^1 \longrightarrow \cdots \xrightarrow{\varphi^n} V^n \longrightarrow 0 \tag{20}$$

in $\mathscr{M}$, where $U \in \mathscr{U}$ and

$$0 \to \mathscr{M}(U,V^1) \to \mathscr{M}(U,V^2) \to \cdots \to \mathscr{M}(U,V^n) \to 0 \tag{21}$$

is exact, for all objects U in $\mathcal{U}$.

By applying the exact functor $P_\bullet$ on the sequence (20), we get the following exact sequence:

$$0 \longrightarrow P_\bullet(U^M) \xrightarrow{P_\bullet(\theta)} P_\bullet(M) \xrightarrow{P_\bullet(\varphi^1)} P_\bullet(V^1) \longrightarrow \cdots \xrightarrow{P_\bullet(\varphi^n)} P_\bullet(V^n) \longrightarrow 0 \tag{22}$$

in $P_\bullet(\mathscr{M})$. To show that $P_\bullet(\mathscr{U})$ is an n-torsion class of $P_\bullet(\mathscr{M})$, it is enough to show that this is the canonical n-exact sequence of $P_\bullet(M)$ with respect to $P_\bullet(\mathscr{U})$.

Since the sequence (22) is exact, we may deduce from [18, Lem. 3.5] that it is an n-exact sequence in $P_\bullet(\mathscr{M})$.

So, it is enough to show that the sequence

$$0 \longrightarrow P_\bullet(\mathscr{M})(P_\bullet(U),P_\bullet(V^1)) \longrightarrow \cdots \longrightarrow P_\bullet(\mathscr{M})(P_\bullet(U),P_\bullet(V^n)) \longrightarrow 0 \tag{23}$$

is exact, for all objects $P_\bullet(U)$ of $P_\bullet(\mathcal{U})$.

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Since the pushdown functor $P_\bullet : \text{mod-}\mathcal{C} \longrightarrow \text{mod-}\mathcal{C}/G$ is G-precovering, there exists the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (P_\bullet(U), P_\bullet(V^1)) & \xrightarrow{P_\bullet(\theta)_*} & (P_\bullet(U), P_\bullet(V^2)) & \longrightarrow \cdots \xrightarrow{P_\bullet(\varphi^n)_*} & (P_\bullet(U), P_\bullet(V^n)) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle P^{(1)}_{\bullet\, U,V^1}} & & \uparrow{\scriptstyle P^{(1)}_{\bullet\, U,V^2}} & & \uparrow{\scriptstyle P^{(1)}_{\bullet\, U,V^n}} & & \\
0 & \longrightarrow & \oplus_{g\in G}\mathscr{M}({}^gU, V^1) & \xrightarrow{\oplus\theta_*} & \oplus_{g\in G}\mathscr{M}({}^gU, V^2) & \longrightarrow \cdots \xrightarrow{\oplus\varphi^n_*} & \oplus_{g\in G}\mathscr{M}({}^gU, V^n) & \longrightarrow & 0,
\end{array}$$

where the vertical maps are $\Bbbk$-isomorphisms (see Definition 2.14).

Since $\mathscr{U}$ is G-equivariant, ${}^gU$ belongs to $\mathscr{U}$ for all $g \in G$. Now, since the sequence (21) is exact, for all $U \in \mathscr{U}$, the bottom row of the above diagram is exact. This implies the exactness of the top row, as desired.

This completes the proof of the theorem. □

In Theorem 3.6, given an n-cluster-tilting subcategory $\mathscr{M}$ of an abelian category $\mathcal{A}$, we characterize the minimal torsion class $T(\mathscr{U})$ of $\mathcal{A}$ containing the n-torsion class $\mathscr{U} \subset \mathscr{M}$. In the following corollary, we compare the minimal torsion class $T(P_\bullet(\mathscr{U}))$ of mod-$\mathcal{C}/G$ containing $P_\bullet(\mathscr{U})$ with the torsion class $P_\bullet(T(\mathscr{U}))$. Recall that for a subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$, the minimal torsion class of $\mathcal{A}$ containing $\mathcal{X}$ is denoted by $T(\mathcal{X})$ and is equal to $\text{Filt}(\text{Fac}(\mathcal{X}))$.

COROLLARY 6.2. *Let the situation be as in Theorem 6.1. Let $\mathcal{U}$ be an n-torsion class of $\mathscr{M}$, and suppose that $T(\mathscr{U})$ is G-equivariant. Then, $P_\bullet(T(\mathscr{U})) = T(P_\bullet(\mathscr{U}))$.*

Proof. By definition, $T(\mathscr{U}) = \text{Filt}(\text{Fac}(\mathscr{U}))$ is the minimal torsion class of mod-$\mathcal{C}$ that contains $\mathscr{U}$. Let $M \in T(\mathscr{U})$. So, there exists a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$$

of M such that $M_i/M_{i-1} \in \text{Fac}(\mathscr{U})$ for all $1 \le i \le t$. Since the pushdown functor $P_\bullet$ is exact, we easily deduce that $P_\bullet(M) \in \text{Filt}(\text{Fac}(P_\bullet(\mathscr{U}))) = T(P_\bullet(\mathscr{U}))$. Therefore, $P_\bullet(T(\mathscr{U})) \subseteq T(P_\bullet(\mathscr{U}))$. For the reverse inclusion, note that the inclusion $\mathscr{U} \subseteq T(\mathscr{U})$ implies that $P_\bullet(\mathscr{U}) \subseteq P_\bullet(T(\mathscr{U}))$. Hence,

$$T(P_\bullet(\mathscr{U})) \subseteq T(P_\bullet(T(\mathscr{U}))).$$

However, $T(P_\bullet(T(\mathscr{U}))) = P_\bullet(T(\mathscr{U}))$, because by Theorem 6.1 we have that the functor $P_\bullet$ preserves torsion classes. The proof is hence complete. □

Note that Theorem 6.1 implies that the functor $P_\bullet : \text{mod-}\mathcal{C} \to \text{mod-}\mathcal{C}/G$ induces a map

$$nP_\bullet : G - n\text{-tors}(\mathscr{M}) \to n\text{-tors}(P_\bullet(\mathscr{M}))$$

from the set $G - n\text{-tors}(\mathscr{M})$ of G-equivariant n-torsion classes of $\mathscr{M} \subset \text{mod-}\mathcal{C}$ to the set $n\text{-tors}(P_\bullet(\mathscr{M}))$ of n-torsion classes of $P_\bullet(\mathscr{M}) \subset \text{mod-}\mathcal{C}/G$. Likewise, $P_\bullet : \text{mod-}\mathcal{C} \to \text{mod-}\mathcal{C}/G$ induces a map

$$P_\bullet : G\text{-tors}(\text{mod-}\mathcal{C}) \to \text{tors}(\text{mod-}\mathcal{C}/G)$$

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from the set G-tors$(\mathscr{M})$ of G-equivariant torsion classes of mod-$\mathcal{C}$ to the set tors$(P_\bullet(\mathscr{M}))$ of torsion classes of mod-$\mathcal{C}/G$. Using this notation, Corollary 6.2 can be restated as follows:

$$\begin{array}{ccc} G\text{-}n\text{-tors}(\mathscr{M}) & \xrightarrow{T(-)} & G\text{-tors}(\text{mod-}\mathcal{C}) \\ \downarrow{\scriptstyle nP_\bullet(-)} & & \downarrow{\scriptstyle P_\bullet(-)} \\ n\text{-tors}(P_\bullet(\mathscr{M})) & \xrightarrow{T(-)} & \text{tors}(\text{mod-}\mathcal{C}/G). \end{array}$$

Suppose that $\delta = \{\mathscr{U}_s : s \in [0,1]\}$ is a chain of G-equivariant n-torsion classes in $\mathscr{M} \subset$ mod-$\mathcal{C}$. Then, Theorem 6.1 implies that $P_\bullet(\delta) := \{P_\bullet(\mathscr{U}_s) : s \in [0,1]\}$ is a chain of n-torsion classes in $P_\bullet(\mathscr{M}) \subset$ mod-$\mathcal{C}/G$.

Now, Theorem 4.6 implies that for every nonzero object $M \in \mathscr{M}$, the chain of G-equivariant n-torsion classes δ induces an n-Harder–Narasimhan filtration, while $P_\bullet(\delta)$ induces an n-Harder–Narasimhan filtration of $P_\bullet(M)$. In the following result, we compare both filtrations.

PROPOSITION 6.3. *Let $\mathcal{C}$ be a locally bounded Krull–Schmidt $\Bbbk$-category with an admissible action of a group G on $\mathcal{C}$ inducing an admissible action on* mod-$\mathcal{C}$. *Let $\mathscr{M}$ be a G-equivariant n-cluster-tilting subcategory of* mod-$\mathcal{C}$ *such that $P_\bullet(\mathscr{M})$ is functorially finite in* mod-$\mathcal{C}/G$. *Let $\delta = \{\mathscr{U}_s : s \in [0,1]\}$ be a chain of G-equivariant n-torsion classes in $\mathscr{M}$, and let M be a nonzero object of $\mathscr{M}$. Then, a filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{r-1} \subsetneq M_r = M$$

is the n-Harder–Narasimhan filtration of M with respect to δ in $\mathscr{M}$ if and only if

$$0 = P_\bullet(M_0) \subsetneq P_\bullet(M_1) \subsetneq \cdots \subsetneq P_\bullet(M_{r-1}) \subsetneq P_\bullet(M_r) = P_\bullet(M)$$

is the n-Harder–Narasimhan filtration of $P_\bullet(M)$ with respect to the chain of n-torsion classes $P_\bullet(\delta)$ in $P_\bullet(\mathscr{M})$.

Proof. Clearly, the union and intersection of G-equivariant sets is itself G-equivariant. This fact together with Proposition 4.3 implies that $\bigcup_{r>s} \mathscr{U}_r$ and $\bigcap_{t<s} \mathscr{U}_t$ are G-equivariant n-torsion classes for every $s \in [0,1]$. Hence, it follows from Theorem 6.1 that $P_\bullet(\bigcup_{r>s} \mathscr{U}_r)$ and $P_\bullet(\bigcap_{t<s} \mathscr{U}_t)$ are n-torsion classes in $P_\bullet(\mathscr{M})$.

We claim that $P_\bullet(\bigcup_{r>s} \mathscr{U}_r) = \bigcup_{r>s} P_\bullet(\mathscr{U}_r)$ and $P_\bullet(\bigcap_{t<s} \mathscr{U}_t) = \bigcap_{t<s} P_\bullet(\mathscr{U}_t)$. We only show the first of these equalities, the proof of the second being similar.

Clearly, $\mathscr{U}_r \subset \bigcup_{r>s} \mathscr{U}_r$ for all $s < r \leq 1$. Then, $P_\bullet(\mathscr{U}_r) \subset P_\bullet(\bigcup_{r>s} \mathscr{U}_r)$ for all $s < r \leq 1$. Hence, $\bigcup_{r>s} P_\bullet(\mathscr{U}_r) \subset P_\bullet(\bigcup_{r>s} \mathscr{U}_r)$. For the reverse inclusion, let $X \in P_\bullet(\bigcup_{r>s} \mathscr{U}_r)$. Then, $X = P_\bullet(Y)$ for some $Y \in \bigcup_{r>s} \mathscr{U}_r$. This implies the existence of a $r \in (s,1]$ such that $Y \in \mathscr{U}_r$. Thus, $X = P_\bullet(Y) \in P_\bullet(\mathscr{U}_r) \subset \bigcup_{r>s} \mathscr{U}_r$, and our claim follows.

Let M be a nonzero object of $\mathscr{M}$, and let

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{r-1} \subsetneq M_r = M$$

be the n-Harder–Narasimhan filtration of M with respect to δ in $\mathscr{M}$, and it follows from Definition 4.4 that M_{r-1} is the n-torsion object of M with respect to the n-torsion class $\bigcup_{r>s_r} \mathscr{U}_r$, where $s_r = \sup\{t \in [0,1] : M \notin \mathscr{U}_t\}$. Then, Theorem 6.1 implies that $P_\bullet(M_{r-1})$ is the torsion object of $P_\bullet(M)$ with respect to the n-torsion class $P_\bullet(\bigcup_{r>s} \mathscr{U}_r) = \bigcup_{r>s} P_\bullet(\mathscr{U}_r)$.

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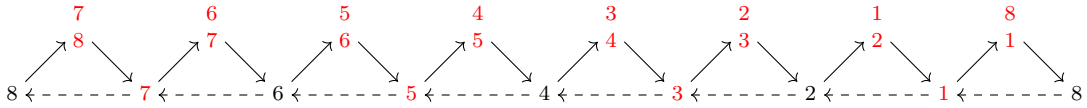


Figure 2.
The Auslander–Reiten quiver of B.

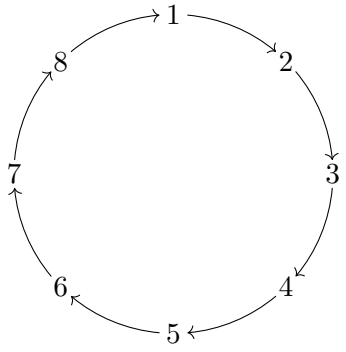
Repeating this argument inductively, we obtain an n-Harder–Narasimhan filtration

$$0 = P_\bullet(M_0) \subsetneq P_\bullet(M_1) \subsetneq \cdots \subsetneq P_\bullet(M_{r-1}) \subsetneq P_\bullet(M_r) = P_\bullet(M)$$

of M with respect to the chain of torsion classes $P_\bullet(\delta)$. Since the n-Harder–Narasimhan filtration is unique up to isomorphism by Theorem 4.6, the proof is finished. □

To finish the paper, we illustrate the results of this section.

EXAMPLE 6.4. Let B be the path algebra of the quiver



modulo its radical squared. The module category mod-B of B has a 2-cluster-tilting subcategory

$$\mathscr{M} = \operatorname{add}\left\{1 \oplus 3 \oplus 5 \oplus 7 \oplus \begin{smallmatrix}8\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}4\\5\end{smallmatrix} \oplus \begin{smallmatrix}5\\6\end{smallmatrix} \oplus \begin{smallmatrix}6\\7\end{smallmatrix} \oplus \begin{smallmatrix}7\\8\end{smallmatrix}\right\}.$$

The Auslander–Reiten quiver of B can be seen in Figure 2. In the figure, the indecomposable objects that belong to $\mathscr{M}$ are indicated in red, and the dashed arrows correspond to the Auslander–Reiten translation in mod-B.

We note that the algebra B can be seen as a $\Bbbk$-linear category $\mathcal{C}_B$ having exactly eight objects $\{e_1, \ldots, e_8\}$ which are pairwise nonisomorphic such that $\operatorname{Hom}_{\mathcal{C}_B}(e_i, e_j)$ is nonempty if and only if $e_i B e_j$ is nonzero. Note that the identity morphism of the object e_i corresponds to the element e_i in $e_i B e_i$ and, moreover, $\operatorname{Hom}(e_i, e_j)$ is one-dimensional corresponding to the one-dimensional vector space $e_j B e_i$ if there is an arrow from j to i or if $i = j$ and $\operatorname{Hom}(e_i, e_j)$ is the zero vector space otherwise. It follows from the definitions that mod-$\mathcal{C}_B$ is equivalent to mod-B.

We also note that there is an admissible $\mathbb{Z}_2$ action g over $\mathcal{C}_B$ which in the objects is defined as $g(e_i) = e_{i+4 \bmod 8}$. In this case, we have that $\mathcal{C}_B/\mathbb{Z}_2$ is a category having exactly

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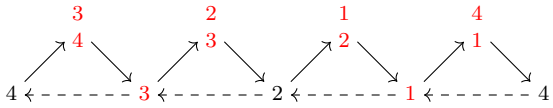
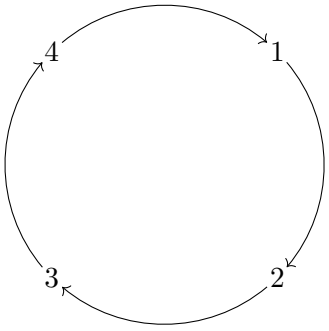


Figure 3.
The Auslander–Reiten quiver of C.

four pairwise nonisomorphic objects and mod-$\mathcal{C}_B/\mathbb{Z}_2$ is equivalent to the module category mod-C, where C is the path algebra of the following quiver:



module the radical squared. In mod-C, there is a 2-cluster-tilting subcategory $\mathscr{M}'$. The Auslander–Reiten quiver of mod-C can be seen in Figure 3. In red, we highlight the indecomposable objects of mod-C that are in $\mathscr{M}'$.

As mentioned in the introduction, there is a natural pushdown functor $P_\bullet : \text{mod-}B \to \text{mod-}C$. Moreover, it follows from the results of [7] that $P_\bullet(\mathscr{M}) = \mathscr{M}'$. Now, we know from Theorem 6.1 that for any $\mathbb{Z}_2$-equivariant 2-torsion class $\mathscr{U}$ of $\mathscr{M}$, $P_\bullet(\mathscr{U})$ is a 2-torsion class in $\mathscr{M}'$. In the following table, we give a complete list of all $\mathbb{Z}_2$-equivariant 2-torsion classes of $\mathscr{M}$ and their respective image under $P_\bullet : \text{mod-}C \to \text{mod-}B$, where we denote the set of $\mathbb{Z}_2$-equivariant 2-torsion classes of $\mathscr{M}$ by $\mathbb{Z}_2$- 2-tors($\mathscr{M}$) and the set of 2-torsion classes of $\mathscr{M}'$ by 2-tors($\mathscr{M}'$).

$\mathscr{U} \in \mathbb{Z}_2$- 2-tors($\mathscr{M}$)	$P_\bullet(\mathscr{U}) \in$ 2-tors($\mathscr{M}'$)
$\mathscr{M}$	$\mathscr{M}'$
$\operatorname{add}\left\{\begin{smallmatrix}6\\7\end{smallmatrix}\oplus\begin{smallmatrix}5\\6\end{smallmatrix}\oplus 5\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1\right\}$	$\operatorname{add}\left\{\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1\right\}$
$\operatorname{add}\left\{\begin{smallmatrix}8\\1\end{smallmatrix}\oplus\begin{smallmatrix}7\\8\end{smallmatrix}\oplus 7\oplus\begin{smallmatrix}4\\5\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3\right\}$	$\operatorname{add}\left\{\begin{smallmatrix}4\\1\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3\right\}$
$\operatorname{add}\left\{\begin{smallmatrix}5\\6\end{smallmatrix}\oplus 5\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1\right\}$	$\operatorname{add}\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1\right\}$
$\operatorname{add}\left\{\begin{smallmatrix}7\\8\end{smallmatrix}\oplus 7\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3\right\}$	$\operatorname{add}\left\{\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3\right\}$
$\operatorname{add}\{5\oplus 1\}$	$\operatorname{add}\{1\}$
$\operatorname{add}\{7\oplus 3\}$	$\operatorname{add}\{3\}$
$\operatorname{add}\{0\}$	$\operatorname{add}\{0\}$

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Now, consider the chain δ of $\mathbb{Z}_2$-equivariant 2-torsion classes of $\mathscr{M}$ defined as follows:

$$\delta = \begin{cases} \mathscr{U}_t = \mathscr{M}, & \text{if } t \in [0,1/3), \\ \mathscr{U}_t = \operatorname{add}\left\{ \begin{smallmatrix} 6 \\ 7 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \oplus 5 \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 1 \right\}, & \text{if } t \in [1/3,2/3), \\ \mathscr{U}_t = \operatorname{add}\{0\}, & \text{if } t \in [2/3,1]. \end{cases}$$

An easy calculation shows that $P_\bullet(\delta)$ is the following chain of 2-torsion classes in $\mathscr{M}'$:

$$P_\bullet(\delta) = \begin{cases} P_\bullet(\mathscr{U}_t) = \mathscr{M}, & \text{if } t \in [0,1/3), \\ P_\bullet(\mathscr{U}_t) = \operatorname{add}\left\{ \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 1 \right\}, & \text{if } t \in [1/3,2/3), \\ P_\bullet(\mathscr{U}_t) = \operatorname{add}\{0\}, & \text{if } t \in [2/3,1]. \end{cases}$$

If we take the object $\begin{smallmatrix} 8 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix}$, one can see that its 2-Harder–Narasimhan filtration with respect to δ is $0 \subset 1 \oplus 5 \subset \begin{smallmatrix} 8 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix}$. Moreover, we have that $P_\bullet\left(\begin{smallmatrix} 8 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 5 \end{smallmatrix}\right) = \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} \in \mathscr{M}'$. Calculating the 2-Harder–Narasimhan filtration of $\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}$ with respect to $P_\bullet(\delta)$, we obtain $0 \subset 1 \subset \begin{smallmatrix} 4 \\ 1 \end{smallmatrix}$, where we can see that $1 = P_\bullet(1 \oplus 5)$, as shown in Proposition 6.3.

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