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RESEARCH ARTICLE

Effective characterization of quasi-abelian surfaces

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Abstract
Let V be a smooth quasi-projective complex surface such that the first three logarithmic plurigenera $\overline{P}_1(V)$, $\overline{P}_2(V)$ and $\overline{P}_3(V)$ are equal to 1 and the logarithmic irregularity $\overline{q}(V)$ is equal to 2. We prove that the quasi-Albanese morphism $a_V : V \to A(V)$ is birational and there exists a finite set S such that a_V is proper over $A(V) \setminus S$, thus giving a sharp effective version of a classical result of Iitaka [12].

Contents

1. Introduction

Let V be a smooth complex quasi-projective variety. By Hironaka's theorem on the resolution of singularities, we can write $V = X \backslash D$, where X is a smooth projective variety and D is a reduced divisor on X with simple normal crossings support (in fact, in the case of surfaces, our main interest here, V is automatically quasi-projective since it is smooth).

Denoting by K_X the canonical divisor of X, one defines the following invariants of V:

- for m a positive integer, the m-th *log-plurigenus* of V is $\overline{P}_m(V) := h^0(X, m(K_X + D))$;
- the *log-Kodaira dimension* of V is $\overline{\kappa}(V) := \kappa(X, K_X + D)$;
- the *log-irregularity* of V is $\overline{q}(V) := h^0(X, \Omega^1_X(\log D))$.

In addition, we say that the *irregularity* $q(V)$ of V is the irregularity of X, that is,

$$q(V) := h^0(X, \Omega^1_X).$$

It easy to see that these invariants do not depend on the choice of the compactification X.

Similarly to what happens for projective varieties, to V we can associate a quasi-abelian variety (i.e., an algebraic group which does not contain $\mathbb{G}_a$) $A(V)$, called the *quasi-Albanese variety* of V. This comes equipped with a morphism $a_V : V \longrightarrow A(V)$ which is called the *quasi-Albanese morphism*. Iitaka in [12] characterizes quasi-abelian surfaces as surfaces of log-Kodaira dimension 0 and log-irregularity 2. More precisely, he proves the following.

Theorem (Iitaka, [12]). *Let V be a smooth complex algebraic surface. Then $\overline{\kappa}(V) = 0$ and $\overline{q}(V) = 2$ if and only if the quasi-Albanese morphism $a_V : V \longrightarrow A(V)$ is birational and there are an open subset $V^0 \subseteq V$ and finitely many points $\{p_1, \dots p_t\} \subseteq A(V)$ such that the restriction $a_{V|_{V^0}} : V^0 \longrightarrow A(V) \backslash \{p_1, \dots p_t\}$ is proper.*

In this paper, we give a characterization of quasi-abelian surfaces using the three first logarithmic plurigenera instead of the logarithmic Kodaira dimension. Our main result is the following Theorem:

Theorem A. *Let V be a smooth complex algebraic surface with $\overline{q}(V) = 2$. Assume that:*

(a) $\overline{P}_1(V) = \overline{P}_2(V) = 1$ *and* $q(V) > 0$, *or*
(b) $\overline{P}_1(V) = \overline{P}_3(V) = 1$ *and* $q(V) = 0$.

Then the quasi-Albanese morphism $a_V : V \longrightarrow A(V)$ is birational. In addition, there are an open subset $V^0 \subseteq V$ and finitely many points $\{p_1, \dots p_t\} \subseteq A(V)$ such that the restriction $a_{V|_{V^0}} : V^0 \longrightarrow A(V) \backslash \{p_1, \dots p_t\}$ is proper.

Using the language of weakly weak proper birational (WWPB) equivalences introduced by Iitaka in [10], Theorem A above can be rephrased in the following manner.

Theorem A*. *Let V be a smooth algebraic surface. Then V is WWPB equivalent to a quasi-abelian variety if and only if either*

(a) $\overline{q}(V) = 2$, $\overline{P}_1(V) = \overline{P}_2(V) = 1$ *and* $q(V) > 0$, *or*
(b) $\overline{q}(V) = 2$, $\overline{P}_1(V) = \overline{P}_3(V) = 1$ *and* $q(V) = 0$.

As WWPB-maps between normal affine varieties are actually isomorphisms (see [10, Corollary on p. 498]), we have:

Corollary B. *A smooth complex affine surface V is isomorphic to $\mathbb{G}_m^2$ if and only if it has $\overline{P}_1(V) = \overline{P}_3(V) = 1$ and $\overline{q}(V) = 2$.*

We remark that Kawamata, in his celebrated work [14], proved in any dimension a weaker form of Iitaka's theorem, showing that the quasi-Albanese morphism of an algebraic variety of log-Kodaira dimension 0 and log-irregularity equal to the dimension is birational. In the compact case, effective versions of this result have been given in [3] and [19]: in [3], it is proven that the Albanese map of a projective variety X is surjective and birational iff $\dim X = q(X)$ and $P_1(X) = P_2(X) = 1$; in [19], the analogous statement is proven for compact Kähler manifolds.

So we are led to formulate the following conjecture:

Conjecture. *Let V a smooth complex quasi-projective variety with $\overline{q}(V) = \dim V$, then there exists a positive integer k, independent of the dimension of V such that $\overline{P}_1(V) = \overline{P}_k(V) = 1$ implies that the*

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quasi-Albanese morphism of V is birational (and there exist an open set $V^0 \subseteq V$ and a closed set $W \subseteq A(V)$ of codimension > 1 such that $a_{V}|_{V^0} : V^0 \to A(V)\backslash W$ is proper).

In view of the results recalled above, one might hope that $k = 2$ is the right bound also in the open setting, but in fact our Theorem A is sharp because there is a surface V with $\overline{P}_1(V) = \overline{P}_2(V) = 1$, $\overline{q}(V) = 2$ and the quasi-Albanese map not dominant (see Example 4.18).

Our argument is completely independent of Iitaka's theorem and its proof. The first step consists in showing that the assumptions on the logarithmic irregularity and on the logarithmic plurigenera imply that the quasi-Albanese map of V is dominant. When $q(V) = 2$, one obtains as an immediate consequence of the work of Chen and Hacon [3] that the quasi-Albanese map of V is birational. When $q(V) = 0$, the lengthy proof uses a fine analysis of the boundary divisor and classical arguments from the theory of surfaces. However, when $q(V) = 1$, we get a considerably shorter proof leveraging on the fact that the Albanese map of a compactification of V is nontrivial, using techniques coming from Green–Lazarsfeld generic vanishing theorems [8] and their more recent extensions to pairs (see [20] and [21]).

Once we know that the quasi-Albanese map of V is dominant, then we get that its extension to the compactifications of V and $A(V)$ respectively is generically finite. Then we use Iitaka's logarithmic ramification formula (see 2.3 for more details) together with some geometric considerations to conclude that the quasi-Albanese map is birational.

The last step of our argument consists in proving that the quasi-Albanese map is proper outside a finite set of points. This is done by showing that the boundary divisor D gets contracted by the quasi-Albanese map. Again, the key tool here is the logarithmic ramification formula together with some more classical arguments exploiting the geometry of the divisor D.

The paper is organized as follows. In §2, we recall the necessary prerequisites; §3 contains some results that hold in arbitrary dimension for $q(V) > 0$ (see in particular Corollary 3.2). Then we focus on surfaces. Section 4 is devoted to proving that the quasi-Albanese map is dominant: here the results of §3 are crucial in case $q(V) > 0$. In §5 we complete the proof of Theorem A.

Notation. We work over the complex numbers. If X is a smooth projective variety, we denote by K_X the canonical class, by $q(X) := h^0(X, \Omega^1_X) = h^1(X, \mathcal{O}_X)$ the *irregularity* and by $p_g(X) := h^0(X, K_X)$ the *geometric genus*.

We identify invertible sheaves and Cartier divisors, and we use the additive and multiplicative notation interchangeably. Linear equivalence is denoted by $\sim$. Given two divisors D_1 and D_2 on X, we write $D_1 \geq D_2$ (respectively $D_1 > D_2$) if the divisor $D_1 - D_2$ is (strictly) effective.

Finally, note that, throughout this paper, we use the term *(-1)-curve* to indicate the total transform E of a point $q \in Y$ via a birational morphism $g\colon X \to Y$ of smooth projective surfaces such that g^{-1} is not defined at q; so one has $E^2 = K_X E = -1$ but E may be reducible and/or nonreduced.

2. Preliminaries

Our proof of Theorem A combines different arguments and techniques; in this section, we recall briefly the necessary prerequisites and set the notation.

2.1. Log varieties

Let X be a smooth complex projective variety of dimension n. A reduced effective divisor $D = \sum D_i$ on X is said to have *simple normal crossings* (in short we say D is *snc*) if all the D_i are smooth and for every $p \in \operatorname{Supp} D$ there are local coordinates $(x_1, \ldots, x_n)$ around p such that D is cut out by the equation $x_1 \cdots x_r = 0$, for some $r \leq n$.

Given a smooth projective n-dimensional variety X together with a snc divisor D, we can define the sheaf of logarithmic 1-forms along D by setting

$$\Omega^1_X(\log D)_p := \sum_{i=1}^{r} \mathcal{O}_{X,p} \frac{dx_i}{x_i} + \sum_{i=r+1}^{n} \mathcal{O}_{X,p} dx_i \subset (\Omega^1_X \otimes k(X))_p$$

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where $(x_1, \ldots, x_n)$ are local coordinates around p such that $D = \{x_1 \cdots x_r = 0\}$. It is a locally free sheaf of rank equal to n. The sheaf of logarithmic m-forms is defined as

$$\Omega^m_X(\log D) := \bigwedge^m \Omega_X(\log D),$$

and, in particular, we have that

$$\Omega^n_X(\log D) \simeq \mathcal{O}_X(K_X + D).$$

We recall the following condition for the existence of logarithmic 1-forms with prescribed poles.

Proposition 2.1. *Let D be a simple normal crossing divisor on a smooth projective variety X, and let $D_1, \ldots D_k$ be a subset of the irreducible components of D.*

There exists $\sigma \in H^0(X, \Omega^1_X(\log D))$ with poles precisely on $D_1 \ldots D_k$ if and only if in $H^2(X, \mathbb{Z})$ there is a relation $\sum_i a_i[D_i] = 0$ between the classes $[D_1], \ldots [D_k]$ such that $a_i \neq 0$ for $i = 1, \ldots k$.

Proof. Let $D_1, \ldots D_t$ be the irreducible components of D; consider the residue sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \to \oplus_{i=1}^t \mathcal{O}_{D_i} \to 0.$$

The claim follows from the fact that the associated coboundary map $\delta \colon \oplus_{i=1}^t H^0(X, \mathcal{O}_{D_i}) \to H^1(X, \Omega^1_X)$ sends $1 \in H^0(X, \mathcal{O}_{D_i})$ to the class $[D_i] \in H^1(X, \Omega^1_X)$. □

Given a smooth quasi-projective variety V, by Hironaka's resolution of singularities, we can embed it in a smooth projective variety X such that $X \backslash V$ is a snc divisor D, called the *boundary of* V. Then we can use the sheaves of logarithmic forms to define logarithmic (in short 'log') invariants on V, as explained in the introduction. In the case when V is a curve, we call $\overline{P}_1(V) = \overline{q}(V)$ the logarithmic genus of V.

While the usual plurigenera and irregularity are birational invariants, this is not the case for logarithmic invariants. For example, an abelian variety X and $X \backslash D$, where $D > 0$ is a smooth ample divisor, are birational but they have different logarithmic Kodaira dimensions. For this reason, Iitaka in [10] introduced the notion of WPB-equivalence (*weakly proper birational* equivalence). It is the equivalence relation generated by the proper birational morphisms and by the open inclusions $V \subset V'$ such that $V' \setminus V$ has codimension at least 2.

WPB-equivalent varieties have the same plurigenera and log irregularity: this is obvious for open immersions as above, and it is proven for proper birational morphisms in [11, Prop. 1]. Therefore, it would seem that WPB-equivalence is the right notion of equivalence when studying the birational geometry of open varieties. However, the set of WPB-maps is not *saturated*; namely, it is possible, for instance, to have rational maps $g \colon U \dashrightarrow V$ and $f \colon V \dashrightarrow W$ such that $f \circ g$ is WPB, but f or g is not. In order to get around this kind of difficulty, Iitaka in [10] defines the notion of WWPB-equivalence, proving that WWPB-equivalent varieties have the same logarithmic plurigenera and irregularity.

2.2. Quasi-abelian varieties and the quasi-Albanese map

Here, we introduce one of the main characters of our story: quasi-abelian varieties, which for many aspects can be thought of as a nonprojective analogue of abelian varieties. We recall briefly the facts that we need.

Definition 2.2. A *quasi-abelian variety*—in some sources also called a *semiabelian variety*—is a connected algebraic group G that is an extension of an abelian variety A by an algebraic torus. More precisely, G sits in the middle of an exact sequence of the form

$$1 \to \mathbb{G}^r_m \longrightarrow G \longrightarrow A \to 0. \tag{2.1}$$

We call A the *compact part* and $\mathbb{G}^r_m$ the *linear part* of G.

Over the complex numbers, $G \simeq \mathbb{C}^{\dim G}/\pi_1(G)$. Observe that $\pi_1(G)$ is a finitely generated free abelian group. When the rank of $\pi_1(G)$ is equal to $2\dim(G)$, then G is an abelian variety.

For later use, we recall the following from [9, §10]:

Proposition 2.3. *Let G be a quasi abelian variety, let A be its compact part and let $r := \dim G - \dim A$. Then there exists a compactification $G \subset Z$ such that:*

(a) *Z is a $\mathbb{P}^r$-bundle over A;*
(b) *$\Delta := Z \setminus G$ is a simple normal crossing divisor and $\Omega^1_Z(\log \Delta)$ is a trivial bundle of rank equal to $\dim G$.*

In particular, $\overline{q}(G) = \dim G$ and $\overline{P}_m(G) = 1$ for all $m > 0$.

We are especially interested in the following particular case of Proposition 2.3:

Corollary 2.4. *Let G be a quasi-abelian variety of dimension 2 with compact part A of dimension 1. Then there is a compactification $G \subset Z$, where $Z = \mathbb{P}(\mathcal{O}_A \oplus L)$ with $L \in \mathrm{Pic}^0(A)$ and the boundary Δ is the disjoint union of two sections Δ_1 and Δ_2 of $Z \to A$.*

Proof. By Proposition 2.3, we may find a compactification Z which is a $\mathbb{P}^1$-bundle over A and such that Δ is a normal crossing divisor. The divisor Δ meets every fiber of $Z \to A$ in exactly two points, so it is a smooth bisection. By Proposition 2.3, there is a nonregular logarithmic 1-form of Z with poles contained in Δ, hence by Proposition 2.1 Δ is reducible and its components satisfy a nontrivial relation in cohomology. Since Δ is a smooth bisection of $Z \to A$, we have $\Delta = \Delta_1 + \Delta_2$ with Δ_i disjoint sections. So $Z = \mathbf{P}(\mathcal{O}_A \oplus L)$ for some $L \in \mathrm{Pic}(A)$. Since in $\mathrm{Pic}(Z)$, we have $\Delta_2 = \Delta_1 + p^*L$, where $p\colon Z \to A$ is the natural projection, Δ_1 and Δ_2 are independent in $H^2(Z, \mathbb{Z})$ unless $p^*L = 0$ in $H^2(Z, \mathbb{Z})$; namely, unless $p^*L \in \mathrm{Pic}^0(Z)$. So Proposition 2.1 implies that L is an element of $\mathrm{Pic}^0(A)$. □

Finally we recall some fundamental properties of abelian varieties that extend verbatim to the quasi-abelian case:

Proposition 2.5. *Let G be a quasi-abelian variety. Then:*

1. *if G' is a quasi-abelian variety and $\phi\colon G' \to G$ is a morphism with $\phi(0) = 0$, then ϕ is a homomorphism;*
2. *if $G' \to G$ is a finite étale cover, then G' is a quasi-abelian variety;*
3. *if $H \subset G$ is closed and $\overline{\kappa}(H) = 0$, then H is a quasi-abelian variety.*

Proof. Item 1. is [17, Thm. 5.1.37], 2. is [7, Thm. 4.2] and 3. is [9, Thm. 4]. □

2.2.1. The quasi-Albanese map

The classical construction of the Albanese variety of a projective variety can be extended to the nonprojective case by replacing regular 1-forms by logarithmic ones and abelian varieties by quasi-abelian ones. The key fact is that by Deligne [4] logarithmic 1-forms are closed (for the details of the construction see [9], [7, Section 3]).

Theorem 2.6. *Let V be a smooth algebraic variety. Then there exists a quasi-abelian variety $A(V)$ and a morphism $a_V : V \to A(V)$ such that:*

1. *if $h\colon V \to G$ is a morphism to a quasi-abelian variety, then h factors through a_V in a unique way;*
2. *if X is a compactification of V with snc boundary D, then we have the following exact sequence:*

$$1 \to \mathbb{G}_m^r \longrightarrow A(V) \longrightarrow A(X) \to 0, \tag{2.2}$$

where $r = \overline{q}(V) - q(V)$.

Proof. Item 1. is proven in [9] in the discussion immediately after Proposition 4. Item 2. is [7, p. 13]. □

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The variety $A(V)$ is called the *quasi-Albanese variety* of V and a_V the *quasi-Albanese map*. Note that the compact part of $A(V)$ is $A(X)$. We note the following logarithmic version of Abel's Theorem:

Proposition 2.7. *Let C be a smooth curve with $\overline{P}_1(C) > 0$. Then the quasi-Albanese map $a_C\colon C \to A(C)$ is an embedding.*

Proof. Denote by $\overline{C}$ the compactification of C. If $g(\overline{C}) > 0$, then the Abel–Jacobi map $\overline{C} \to A(\overline{C})$ factorizes through a_C, so the claim follows by Abel's theorem. If $g(\overline{C}) = 0$, we have $C := \overline{C} \setminus \{p_0, \dots p_k\}$, where $k := \overline{P}_1(C)$. By the universal property, the inclusion $C \to \overline{C} \setminus \{p_0, p_1\} \simeq \mathbb{G}_m$ factorizes through a_C, which therefore is an isomorphism. □

2.3. Logarithmic ramification formula

Let V, W be smooth varieties of dimension n, and let $h\colon V \to W$ be a dominant morphism. Let $g\colon X \to Z$ be a morphism extending h, where X, Z are smooth compactifications of V, respectively W, such that $D := X \setminus V$ and $\Delta := Z \setminus W$ are snc divisors. Then the pullback of a logarithmic n-form on Z is a logarithmic n-form on X, and a local computation shows that there is an effective divisor $\overline{R}_g$ of X—the *logarithmic ramification divisor*—such that the following linear equivalence holds:

$$K_X + D \sim g^*(K_Z + \Delta) + \overline{R}_g. \tag{2.3}$$

Equation (2.3) is called the *logarithmic ramification formula* (cf. [13, §11.4]).

We note the following for later use:

Lemma 2.8. *In the above setup, denote by R_g the (usual) ramification divisor of g. Let Γ be an irreducible divisor such that $g(\Gamma) \nsubseteq \Delta$. Then Γ is a component of $\overline{R}_g$ if and only if Γ is a component of $D + R_g$.*

Proof. Let $x \in \Gamma$ be a general point so that $y = g(x)$ does not lie on Δ. Then $K_Z + \Delta$ is generated locally near y by a nowhere vanishing regular n-form, while $K_X + D$ is generated locally near x either by an n-form with a logarithmic pole on Γ or by a regular n-form, according to whether Γ is a component of D or not. In the former case, Γ is always a component of $\overline{R}_g$; in the latter case, it is a component of $\overline{R}_g$ if and only if g is ramified along Γ. □

2.4. Generic vanishing

The theory of generic vanishing was introduced by Green–Lazarsfeld in [8]. It has since developed in a powerful tool to study the geometry of projective varieties via their Albanese morphism (see, for example, [18] for a nice survey). We are going to see in Section 3 that these techniques can be useful instruments also in the quasi-projective setting.

Let X be a smooth projective variety. Given a coherent sheaf $\mathcal{F}$ on X, the *cohomological support loci* of $\mathcal{F}$ are the subsets

$$V^i(X, \mathcal{F}) := \{\alpha \in \operatorname{Pic}^0(X) \mid h^i(X, \mathcal{F} \otimes \alpha) \neq 0\} \subseteq \operatorname{Pic}^0(X).$$

We say that $\mathcal{F}$ is a *GV-sheaf* if, for every $i > 0$, we have that

$$\operatorname{codim}_{\operatorname{Pic}^0(X)} V^i(X, \mathcal{F}) \geq i.$$

When all the $V^i(X, \mathcal{F})$ are empty for $i > 0$, one says, following the original terminology of Mukai [16], that $\mathcal{F}$ is a *IT(0)-sheaf*.

We recall the following well-known useful observation:

Lemma 2.9. *Let X be a smooth projective variety and L a line bundle of X.*

If $V^0(X, L) \cap (-V^0(X, L))$ has positive dimension, then $h^0(X, 2L) \geq 2$.

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Proof. For $\alpha \in V^0(X, L) \cap (-V^0(X, L))$, consider the multiplication map

$$H^0(X, L \otimes \alpha) \otimes H^0(X, L \otimes \alpha^{-1}) \longrightarrow H^0(X, 2L).$$

As α varies, the image of the map must vary since a divisor can be written as the sum of two effective divisors only in finitely many ways. As a consequence, $h^0(X, 2L) \geq 2$. □

2.5. *Curves on smooth surfaces*

Let $D > 0$ be a divisor (a 'curve') on a smooth projective surface X. The *arithmetic genus* of D is $p_a(D) := 1 - \chi(\mathcal{O}_D)$ and can be computed by means of the *adjunction formula*

$$p_a(D) - 1 = \frac{1}{2} D(K_X + D). \tag{2.4}$$

By Serre duality, one also has $p_a(D) - 1 = h^0(D, \omega_D) - h^0(D, \mathcal{O}_D)$, hence $h^0(D, \omega_D) \geq p_a(D)$.

For $m \in \mathbb{N}$, we say that D is *m-connected* if, for every decomposition $D = D_1 + D_2$, with D_1 and D_2 effective, one has $D_1 D_2 \geq m$.

We recall some well-known facts.

Lemma 2.10. *Let D be a* 1*-connected divisor, then:*

1. $h^0(D, \mathcal{O}_D) = 1$ *(so in particular,* $p_a(D) = h^0(D, \omega_D) \geq 0$*);*
2. *if L is a line bundle on D that has degree 0 on every component of D, then* $h^0(D, L) \leq 1$*, and* $h^0(D, L) = 1$ *if and only if* $L = \mathcal{O}_D$.

Proof. See [2, Lem. II 12.2]. □

Lemma 2.11. *Let* D_1*,* D_2 *be two effective nonzero divisors on a smooth projective surface X. Then*

1. $p_a(D_1 + D_2) = p_a(D_1) + p_a(D_2) + D_1 D_2 - 1$*;*
2. *if* $D_1 < D_2$*, then* $h^0(D_1, \omega_{D_1}) \leq h^0(D_2, \omega_{D_2})$*.*

Proof.

1. Use the adjunction formula (2.4).
2. Write $D_2 = D_1 + A$, and consider the decomposition sequence:

$$0 \to \mathcal{O}_{D_1}(-A) \to \mathcal{O}_{D_2} \to \mathcal{O}_A \to 0.$$

Twisting by $K_X + D_2$ and taking cohomology gives the result. □

Lemma 2.12. *Let D be an effective nonzero divisor on a smooth projective surface X. Then we have the following formulae:*

1. $h^0(X, K_X + D) = p_a(D) + p_g(X) - q(X) + h^1(X, -D)$*;*
2. $h^0(X, K_X + D) = h^0(D, \omega_D) + p_g(X) - q(X) + d$*,*
   where d is the dimension of the kernel of the restriction map $H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$*.*

Proof. 1. follows from the Riemann–Roch theorem, the adjunction formula (2.4) and Serre duality.

Taking cohomology in the restriction sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0,$$

one obtains $h^1(X, -D) = h^0(D, \mathcal{O}_D) - 1 + d$. Plugging this relation and the equality $p_a(D) - 1 = h^0(D, \omega_D) - h^0(D, \mathcal{O}_D)$ in 1. one obtains 2. □

Remark 2.13. Note that by Serre duality d is exactly the codimension of the image of the map $t \colon H^0(D, \omega_D) \to H^1(X, K_X)$.

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3. Results in arbitrary dimension via generic vanishing

Let V be a smooth quasi-projective variety, and let X be a smooth compactification of V with snc boundary. In this section, we investigate the geometric constraints imposed on the Albanese map a_X of X, and on the image of V via a_X, by the fact the logarithmic plurigenera of V are small. Our main tool will be the theory of generic vanishing (§2.4), and its recent extensions to the 'log-canonical' setting by Popa–Schnell [20] and Shibata [21].

Proposition 3.1. *Let V be a smooth quasi-projective variety, and denote by X a compactification of V with simple normal crossing boundary divisor D.*

If $\overline{P}_1(V) = \overline{P}_2(V) = 1$, then the Albanese morphism $a_X : X \to A(X)$ is surjective.

Proof. Since $V^0(X, \mathcal{O}_X(K_X + D))$ is a union of translates of subtori of $A(X)$ (cf. [21, Theorem 1.3]), Lemma 2.9 implies that $\mathcal{O}_X$ is an isolated point of $V^0(X, \mathcal{O}_X(K_X + D))$. Thus, by the projection formula $\mathcal{O}_{A(X)}$ is an isolated point of $V^0(A(X), a_{X,*}\mathcal{O}_X(K_X + D))$. Thanks to [20, Variant 5.5], we know that $a_{X,*}\mathcal{O}_X(K_X + D)$ is a GV-sheaf on $A(X)$, and therefore by [18, Lemma 1.8] we know that $\mathcal{O}_{A(X)}$ is a component of $V^{q(X)}(A(X), a_{X,*}\mathcal{O}_X(K_X + D))$. In particular, $h^{q(X)}(A(X), a_{X,*}\mathcal{O}_X(K_X + D)) \neq 0$, and we deduce that the dimension of the Albanese image of X is equal the dimension of $A(X)$. We conclude because X and $A(X)$ are projective. □

As an immediate consequence we have:

Corollary 3.2. *Let V an n-dimensional smooth quasi-projective variety, X a compactification of V such that $D := X \setminus V$ is a snc divisor. If $\overline{P}_1(V) = \overline{P}_2(V) = 1$ and $\overline{q}(V) = q(X) = n$, then the quasi-Albanese morphism $a_V : V \to A(V)$ is birational.*

Proof. By Proposition 3.1, we know that a_X is generically finite, and so X is of maximal Albanese dimension. In particular, we have that

$$0 < h^0(X, K_X) \leq h^0(X, K_X + D) = 1.$$

We conclude that $h^0(X, K_X) = 1$. Similarly, we see that $h^0(X, 2K_X) = 1$, and so by the characterization Theorem of Chen–Hacon [3] we conclude that a_X is a birational morphism. In addition, by Theorem 2.6 there is a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\ a_V\ } & A(V) \\ \big\downarrow & & \big\downarrow \simeq \\ X & \xrightarrow{\ a_X\ } & A(X) \end{array}$$

and so also a_V is birational. □

The next result refines Proposition 3.1:

Proposition 3.3. *Let V be a smooth quasi-projective variety, X a compactification of V such that $D := X \setminus V$ is a snc divisor. Let $a_X : X \to A(X)$ be the Albanese morphism, and let $E > 0$ be a divisor on $A(X)$.*

If $\overline{P}_1(V) > 0$ and D contains the support of a_X^*E, then $\overline{P}_2(V) \geq 2$.

Proof. The divisor E is of the form π^*H, where $\pi : A \to B$ is a morphism onto a positive dimensional abelian variety, and H is an ample divisor on B. Set $f := \pi \circ a_X$; by assumption there is a positive integer N such that $ND \geq f^*H$. Set $\Delta := D - \frac{1}{N} f^*H$: by assumption the $\mathbb{Q}$-divisor Δ has snc support and $\Delta = \sum_i d_i \Delta_i$, with Δ_i irreducible divisors and $0 \leq d_i \leq 1$. Given $\alpha \in \mathrm{Pic}^0(B)$, the divisor

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$K_X + D + f^*(\alpha)$ is $\mathbb{Q}$-linearly equivalent to $K_X + \Delta + f^*(\frac{1}{N}H + \alpha)$. Since $\frac{1}{N}H + \alpha$ is ample if H is, the assumptions of [6, Thm. 6.3] or [1, Thm. 3.2] are satisfied and we have

$$H^i(B, f_*\mathcal{O}_X(K_X + D) \otimes \alpha) = 0, \quad \text{for all } i > 0, \quad \text{and all } \alpha \in \operatorname{Pic}^0(B).$$

So $f_*(K_X + D)$ is an IT(0)-sheaf, and therefore for all $\alpha \in \operatorname{Pic}^0(B)$ we have $h^0(B, f_*\mathcal{O}_X(K_X + D) \otimes \alpha) = h^0(B, f_*(K_X + D)) = \overline{P}_1(V) > 0$. So $V^0(X, K_X + D)$ contains the positive dimensional abelian subvariety $f^* \operatorname{Pic}^0(B) = \pi^* \operatorname{Pic}^0(B)$ and $\overline{P}_2(V) \geq 2$ by Lemma 2.9. □

When $q(V)$ is equal to 1, Proposition 3.3 gives:

Corollary 3.4. *Let V be a smooth quasi-projective variety, X a compactification of V such that $D := X \setminus V$ is a snc divisor. Let $a_X : X \longrightarrow A(X)$ be the Albanese morphism.*

If $q(V) = \overline{P}_1(V) = \overline{P}_2(V) = 1$, then $a_X(V) = A(X)$.

Remark 3.5. Observe that, by Proposition 3.1, we can deduce that if $\overline{P}_1(V) = \overline{P}_2(V) = 1$ then the map $a_X|_V$ is dominant. In addition, as a consequence of Proposition 3.3, we know that the complement of $a_X(V)$ in $A(X)$ does not contain a divisor. Notice that, if $q(X) > 1$, this does not mean that the complement of $a_X(V)$ has codimension > 1, since the image of V is not necessarily open in $A(X)$.

4. The geometry of the quasi-Albanese morphism

The aim of this section is to establish the following fundamental step in the proof of Theorem A:

Proposition 4.1. *Let V be a smooth complex algebraic surface with $\overline{q}(V) = 2$. Assume that:*

(a) $\overline{P}_1(V) = \overline{P}_2(V) = 1$ *and* $q(V) > 0$*; or*
(b) $\overline{P}_1(V) = \overline{P}_3(V) = 1$ *and* $q(V) = 0$.

Then the quasi-Albanese morphism $a_V : V \longrightarrow A(V)$ is dominant.

We use different approaches for the case $q(V) = 0$ and $q(V) > 0$, so we treat them separately. In fact, Proposition 4.1 is just the combination of Propositions 4.2 and 4.3 below.

4.1. The quasi-Albanese map when $q(V) \geq 1$

Proposition 4.2. *Let V be a smooth algebraic surface such that $\overline{P}_1(V) = \overline{P}_2(V) = 1$, $\overline{q}(V) = 2$ and $q(V) \geq 1$. Then the quasi-Albanese morphism of V is dominant and, hence, generically finite.*

Proof. When $q(V) = \overline{q}(V) = 2$, the map a_V is dominant by Corollary 3.2.

Assume now that $q(V) = 1$ and, by contradiction, that the image of a_V is a curve C. Denote by $\overline{C}$ the smooth projective model of C; since $\overline{C}$ dominates the elliptic curve $A(X)$, the curve $\overline{C}$ has positive genus and a_V extends to a morphism $f : X \longrightarrow \overline{C}$ such that the Albanese morphism factorizes through $\overline{C}$. By the universal property of the Albanese map, the map $\overline{C} \longrightarrow A(X)$ is an isomorphism. Now, Corollary 3.4 implies that $C = \overline{C}$. In particular, by Proposition 2.5 up to translation C is an algebraic subgroup of $A(V)$. On the other hand, C must generate $A(V)$ as an algebraic group, so it follows $C = A(V)$, a contradiction. □

4.2. The quasi-Albanese map when $q(V) = 0$

In this section, we prove the following

Proposition 4.3. *Let V be a smooth algebraic surface such that $\overline{P}_1(V) = \overline{P}_3(V) = 1$, $\overline{q}(V) = 2$, and $q(V) = 0$. Then the quasi-Albanese morphism of V is dominant and, hence, generically finite.*

The proof is based on numerical arguments and is quite intricate, so we break it into several steps. We start by proving two results on curves on smooth projective surfaces.

Lemma 4.4. *Let X be a nonsingular projective surface and A a 1-connected effective divisor with $p_a(A) = 1$.*

Then A contains a 2-connected divisor B such that $p_a(B) = 1$.

Proof. If A is 2-connected, there is of course nothing to prove. Otherwise, take any decomposition $A = A_1 + A_2$ with $A_1A_2 = 1$ and A_1 minimal with respect to $A_1A_2 = 1$. A_2 is 1-connected, A_1 is 2-connected, and by Lemma 2.11 $p_a(A_1) + p_a(A_2) = 1$, $p_a(A_1) \geq 0$, $p_a(A_2) \geq 0$. If $p_a(A_1) = 1$, we have proved the statement; if not, we repeat the argument on A_2. □

Lemma 4.5. *Let X be a nonsingular projective surface and B a 2-connected effective divisor with $p_a(B) = 1$. Then:*

1. *if B is not irreducible, then every component Γ of B is smooth rational and satisfies $(K_X + B)\Gamma = 0$;*
2. *$\omega_B = \mathcal{O}_B$;*
3. *if Γ is an irreducible component of B and $B - \Gamma > 0$, then $B - \Gamma$ is 1-connected.*

Proof. 1. Since $p_a(B) = 1$, $(K_X + B)B = 0$. On the other hand, for every irreducible component Γ of B, one has $(K_X+B)\Gamma = (K_X+\Gamma)\Gamma+(B-\Gamma)\Gamma \geq 0$ because $(K_X+\Gamma)\Gamma \geq -2$ by adjunction and $(B-\Gamma)\Gamma \geq 2$ since B is 2-connected and reducible. So necessarily $(K_X + B)\Gamma = 0$, and Γ is smooth rational.

2. Since, by 1. ω_B has degree 0 on every component of B and $h^0(B, \omega_B) = 1$, by Lemma 2.10 $\omega_B = \mathcal{O}_B$.

3. By the proof of 1. one has $\Gamma(B - \Gamma) = 2$. Let $B - \Gamma = A_1 + A_2$ with $A_1 > 0$, $A_2 > 0$. Then because B is 2-connected, $A_i(A_j + \Gamma) \geq 2$ for $\{i, j\} = \{1, 2\}$. Since $(A_1 + A_2)\Gamma = 2$ necessarily $A_1A_2 \geq 1$ and so $B - \Gamma$ is 1-connected. □

Lemma 4.6. *Let X be a nonsingular projective surface with $q(X) = 0$, and let B be an effective 2-connected divisor satisfying $p_a(B) = 1$. Then one of the following occurs:*

(a) *$h^0(X, 2K_X + 2B) \geq 2$;*
(b) *X is rational, and there is a blow-down morphism $\rho\colon X \to T$ with exceptional divisor $\sum_{j=1}^n E_j$ (where the E_j are -1-curves) such that $K_X + B \sim \sum_{j=1}^n E_j$ and B is disjoint from $\sum_{j=1}^n E_j$.*

Proof. From Lemma 2.12, we obtain $h^0(X, K_X + B) = p_g(X) + 1 \geq 1$.

Assume that $K_X + B$ is nef. Then since $(K_X + B)B = 0$, we have $K_X(K_X + B) = (K_X + B)^2 \geq 0$ and $(2K_X + B)(K_X + B) = 2K_X(K_X + B) \geq 0$. So we have two possibilities: either $K_X + B \sim 0$ (and $p_g(X) = 0$) or there is a nonzero effective divisor B_1 in $|K_X + B|$, and $p_a(B_1) \geq 1$ implying by Lemma 2.12 that $h^0(X, K_X + B_1) \geq 1 + p_g$, that is, $h^0(X, 2K_X + B) \geq 1$.

In the second case, the restriction map $H^0(X, K_X + B) \to H^0(B, \omega_B)$ is surjective because $q(X) = 0$. Since $\omega_B = \mathcal{O}_B$, the map $H^0(X, 2K_X + 2B) \to H^0(B, \omega_B^{\otimes 2})$ is also nonzero. So the exact sequence

$$0 \to H^0(X, 2K_X + B) \to H^0(X, 2K_X + 2B) \to H^0(B, \omega_B^{\otimes 2})$$

gives $h^0(X, 2K_X + 2B) \geq 2$.

Assume now that B_1 is not nef, and let θ be an irreducible curve with $B_1\theta < 0$. Since B_1 is effective, θ is a component of B_1 with $\theta^2 < 0$. In addition, for every component Γ of B, we have, by Lemma 4.5, $B_1\Gamma = 0$, so θ is not a component of B, and thus $B\theta \geq 0$. Then the only possibility is that θ is an irreducible -1-curve disjoint from B. We contract θ and replace B by its image under the contraction; repeating this process, we eventually end up with a birational morphism of smooth surfaces $\rho\colon X \to T$ such that $K_T + \rho(B)$ is nef, and $\rho(B)$ is still a 2-connected divisor with $p_a = 1$.

By the discussion in the previous case, we see that either $h^0(T, 2K_T + 2\rho(B)) \geq 2$ or $K_T + \rho(B) \sim 0$. In the former case $h^0(X, 2K_X + 2B) \geq h^0(T, 2K_T + 2\rho(B)) \geq 2$. In the latter case, taking pullbacks we get $\rho^*(K_T) + B = K_X - \sum_{j=1}^n E_j + B \sim 0$. So, if L is an ample divisor on T, then ρ^*L is nef and big and it satisfies $K_X\rho^*L = -B\rho^*L < 0$, so $\kappa(X) = -\infty$. Since $q(X) = 0$, the surface X is rational. □

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For the rest of the section, we refer to the following situation:

Setting 4.7. We let V be a smooth open algebraic surface with $\overline{q}(V) = 2$, $q(V) = 0$ and $\overline{P}_1(V) = \overline{P}_3(V) = 1$. We consider the standard compactification $Z = \mathbb{P}^1 \times \mathbb{P}^1$ of $A(V) = \mathbb{G}_m^2$, and we denote by $\Delta := Z \setminus A(V)$ the boundary. Finally, we fix a compactification X of V with snc boundary D such that the quasi-Albanese map $a_V : V \to A(V)$ extends to a morphism $g\colon X \to Z$.

We exploit the previous results to gain more information on the pair (X, D).

Lemma 4.8. *There is a blow-down morphism $\rho\colon X \to T$ with exceptional divisor $\sum_{j=1}^n E_j$ (where the E_j are -1-curves) such that one of the following holds:*

(a) *$p_g(X) = 0$, $h^0(D, \omega_D) = 1$, X is a rational surface, and there is a 2-connected divisor $B \leq D$ such that $K_X + B \sim \sum_{j=1}^n E_j$ and B is disjoint from $\sum_{j=1}^n E_j$;*

(b) *$p_g(X) = 1$, $h^0(D, \omega_D) = 0$, and T is a K3 surface.*

In either case, there is a divisor $B \geq 0$ such that $K_X + B \sim \sum_{j=1}^n E_j$.

Proof. The assumption $\overline{P}_1(V) = 1$ implies that either $p_g(X) = 0$ or $p_g(X) = 1$; in either case, the value of $h^0(D, \omega_D)$ can be computed by Lemma 2.12.

Assume first $p_g(X) = 0$. Then, by Lemma 2.12, $q(X) = 0$ implies that $h^0(D, \omega_D) = 1$. So there is a connected component A of D such that $p_a(A) = h^0(A, \omega_A) = 1$, and by Lemma 4.4 there is a 2-connected divisor $B \leq A$ with $p_a(B) = 1$. By Lemma 4.6 and the hypothesis $\overline{P}_2(V) = 1$, X is rational and there is a blow-down morphism $\rho\colon X \to T$ as in (a).

Assume now $p_g(X) = 1$. In this case, X has nonnegative Kodaira dimension and we take $\rho\colon X \to T$ to be the morphism to the minimal model. If X is of general type, $K_T^2 > 0$ and $h^0(X, 2K_X) = h^0(T, 2K_T) = K_T^2 + \chi(\mathcal{O}_X) \geq 2$, contradicting $\overline{P}_2(V) = 1$.

If X is properly elliptic, then, because $p_g(T) = 1$ and $K_T \neq \mathcal{O}_T$, $h^0(T, -K_T) = 0$, and so by duality $h^2(T, 2K_T) = 0$. Since $K_T^2 = 0$, by the Riemann–Roch theorem, we obtain $h^0(T, 2K_T) = h^1(T, 2K_T) + \chi(\mathcal{O}_X) \geq 2$.

So $\kappa(X) = 0$ and by the classification of projective surfaces we conclude that T is a $K3$-surface. We have $K_X = \sum_{j=1}^n E_j$, so in this case the last claim holds with $B = 0$. □

Lemma 4.9. *One has:*

1. *$h^0(X, 2K_X + D) = h^0(X, 3K_X + 2D) = p_g(X)$;*
2. *if D_i is the unique divisor in $|i(K_X + D)|$, $i = 1, 2$, then $h^0(D_i, \omega_{D_i}) = 0$.*

Proof. 1. Since $D > 0$ and $\overline{P}_i(V) = 1$ for $i = 1, 2, 3$ by assumption, we have $p_g(X) \leq h^0(X, 2K_X + D) \leq h^0(X, 3K_X + 2D) \leq 1$. So if $p_g(X) = 1$ the assertion is trivial.

For $p_g(X) = 0$ and $m \geq 1$, consider the exact sequence:

$$0 \to H^0(X, mK_X + (m-1)D) \to H^0(X, m(K_X + D)) \to H^0(D, \omega_D^{\otimes m}). \tag{4.1}$$

The second map in equation (4.1) is an isomorphism for $m = 1$ since $q(X) = 0$, so it is nonzero for all $m \geq 1$ since D is a reduced divisor. Then $\overline{P}_2(V) = 1$ and $\overline{P}_3(V) = 1$ imply that $h^0(X, 2K_X + D) = h^0(X, 3K_X + 2D) = 0$.

2. is an immediate consequence of $q(X) = 0$ and of 1. (see Lemma 2.12). □

We now turn to the study of the quasi-Albanese map.

Lemma 4.10. *If the image of a_V is a curve, then it is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the general fiber of a_V is connected.*

Proof. Let g as in Setting 4.7. By assumption, the image Γ of g is a curve. Let $X \to \widetilde{\Gamma} \to \Gamma$ be the Stein factorization of g, and let $\Gamma_0 \subset \widetilde{\Gamma}$ be the image of V. Then by the universal property of the quasi-Albanese map, $A(V)$ is isomorphic to $A(\Gamma_0)$. In addition, the image of a_V is isomorphic to Γ_0 by Proposition 2.7.

It follows that g has connected fibers; hence, the general fiber of a_V is also connected. Finally, Γ_0 is rational, since $q(X) = 0$, and has logarithmic genus 2; hence, it is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. □

We get immediately:

Corollary 4.11. *If the image of a_V is a curve, then there is a fibration $f : X \to \mathbb{P}^1$ such that D contains the supports F_1^s, F_2^s and F_3^s of three distinct fibers F_1, F_2 and F_3 of f.*

Lemma 4.12. *Let E_j, $j = 1, \dots n$, be the -1-curves contracted by the blow-down morphism $\rho : X \to T$ of Lemma 4.8.*

Then $E_j F_i^s \geq 0$ for all j, i.

Proof. Assume by contradiction that $E_j F_1^s < 0$. Then E_j and F_1^s have common components. Write $E_j = A + C_1$ and $F_1^s = A + C_2$, where $A > 0$ and $C_1, C_2 \geq 0$ have no common components. Note that $C_2 > 0$ otherwise blowing down E_j we would contract the whole fiber. Now, $E_j F_1^s < 0$ yields $A^2 + AC_1 + AC_2 + C_1C_2 < 0$. But, because E_j is a (-1)-curve, $E_j A = A^2 + AC_1 \geq -1$ (see [15, Prop. 3.2]) and because F_1^s is 1-connected $AC_2 \geq 1$. Since $C_1C_2 \geq 0$, we have a contradiction. □

Lemma 4.13. *Let E_j, $j = 1, \dots n$, be the -1-curves contracted by the blow down-morphism $\rho : X \to T$ of Lemma 4.8.*

Then $\sum_{j=1}^n E_j + D$ has at most one component transversal to f.

Proof. Let M_1, M_2 be two distinct components of $\sum_{j=1}^n E_j + D$ tranversal to f. Then because F_i^s is the support of a full fiber $M_k(\sum_{i=1}^3 F_i^s) \geq 3$ for $k = 1, 2$ yielding $p_a(M_1 + M_2 + \sum_{i=1}^3 F_i^s) \geq 2$ and so, by Lemma 2.12, $h^0(X, K_X + M_1 + M_2 + \sum_{i=1}^3 F_i^s) \geq 2$.

By Lemma 4.8, there is a divisor $0 \leq B \leq D$ such that $K_X + B \sim \sum_j E_j$. Since $M_1 + M_2 + \sum_{i=1}^3 F_i^s \leq \sum_{j=1}^n E_j + D \in |K_X + B + D|$, we obtain $h^0(X, 2K_X + B + D) \geq 2$, which contradicts $\overline{P}_2(V) = 1$. □

Corollary 4.14. *Assume $p_g(X) = 0$, and let $0 < B \leq D$ be the divisor such that $K_X + B \sim \sum_j E_j$ (cf. Lemma 4.8). If B has components in common with $\sum_{i=1}^3 F_i^s$, then one of the following happens:*

(a) *$B \leq F_i^s$ for some $i \in \{1, 2, 3\}$, or*
(b) *B has a unique component H transversal to f and $B - H$ is contained in, say, F_1^s. Furthermore, $HF_1^s = 2$ and $HF_i^s = 1$ for $i = 2, 3$.*

Proof. Suppose no component of B is transversal to f. Then because B is connected and we are assuming that B has common components with $\sum F_i^s$, we have statement (a).

If there is a component H of B transversal to f, the assumption that B has common components with $\sum_{i=1}^3 F_i^s$ implies $B - H \neq 0$. Recall that, by Lemma 4.8, the divisor B is 2-connected and $p_a(B) = 1$, so, by Lemma 4.5, $H \simeq \mathbb{P}^1$ and $B - H$ is connected. From Lemma 4.13, $B - H$ is contained in fibers of f, and since $B - H$ is connected, we obtain $B - H$ contained in F_i^s for one of $i = 1, 2, 3$, say F_1^s.

In this case, $HF_1^s \geq 2$ (since B is 2-connected) and $HF_i^s \geq 1$ for $i = 2, 3$ because F_i^s is the support of a full fiber. Now, $p_a(H + \sum_{i=1}^3 F_i^s) \geq p_a(H) + \sum_{i=1}^3 p_a(F_i^s) + H(\sum_{i=1}^3 F_i^s) - 3$. Since $h^0(D, \omega_D) = 1$ (cf. Lemma 4.8), D cannot contain an effective divisor with $p_a \geq 2$ and so necessarily $HF_1^s = 2$ and $HF_i^s = 1$ for $i = 2, 3$. □

Lemma 4.15. *Let E_j, $j = 1, \dots n$, be the -1-curves contracted by the blow-down morphism $\rho : X \to T$, and let $B \leq D$ be the divisor such that $K_X + B \sim \sum_j E_j$ (cf. Lemma 4.8):*

1. *if $F_i^s \leq D - B$, then $E_j F_i^s \leq 1$ for all E_j;*
2. *the general fiber F of f satisfies $F \sum_{j=1}^n E_j = 0$.*

Proof. 1. Assume that $E_j F_i^s \geq 2$. Then $h^0(E_j + F_i^s, \omega_{E_j + F_i^s}) = p_a(E_j + F_i^s) \geq p_a(F_i^s) + 1 \geq 1$, and so since by the assumption $F_i^s \leq D - B$, we have $E_j + F_i^s \leq D_1 \in |K_X + D|$, contradicting Lemma 4.9 (cf. Lemma 2.11).

2. Suppose otherwise, that is, that there is an irreducible curve $M \leq \sum_{j=1}^n E_j$ transversal to the fibration f. Since B and $\sum_{j=1}^n E_j$ are disjoint (cf. Lemma 4.8), by Lemma 4.13 the divisor B must be contained in a fiber of f.

Since each F_i^s is the support of a whole fiber, we have $MF_i^s \geq 1$. Since $M \leq \sum_{j=1}^n E_j$, we have $M^2 = -l < 0$. Then among the E_j's, there are $E_{p_1}, \dots, E_{p_{l-1}}$ such that $E_m = M + \sum_{k=1}^{l-1} E_{p_k}$ is one of the E_j. This is true by [15, Lemma 3.2] if $l = 1$, by [15, Prop. 4.1] if $l = 2$ and by [15, Prop. 4.2] if $l > 2$.

Since $E_{p_k} F_i^s \geq 0$ for all $k = 1, \dots l-1$ and $i = 1, 2, 3$ by Lemma 4.12, we obtain $E_m(\sum F_i^s) \geq 3$. Since $E_m(E_m + \sum_{i=1}^3 F_i^s) \geq 2$, one obtains

(a) $p_a(2E_m + \sum_{i=1}^3 F_i^s) \geq 2$ if $p_g(X) = 0$ and $B \neq 0$ is contained in one of the F_i^s.
(b) $p_a(2E_m + \sum_{i=1}^3 F_i^s) \geq 1$, if $p_g(X) = 1$ or $p_g(X) = 0$ and B is not contained in $\sum_{i=1}^3 F_i^s$.

Since $E_m \leq K_X + B$, we have $2E_m + \sum F_i^s \leq 2K_X + 2B + D$, which in case (a) yields $h^0(X, 3K_X + 2B + D) \geq 2$, contradicting $\overline{P}_3(V) = 1$. In case (b), we have $2E_m + \sum F_i^s \leq 2(K_X + B) + (D - B) = 2K_X + B + D$, yielding $h^0(X, 3K_X + B + D) \geq p_g(X) + 1$ and contradicting Lemma 4.9. □

Corollary 4.16. *The fibration $f\colon X \longrightarrow \mathbb{P}^1$ descends to a fibration $\bar{f}\colon T \longrightarrow \mathbb{P}^1$.*

Lemma 4.17. *The general fiber F of $\bar{f}$ has genus* 0.

Proof. Clearly, f and $\bar{f}$ have the same general fiber by construction. Since $K_T + \rho(B) = 0$ and F is nef, we have $FK_T = -F\rho(B) \leq 0$, so either $FK_T = 0$ and F has genus 1, or $K_T F = -2$ and F is smooth rational.

Assume by contradiction that F has genus 1. In case $p_g(X) = 0$, all the components of B are contracted by f since $F\rho(B) = 0$, so B, being connected, is contained in a fiber of f and we may assume that B and $F_2^s + F_3^s$ are disjoint. The divisor $D_0 = B + F_2^s + F_3^s$ satisfies $h^0(D_0, \omega_{D_0}) = p_a(B) + p_a(F_2^s) + p_a(F_3^s) = 1 + p_a(F_2^s) + p_a(F_3^s)$. Since $h^0(D_0, \omega_{D_0}) \leq h^0(D, \omega_D) = \overline{P}_1(V) = 1$ (cf. Lemma 2.12), we conclude that $p_a(F_2^s) = p_a(F_3^s) = 0$. If $p_g(X) = 1$, applying the same argument to $D_0 := F_1^s + F_2^s + F_3^s$, we obtain $p_a(F_i^s) = 0$ for $i = 1, 2, 3$.

Denote by $\overline{F}_i$ the full fiber of $\bar{f}$ corresponding to F_i^s for $i = 1, 2, 3$. If $p_g(X) = 1$, then T is minimal by Lemma 4.8; hence, $\overline{F}_i$ does not contain -1-curves for $i = 1, 2, 3$. If $p_g(X) = 0$ and E is an irreducible -1 curve of T, then $\rho(B)E = -K_T E = 1$; namely, E meets $\rho(B)$. So, in this case, there is no -1-curve contained in $\overline{F}_2 + \overline{F}_3$.

By Lemmas 4.12 and 4.15, we know that $0 \leq F_i^s E_j \leq 1$ for $i = 2, 3$ and also for $i = 1$ if $p_g(X) = 1$. Therefore, $\rho^*(\rho(F_i^s)) \leq F_i^s + \sum_{j=1}^n E_j$. So the reduced divisor $\rho(F_i^s)$ still has $p_a = 0$ for $i = 1, 2, 3$ in the case $p_g(X) = 1$ and for $i = 2, 3$ in the case $p_g(X) = 0$.

For $i = 1, 2, 3$ in the case $p_g(X) = 1$ and for $i = 2, 3$ in the case $p_g(X) = 0$, the elliptic fibers $\overline{F}_i$, since they do not contain -1-curves and have support with $p_a = 0$, must be of type * (see [2, Chp.V, §7]). Note that, in particular, the $\overline{F}_i$ cannot be multiple fibers of $\bar{f}$ (see [2, Chp.V, §7]). So, if $\overline{F}$ is a fiber of type * with support F_0, we have $2F_0 \geq \overline{F}$ if $\overline{F}$ is of type I_b^* $(= \tilde{D}_{4+b})$ and $3F_0 \geq \overline{F}$ if $\overline{F}$ is of type IV^* $(= \tilde{E}_6)$.

On the other hand, if $p_g(X) = 0$, note that the sum of the Euler numbers $e(\overline{F}_2) + e(\overline{F}_3)$ cannot exceed 12 since a relatively minimal elliptic fibration on a rational surface has $c_2 = 12$. Since the Euler numbers of fibers of type * are always bigger than 6 except for type I_0^*, $\overline{F}_2$ and $\overline{F}_3$ must be of type I_0^*, and so, for instance, $2(F_2^s + \sum_{j=1}^n E_j)$ contains $\rho^*(\overline{F}_2) = F_2$. Since $h^0(X, F_2) = 2$, we obtain $h^0(X, 2K_X + 2B + 2F_2^s) \geq 2$, contradicting $h^0(X, 2K_X + 2D) = 1$.

Similarly, if $p_g(X) = 1$, then the sum of the Euler numbers $e(\overline{F}_1) + e(\overline{F}_2) + e(\overline{F}_3)$ cannot exceed 24. If one of the $\overline{F}_i$ is of type I_b^*, we have a contradiction as above, and if one of the $\overline{F}_i$ is of type IV^* $(= \tilde{E}_6)$, then $F_i \leq 3F_i^s + 3\sum_{j=1}^n E_j \leq 3K_X + 3D$, contradicting $h^0(X, 3K_X + 3D) = 1$. On the other hand if all the $\overline{F}_i$ are of type II^* $(= \tilde{E}_8)$ or type III^* $(= \tilde{E}_7)$, the sum of the Euler numbers is larger than 24. □

Proof of Proposition 4.3. By Lemma 4.17, the general fiber F of f has genus 0, and in particular, we have $p_g(X) = 0$. Since $K_X F = -2$, we have $BF = 2$, and so B has a component H transversal to f. If B

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has no common component with F_i^s, $i = 1, 2, 3$, then we set $D_0 = B + F_1^s + F_2^s + F_3^s$ and we compute

$$p_a(D_0) = p_a(B) + p_a(F_1^s) + p_a(F_2^s) + p_a(F_3^s) + B(F_1^s + F_2^s + F_3^s) - 3.$$

Since D_0 is reduced and connected, we have $p_a(D_0) = h^0(D_0, \omega_{D_0}) \le h^0(D, \omega_D) = \overline{P}_1(V) = 1$ (cf. Lemma 2.11 and 2.12) and we conclude that $p_a(F_i^s) = 0$, $BF_i^s = 1$ for $i = 1, 2, 3$. On the other hand, if B and $F_1^s + F_2^s + F_3^s$ have common components, then by Corollary 4.14 at least two of the F_i^s, say F_2^s and F_3^s, have no common components with B and satisfy $F_i^s B = 1$ for $i = 2, 3$.

So in any case, F_2^s contains a unique irreducible curve Γ such that $\Gamma B \neq 0$. Since $BF = 2$ for a general fiber of F and $BF_2^s = 1$, Γ appears with multiplicity 2 in the full fiber F_2 containing F_2^s. The curve Γ is not contracted by ρ since $B\Gamma = 1$ and B does not meet the ρ-exceptional curves.

Write $\rho^*(\rho(\Gamma)) = \Gamma + Z$, with Z an exceptional divisor. Since $B = \rho^*(\rho(B))$, the projection formula gives

$$1 = B\Gamma = \rho^*(\rho(B))\Gamma = \rho^*(\rho(B))(\Gamma + Z) = \rho^*(\rho(B))\rho^*(\Gamma) = \rho(B)\rho(\Gamma).$$

Since $K_T + \rho(B) = 0$, we have $K_T\rho(\Gamma) = -1$. The curve $\rho(\Gamma)$ is contained in a fiber of $\bar{f}$, so $\rho(\Gamma)^2 \le 0$ and $\rho(\Gamma)$ is a -1-curve by the adjunction formula. So $\overline{F}_2 = 2\rho(\Gamma) + C$, where C does not contain $\rho(\Gamma)$. The components of C do not meet $\rho(B) = -K_T$; hence, they are all -2-curves. From $\rho(\Gamma)\overline{F}_2 = 0$, we get $C\rho(\Gamma) = 2$. If there are two distinct components N_1 and N_2 of C with $\rho(\Gamma)N_i = 1$, then N_1 and N_2 are disjoint since the dual graph of $\overline{F}_2$ is a tree. So $(2\rho(\Gamma)+N_1+N_2)^2 = 0$ and therefore $\overline{F}_2 = 2\rho(\Gamma)+N_1+N_2$. If there is only a component N of G with $N\rho(\Gamma) = 1$, then N appears in $\overline{F}_2$ with multiplicity 2 and $M_1 := \rho(\Gamma) + N$ is a (reducible and reduced) -1-curve such that $\overline{F}_2 = 2M_1 + C_1$ and C_1 and M_1 have no common component. So we may repeat the previous argument and either write $\overline{F}_2 = 2M_1 + N_1 + N_2$, where N_1, N_2 are disjoint -2-curves contained in C_1 with $N_iM_1 = 1$, or $\overline{F}_2 = 2(M_1 + N) + C_2$, where N is a component of C such that $M_2 := M_1 + N$ is a -1-curve and C_2 and M_2 have no common component. This process must of course terminate, showing that $\overline{F}_2 = 2M_0 + N_1 + N_2$, where M_0 is a reduced -1-curve, and N_1 and N_2 are disjoint -2-curves not contained in M_0. In particular, we have shown that $\overline{F}_2$ has no component of multiplicity > 2.

Since (as in the proof of Lemma 4.17) $\rho^*(\rho(F_2^s)) \le F_2^s + \sum_{j=1}^n E_j$, we conclude that $2F_2^s + 2\sum_{j=1}^n E_j$ contains the full fiber F_2 of f, and as in the in the proof of Lemma 4.17, we obtain $h^0(X, 2K_X + 2B + 2F_2^s) \ge 2$, contradicting $h^0(X, 2K_X + 2D) = \overline{P}_2(V) = 1$.

So we have excluded all the possibilities for the genus of the general fiber F of the fibration induced by a_V if a_V is not dominant, and the proof is complete. □

Example 4.18. The hypothesis $\overline{P}_3(V) = 1$ in Proposition 4.3 is necessary. A K3 surface with an elliptic fibration with three singular fibers of type IV^*(cf. the proof of Lemma 4.17) does exist. Take for instance the elliptic curve C with an automorphism h of order 3. Let $\mathbb{Z}/3$ act on $C \times C$ by $(x, y) \mapsto (hx, h^2y)$, and denote by X_0 the quotient surface. Then X_0 has nine singular points of type A_2, and the first projection $C \times C \to C$ descends to an isotrivial elliptic fibration $X_0 \to C/\mathbb{Z}_3 \cong \mathbb{P}^1$ with three 'triple' fibers, each containing three of the nine singular points. The minimal resolution X of X_0 is a K3 surface with an isotrivial elliptic fibration with three fibers of type IV^*. Alternatively, X can be constructed as the minimal resolution of a simple $\mathbb{Z}_3$-cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over three fibers of one of the fibrations plus three fibers of the other.

Consider the surface $V := X \backslash \{F_1^s,\ F_2^s,\ F_3^s\}$, where the F_i^s denote the supports of the three fibers of type IV*. Then V has $\overline{P}_1(V) = \overline{P}_2(V) = 1$ and $\overline{q}(V) = 2$ (see Proposition 2.1), and its quasi-Albanese map is the restriction of the elliptic fibration $X \to \mathbb{P}^1$, and so it is not dominant.

5. Proof of Theorem A

Thanks to what we have proven in the previous section, we know that in the assumptions of Theorem A the quasi-Albanese map of V is dominant. Since V and $\mathrm{Alb}(V)$ have the same dimension, we can conclude that a_V is generically finite. In particular, there is a generically finite morphism $g : X \to Z$, where Z is the compactification of $\mathrm{Alb}(V)$ described in Proposition 2.3. Now, our proof boils down to the following two facts

1. the morphism g has degree 1;
2. all the components of D that are not mapped to the boundary Δ of $\mathrm{Alb}(V)$ are contracted by g.

When $q(V) = 2$, the first assertion is Corollary 3.2. The second statement can be proven by contradiction (see §5.2).

The situation is more involved when $q(V) < 2$. We start by proving a slightly weaker version of assertion 2 (Lemma 5.5), and we use it to show that the finite part of the Stein factorization of g is étale over $A(V)$ (in case $q(V) = 1$ this requires also a topological argument). Now, the universal property of the quasi-Albanese map implies that g has indeed degree 1. Finally, we complete the proof of assertion 2 by means of a local computation.

5.1. *Preliminary steps*

Notation 5.1. We let V be a smooth open algebraic surface V with $\overline{q}(V) = 2$. We assume $\overline{P}_1(V) = \overline{P}_2(V) = 1$ if $q(V) \geq 1$ and $\overline{P}_1(V) = \overline{P}_3(V) = 1$ if $q(V) = 0$.

If $q(V) \leq 1$ fix a compactification Z with boundary Δ of the quasi-Albanese variety $A(V)$ of V as follows:

- if $q(V) = 1$, we take Z a $\mathbb{P}^1$-bundle over the compact part A of $A(V)$ as in Corollary 2.4, and we write $\Delta = \Delta_1 + \Delta_2$, where Δ_i are disjoint sections of $Z \to A$;
- if $q(V) = 0$ (and thus $A(V) = \mathbb{G}_m^2$), we take $Z = \mathbb{P}^1 \times \mathbb{P}^1$ with the obvious choice of the boundary Δ.

We fix a compactification X of V with snc boundary D such that the quasi-Albanese map $a_V : V \to A(V)$ extends to a morphism $g \colon X \to Z$. In addition, we write $H := g^{-1}(\Delta)$ (set-theoretic inverse image), and in case $q(V) = 1$, we also write $H_i = g^{-1}(\Delta_i)$, $i = 1, 2$. Note that $D \geq H$ by construction. When $q(V) = 1$, we denote by $a_X : X \to A = A(X)$ the Albanese map of X.

The proof is quite involved, so we break it into several smaller steps, and we examine the case $q(V) \leq 1$ since, by Corollary 3.2, for $q(V) = 2$ we already know that a_V is birational.

Lemma 5.2. *If $q(V) \leq 1$, then the divisor of poles of a generator of $H^0(X, K_X + D)$ is a nonzero subdivisor of H. In particular, $p_g(X) = 0$ and $h^0(X, K_X + H) = 1$.*

Proof. The vector space $H^0(Z, \Omega^1_Z(\log \Delta))$ is generated by two logarithmic 1-forms τ_1 and τ_2 such that $\omega := \tau_1 \wedge \tau_2$ vanishes nowhere on $A(V)$ (cf. Proposition 2.3) and has poles exactly on Δ. More precisely, if $q(V) = 1$, then we can take τ_1 to be the pullback of a nonzero regular 1-form on $A = A(X)$ via the projection $Z \to A$ and τ_2 a logarithmic 1-form with poles on Δ_1 and Δ_2 (cf. proof of Corollary 2.4); if $q(V) = 0$, then we can take τ_1 and τ_2 to be pullbacks of nonzero logarithmic forms via the two projections $\mathbb{G}_m^2 \to \mathbb{G}_m$.

Since g is surjective by Propositions 4.2 and 4.3, $g^*\omega$ is a nonzero logarithmic form on X and a local computation shows that, if Γ is an irreducible component of H not contracted by g, then $g^*\omega$ has a pole along Γ. On the other hand, the poles of ω are contained in H by construction.

Since $\overline{P}_1(V) = h^0(X, K_X + D) = 1$, it follows immediately that $p_g(X) = 0$ and $h^0(X, K_X + H) = 1$. □

Lemma 5.3.

1. *If $q(V) = 0$, then H is connected and $h^0(\omega_H) = 1$;*
2. *if $q(V) = 1$, then for $i = 1, 2$ the divisor H_i is connected and $h^0(\omega_{H_i}) = 1$.*

Proof. 1. The divisor H is connected since it is the support of the nef and big divisor $g^*\Delta$. Lemma 2.12 gives $h^0(H, \omega_H) = 1$ since $p_g(X) = 0$ by Lemma 5.2.

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2. Consider the exact sequence

$$H^1(X,-H) \to H^1(X,\mathcal{O}_X) \to H^1(H,\mathcal{O}_H).$$

The second map in the sequence is nonzero, since H is mapped onto A by the Albanese map a_X, so in this case Lemma 2.12 gives $h^0(H,\omega_H) = 2$. The divisor H is the disjoint union of H_1 and H_2, so $2 = h^0(H,\omega_H) = h^0(H_1,\omega_{H_1}) + h^0(H_2,\omega_{H_2})$. For $i = 1,2$, let S_i be a component of H_i such that $g(S_i) = \Delta_i$. Since Δ_i has geometric genus 1, by Lemma 2.11 we have $1 \leq h^0(S_i,\omega_{S_i}) \leq h^0(H_i,\omega_{H_i})$. So we have $h^0(S_i,\omega_{S_i}) = h^0(H_i,\omega_{H_i}) = 1$ for $i = 1,2$ and S_i is the only component of H with $p_a > 0$. Assume now by contradiction that $H_i = B_i + C_i$, where B_i and C_i are disjoint nonzero effective divisors and $S_i \leq B_i$. Then all the components of C_i are rational, and so their images via g are contained in fibers of $Z \to A$. Since $g(C_i) \subset \Delta_i$, it follows that $g(C_i)$ is a finite set. So the intersection form on the set of components of C_i is negative definite, contradicting the fact that $B_i + C_i$ is the support of the nef divisor $g^*\Delta_i$. □

Since $K_Z + \Delta = 0$, the logarithmic ramification formula (2.3) gives $K_X + D \sim \overline{R}_g$. We aim to show that the components of $\overline{R}_g$ not contained in H are contracted to points. We begin with a simple observation:

Lemma 5.4. *If Γ is an irreducible component of $\overline{R}_g$, then $h^0(X, K_X + H + \Gamma) \leq 1$.*

Proof. We have $(K_X + H) + \Gamma \leq (K_X + D) + \overline{R}_g = 2(K_X + D)$. So $h^0(X, K_X + H + \Gamma) \leq \overline{P}_2(V) = 1$. □

Lemma 5.5. *Let C be the union of all the components of $\overline{R}_g$ that are not contained in H and are not contracted by g. Then:*

1. *if $q(V) = 0$, then $C = 0$;*
2. *if $q(V) = 1$ and $C > 0$, then C is irreducible and $g(C)$ is a ruling of Z.*

Proof. 1. Let Γ be an irreducible component of C. Since $g(\Gamma)$ is a curve not contained in Δ and since Δ is the union of two fibers of the first projection $\mathbb{G}_m^2 \to \mathbb{G}_m$ and two fibers of the second projection, $g(\Gamma) \cap \Delta$ contains at least two distinct points. So $\Gamma \cap H$ also contains at least two distinct points, and therefore, $H\Gamma \geq 2$ and $p_a(H + \Gamma) = p_a(H) + p_a(\Gamma) + H\Gamma - 1 \geq p_a(H) + 1$. Since both the reduced divisors H and $H + \Gamma$ are connected by Lemma 5.3, we have $p_a(H) = h^0(H,\omega_H)$ and $h^0(H + \Gamma, \omega_{H+\Gamma}) = p_a(H + \Gamma) \geq h^0(H,\omega_H) + 1 = 2$, where the last equality follows from Lemma 5.3. Now, Lemma 2.12 gives $h^0(X, K_X + H + \Gamma) = h^0(H + \Gamma, \omega_{H+\Gamma}) \geq 2$ (recall that $p_g(X) = 0$ by Lemma 5.2), contradicting Lemma 5.4.

2. Let again Γ be a component of C such that $g(\Gamma)$ is not a ruling of Z. Then Γ dominates A and therefore $p_a(\Gamma) \geq 1$. If Γ is disjoint from H, then $h^0(H + \Gamma, \omega_{H+\Gamma}) = h^0(H,\omega_H) + h^0(\Gamma,\omega_\Gamma) \geq 3$ by Lemma 5.3. If Γ intersects H, then it intersects both H_1 and H_2, since they support the numerically equivalent divisors $g^*\Delta_1$ and $g^*\Delta_2$. So $\Gamma + H$ is connected and $h^0(\Gamma + H, \omega_{\Gamma+H}) = p_a(\Gamma + H) \geq p_a(\Gamma) + p_a(H) + 1 = 3$. In either case, Lemma 2.12 gives $h^0(X, K_X + H + \Gamma) \geq 2$, contradicting Lemma 5.4. So we conclude that $g(\Gamma)$ is a ruling of Z. Assume that C has at least two components Γ_1 and Γ_2: then the same argument as above gives $h^0(X, K_X + H + \Gamma_1 + \Gamma_2) \geq 2$, contradicting Lemma 5.4 again. □

Now, we consider the Stein factorization $X \xrightarrow{\nu} \overline{X} \xrightarrow{\overline{g}} Z$ of g.

Lemma 5.6. *The morphism $\overline{g}$ is étale over $A(V)$.*

Proof. Let $m := \deg g$; if $m = 1$, then $\overline{g}$ is an isomorphism and the claim is of course true. So we may assume $m > 1$.

The map $\overline{g}$ is finite by construction, so by purity of the branch locus it is enough to show that there is no component of the (usual) ramification divisor of g that is not contained in H and is not contracted to a point. By Lemma 2.8, such a curve is a component of the logarithmic ramification divisor $\overline{R}_g$. So

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if $q(V) = 0$, the statement follows directly from Lemma 5.5. Therefore, we assume for the rest of the proof that $q(V) = 1$.

Again, by Lemma 5.5, if $\overline{g}$ is not étale, then there is exactly one irreducible curve Γ in R_g such that Γ is not contained in H and is not contracted by g, and the image of Γ is a ruling Φ of Z. So $\overline{g}$ restricts to a connected étale cover $q\colon \widetilde{W} \to W := A(V) \setminus \Phi$. The preimage in X of a ruling of Z is a fiber of the Albanese map $a_X\colon X \to A(V)$, and so it is connected. So q restricts to a connected cover of $\mathbb{G}_m$.

Since algebraically trivial line bundles are topologically trivial, Corollary 2.4 implies that Z is homeomorphic to $A \times \mathbb{P}^1$, $A(V)$ is homeomorphic to $A \times \mathbb{G}_m$ and W is homeomorphic to $(A \setminus \{a\}) \times \mathbb{P}^1$, where $a \in A$ is a point. Fix base points in $\widetilde{W}$ and W, and denote by N the subgroup of index m of $\pi_1(W) \simeq \pi_1(A \setminus \{a\}) \times \pi_1(\mathbb{G}_m)$ corresponding to the cover q. If γ is a generator of $\pi_1(\mathbb{G}_m)$, we have $N \cap \pi_1(\mathbb{G}_m) =< \gamma^m >$. So the elements $1, \gamma, \dots \gamma^{m-1}$ represent distinct left cosets of $\pi_1(W)$ modulo N. Since $\pi_1(\mathbb{G}_m)$ is a central subgroup of $\pi_1(W)$, it follows that left and right cosets modulo N coincide; namely, N is a normal subgroup of $\pi_1(W)$ and the quotient $\pi_1(W)/N$ is cyclic of order m. Since the map $\pi_1(W) \to \pi_1(A(V))$ is just abelianization, N is the preimage of an index m subgroup $\overline{N} < \pi_1(A(V))$. This shows that q extends to an étale cover $q'\colon W' \to A(V)$. Since by [5, Thm. 3.4] both q and q' extend uniquely to an analytically branched covering of Z, it follows that $\overline{g}\colon \overline{X} \to Z$ extends q', that is, $\overline{g}$ is étale over $A(V)$ as claimed. □

5.2. *Conclusion*

We are finally ready to complete the proof of Theorem A.

Proof of Theorem A. Consider the case $q(V) = 2$ first. By Corollary 3.2, we only need to show that all components of D are contracted by a_X. So assume for contradiction that there is an irreducible component Γ of D that is not contracted by a_X, and denote by $\overline{\Gamma}$ its image in $A(X)$; note that the map $\Gamma \to \overline{\Gamma}$ is birational. The geometric genus of $\overline{\Gamma}$ is positive, and if it is equal to 1, then $\overline{\Gamma}$ is a translate of an abelian subvariety of $A(X)$, so in particular, it is smooth.

Assume first that $p_a(\Gamma) = 1$: then Γ is smooth of genus 1 and $\Gamma \to \overline{\Gamma}$ is an isomorphism. It follows that $a_X^*\overline{\Gamma} \preceq K_X + \Gamma \preceq K_X + D$. Since $h^0(A(X), 2\overline{\Gamma}) = 2$, we have a contradiction to $\overline{P}_2(V) = 2$. If $p_a(\Gamma) \geq 2$, then by Serre duality and Riemann–Roch for all $\alpha \in \mathrm{Pic}^0(X)$ we have $h^0(X, K_X + \Gamma + \alpha) \geq \chi(K_X + \Gamma) = p_a(\Gamma) - 1 \geq 1$. Lemma 2.9 gives $h^0(X, 2(K_X + \Gamma)) \geq 2$, contradicting again the assumption $\overline{P}_2(V) = 1$. So all the components of D are a_X-exceptional.

From now on, we assume $q(V) \leq 1$. By Lemma 5.6, the quasi-Albanese map $a_V : V \to A(V)$ factors through an étale cover of degree $m := \deg g$. Since a finite étale cover of a quasi-abelian variety is also a quasi-abelian variety (Proposition 2.5) by the universal property of a_V (cf. Theorem 2.6), we have $m = 1$; namely, g is birational. To complete the proof, we need to show that all the components of $D - H$ are contracted by g. Since $D - H \preceq \overline{R}_g$ by Lemma 2.8, in case $q(V) = 0$ the claim follows by Lemma 5.5.

Assume $q(V) = 1$. By Lemma 5.5, there is at most an irreducible curve $\Gamma \preceq D - H$ such that Γ is not contracted by g, and $g(\Gamma)$ is a ruling Φ of Z. We are going to show that $g^*\Phi \preceq \overline{R}_g$, hence $g^*(2\Phi) \preceq 2\overline{R}_g \sim 2(K_X + D)$. Since $h^0(Z, 2\Phi) = 2$, this contradicts the assumption $\overline{P}_2(V) = 1$.

Write $g^*\Phi = \Gamma + \sum_i \alpha_i C_i$, where the C_i are distinct irreducible curves contracted by g and $\alpha_i \in \mathbb{N}_{>0}$. Let u, v be local coordinates on Z centered at the point $P_i := g(C_i)$ such that $u = 0$ is the ruling Φ of Z; if, in addition $P_i \in \Delta$, we assume that v is a local equation of Δ. At a general point of C_i, we have $u = x^{\alpha_i} a$, $v = x^{\beta_i} b$, where x is a local equation for C_i, a, b are nonzero regular functions and $\beta_i > 0$ is an integer. A simple computation gives

$$g^*\frac{dv}{v} = \beta_i \frac{dx}{x} + \frac{db}{b}, \qquad g^*\left(\frac{du}{u} \wedge \frac{dv}{v}\right) = \frac{dx}{x} \wedge \left(\alpha_i \frac{db}{b} - \beta_i \frac{da}{a}\right). \tag{5.1}$$

Note that $\alpha_i \frac{db}{b} - \beta_i \frac{da}{a}$ is a regular 1-form. So if $P_i \in \Delta$, then $C_i \preceq D$, $K_Z + \Delta$ is locally generated by $u \wedge \frac{dv}{v}$, $K_X + D$ is locally generated by $\frac{dx}{x} \wedge dy$ and C_i appears in $\overline{R}_g$ with multiplicity $\geq \alpha_i$.

If $P_i \notin \Delta$, then $K_Z + \Delta$ is locally generated by $du \wedge dv$, $K_X + D$ is locally generated by $\frac{dx}{x} \wedge dy$ or by $dx \wedge dy$ according to whether $C_i \leq D$ or not; in either case, C_i appears in $\overline{R}_g$ with multiplicity $\geq \alpha_i + \beta_i - 1 \geq \alpha_i$. Summing up, we have shown $g^*\Phi \leq \overline{R}_g$, as promised. □

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