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UNIFORM APPROXIMATION TO MAHLER'S MEASURE IN SEVERAL VARIABLES

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ABSTRACT. If $f(x_1,\ldots,x_k)$ is a polynomial with complex coefficients, the Mahler measure of f, $M(f)$ is defined to be the geometric mean of $|f|$ over the k-torus $\mathbb{T}^k$. We construct a sequence of approximations $M_n(f)$ which satisfy $-d2^{-n}\log 2+\log M_n(f)\le \log M(f)\le \log M_n(f)$. We use these to prove that $M(f)$ is a continuous function of the coefficients of f for polynomials of fixed total degree d. Since $M_n(f)$ can be computed in a finite number of arithmetic operations from the coefficients of f this also demonstrates an effective (but impractical) method for computing $M(f)$ to arbitrary accuracy.

Introduction. Let $f(x_1,\ldots,x_k)$ be a polynomial with complex coefficients. If f is not identically zero, the Mahler measure of f is defined by

$$M(f)=\exp\Big(\int_0^1\cdots\int_0^1 \log\Big|f\big(\exp(2\pi i t_1),\ldots,\exp(2\pi i t_k)\big)\Big|\,dt_1\cdots dt_k\Big). \tag{1}$$

So $M(f)$ is the geometric mean of $|f|$ over the k-torus, $\mathbb{T}^k$. We define $M(0)=0$. An obvious but important property of M is that

$$M(fg)=M(f)M(g). \tag{2}$$

Mahler [4] used $M(f)$ as a tool in a simple proof of the "Gelfond–Mahler inequality", a result of importance in transcendence theory. In [2], we indicate a more intrinsic reason for an interest in $M(f)$ for polynomials with integer coefficients.

In this paper, we give a rather simple answer to two basic questions concerning $M(f)$. The first of these was posed by Andrzej Schinzel who recently asked the author if $\lim_{a\to 0} M(f+a)=M(f)$, where a is a complex variable. This is equivalent to asking whether $M(f)$ is a continuous function of the coefficients of f. The second is the question of the effective computation of $M(f)$ to arbitrary accuracy, a question recently considered by Everest [3] for polynomials with integer coefficients.

Both of these questions are easily answered if the polynomial f does not vanish on the torus $\mathbb{T}^k$. For then $F=\log|f|$ is a smooth function of $(x_1,\ldots,x_k)$ on the torus. In particular, F is bounded on $\mathbb{T}^k$ and a continuous function of the coefficients of f so the continuity of $M(f)$ follows by Lebesgue's bounded convergence theorem. Similarly, the effective computation of $M(f)$ from the definition (1) can, in this case, be accomplished by any of the standard methods of numerical quadrature.

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However, if f does vanish on the torus, both questions become more delicate since $F = \log|f|$ has logarithmic discontinuities on the zero set of f, a set which can be quite complicated. Our approach, which relies on a basic inequality of Mahler [4], entirely avoids the consideration of the zero set of f.

To be more precise about what we will prove, fix k, fix a vector of positive integers $(d_1, \ldots, d_k)$ and consider f of the form

$$f(x_1, \ldots, x_k) = \sum_{i_1=0}^{d_1} \cdots \sum_{i_k=0}^{d_k} a(i_1, \ldots, i_k) x_1^{i_1} \cdots x_k^{i_k}. \tag{3}$$

Thus the degree of f as a polynomial in x_j satisfies $\deg_j(f) \le d_j$. Then $M(f)$ becomes a function of the $D = (d_1 + 1) \cdots (d_k + 1)$ variables $\mathbf{a} = \{a(i_1, \ldots, i_k)\}$. We make no restrictions on the vanishing of any of these coefficients so possibly $\deg_j(f) < d_j$. We will show that in this situation $M(f)$ is a continuous function of $\mathbf{a}$ on $\mathbb{C}^D$. In this case we will say that *$M(f)$ is a continuous function of f*.

Some approximations to Mahler's measures. In this section, we will construct a sequence of approximations to $M(f)$ using an obvious generalization of Graeffe's root squaring method. A crucial ingredient of the construction is the following inequality of Mahler [4]. Write $\deg(f) = \sum_{j=1}^{k} \deg_j(f)$ for the *total degree* of f and $d = \sum_{j=1}^{k} d_j$. Let $L(f)$ denote the sum of the absolute values of the coefficients of f (the *length* of f). If $\deg(f) \le d$, then

$$2^{-d} L(f) \le M(f) \le L(f). \tag{4}$$

We begin by defining an operation G on polynomials of k variables by the following formula:

$$(Gf)(x_1, \ldots, x_k) := \prod f(\pm x_1^{1/2}, \ldots, \pm x_k^{1/2}), \tag{5}$$

where the product is over all 2^k choices of the signs. To see that Gf is a polynomial, note that the expression $g(y_1, \ldots, y_k) = \prod f(\pm y_1, \ldots, \pm y_k)$ is a polynomial in $y_1, \ldots, y_k$ which is invariant under any of the changes of variable $y_j \longrightarrow -y_j$. Hence $g(y_1, \ldots, y_k)$ is an even function of each y_j, and thus is a polynomial, say Gf, in $y_1^2, \ldots, y_k^2$.

It is clear that $\deg_j(Gf) = 2^{k-1} \deg_j(f)$ so $\deg(Gf) = 2^{k-1} \deg(f)$. Also, $M(Gf) = M(g) = M(f)^{2^k}$, from (2), since each of the polynomials $f(\pm y_1, \ldots, \pm y_k)$ has measure $M(f)$.

For a given f, define $f_n = G^n f$ to be the result of n applications of the operation G to f. So $M(f_n) = M(f)^{2^{kn}}$ and $\deg(f_n) = 2^{(k-1)n} \deg(f)$. Now define

$$M_n(f) := L(f_n)^{2^{-kn}}. \tag{6}$$

From Mahler's inequality (4),

$$2^{-2^{(k-1)n} d} L(f_n) \le M(f_n) \le L(f_n). \tag{7}$$

Raising (7) to the power 2^{-kn} and using the definition (6) produces the inequality

$$2^{-d2^{-n}} M_n(f) \le M(f) \le M_n(f). \tag{8}$$

THEOREM. *For polynomials of bounded degree $M(f)$ is a continuous function of f.*

PROOF. Clearly each of the coefficients of f_n is a polynomial in the coefficients of f so $L(f_n)$ and hence $M_n(f)$ are continuous functions of f. Define $\epsilon_n = 2^{d2^{-n}} - 1$ so $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then (8) and (4) imply that

$$0 \le M(f) - M_n(f) \le \epsilon_n M(f) \le \epsilon_n L(f).$$

Thus $M_n(f)$ converges uniformly to $M(f)$ on any set of the form $L(f) \le B$ and hence $M(f)$ is a continuous function of f. ■

REMARK. In the case of polynomials in one variable, one can base a proof of a weaker result on an alternative expression for the measure. If $f(x) = \sum_{j=0}^{d} a_j x^j = a_d \prod_{j=1}^{d}(x - \alpha_j)$, Jensen's formula gives

$$M(f) = |a_d| \prod_{j=1}^{d} \max(|\alpha_j|, 1). \tag{9}$$

Since the zeros of a polynomial are continuous functions of the coefficients, provided the leading coefficient does not vanish, it follows from (9) that, for $k = 1$, $M(f)$ is a continuous function of $(a_0, a_1, \ldots, a_d)$ on the subset of $\mathbb{C}^{d+1}$ where $a_d \ne 0$. Our theorem does not require the latter restriction.

Effective computation of Mahler's measure. The natural approach to the computation of $M(f)$ is to use numerical integration to evaluate the integral in (1). While this approach seems to work well in practice for small values of k, it is difficult to give a general estimate of the error in such an approximation if f vanishes on $\mathbb{T}^k$ since then the integral is improper. For an example of the difficulties in such an approach, see the paper of Everest [3] where an estimate is given for the error in approximating (1) by certain Riemann sums for polynomials with integer coefficients. His proof uses some deep results from the theory of linear forms in logarithms and discrepancy theory.

In contrast, the computation of $M_n(f)$ is clearly effective since it involves only the computation of products and sums of the coefficients of f followed by a final extraction of a 2^{kn}-th root. The estimate (8) is uniform for f of a fixed total degree, being independent even of the number of variables. However, it must be admitted that the method is totally impractical even for $k = 2$ because the total number of terms and the size of the coefficients in the polynomial f_n grow exponentially. For $k = 1$ the method reduces to one used in [1]. In this case, the number of terms in f_n is fixed so the method is less impractical and indeed is very useful for obtaining rough estimates of $M(f)$. But there are other reasons discussed in [1] for preferring other methods for highly accurate estimates of $M(f)$.

To illustrate the practical difficulties with the effective compuation of $M(f)$ using $M_n(f)$, consider the polynomial $f(x,y) = 1+x+y$. This f vanishes on $\mathbb{T}^k$ at the two points (ω,ω^2) and (ω^2,ω), with $\omega = \exp(2\pi i/3)$. It is known to have measure

$$M(1+x+y) = 1.3813564445184977933\cdots.$$

This is easily computed by first reducing to a single integral by Jensen's formula and then numerically integrating $\log\bigl(\max(|1+x|,1)\bigr)$ over the circle.

We can obtain two-sided estimates of $M(f)$ from (8), but we obtain a better upper bound by using the fact that $M(f) \le \|f\| \le L(f)$, where $\|f\|$ is the ℓ^2 norm of f. If $N_n(f) = \|f_n\|^{2^{-kn}}$ then $M(f) \le N_n(f) \le M_n(f)$. Using the Maple computer algebra system, the results of the Table are computed fairly quickly, but at high memory cost. In the table, $M_n^-(f) = 2^{-d2^{-n}} M_n(f)$ and "terms" denotes the number of terms in the polynomial f_n. For example, the most accurate estimate, $N_5(f)$ is $K^{1/2048}$, where K is an integer with 288 decimal digits.

One could obtain better lower bounds by using Mahler's estimates for the individual coefficients of f_n in terms of $M(f_n)$, but it should be clear that the method should not be considered as a practical method for computing estimates of $M(f)$ of high accuracy. Rather, its interest lies in its generality.

n	$M_n^-(f)$	$N_n(f)$	$M_n(f)$	terms
0	.7500000000	1.732050808	3	3
1	.8660254040	1.402850552	1.732050808	6
2	1.030964012	1.398974059	1.458003288	15
3	1.177248070	1.383252123	1.399991780	45
4	1.271001650	1.382177945	1.386037128	153
5	1.323901724	1.381514320	1.382515861	561

TABLE: ESTIMATES OF $M(1+x+y)$

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