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A NOTE ON A CLASS OF SUBMULTIPLICATIVE FUNCTIONS

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Abstract. In 1989, Alladi, Erdös and Vaaler confirmed their own conjecture about a class of multiplicative functions by means of a deep result of Baranyai on hypergraphs. In this note we give a simple direct proof of the result which is derived in their proof as a consequence of the above mentioned graph theoretic result.

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1. Introduction. In 1984, Alladi, Erdös and Vaaler considered the following conjecture.

CONJECTURE. *Let n be a square-free integer and h a multiplicative function satisfying $0 \leq h(p) \leq 1/(k-1)$ on primes p, where k is a natural number. Then*

$$\sum_{d|n} h(d) \leq c_k \sum_{d|n, d \leq n^{1/k}} h(d),$$

where c_k denotes a constant depending only on k.

Later in [**1**], they proved the above conjecture, using a result (namely, the following Proposition) which is a special case of a theorem of Baranyai on hypergraphs [**2**]. Thus, in view of [**3**], the above statement *automatically* holds even when h is a submultiplicative function. In the sequel, we use p (with or without suffixes) to denote primes.

PROPOSITION. *Let $k(\geq 1)$ and $\ell(\geq 0)$ be given integers. Suppose $N = p_1p_2p_3 \ldots p_{k\ell}$, with $p_1 < p_2 < p_3 < \ldots < p_{k\ell}$. Then the number of d, such that $d \mid N$, $d \leq N^{\frac{1}{k}}$ and having exactly ℓ prime divisors, is at least*

$$\frac{1}{k}\binom{k\ell}{\ell}.$$

2. Proof of the Proposition. For $\ell = 0$, the Proposition holds trivially. Let $S_{k\ell} = \{1, 2, 3, \ldots, k\ell\}$. For any permutation $\pi = \{\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_{k\ell}\}$ of $S_{k\ell}$, set $\xi_\pi = \{A_1, A_2, A_3, \ldots A_k\}$ where $A_j = \{\sigma_{(j-1)\ell+1}, \ldots \sigma_{j\ell}\}$. For every B with $|B| = \ell$, let $\delta_\pi(B)$ denote 1 if $B \in \xi_\pi$, or 0 otherwise. For each subset A of $S_{k\ell}$, there is an associated divisor d_A of N given as the product of all primes p_i, with $i \in A$, and this association is a bijection. Let ζ_1 be the collection of all subsets A of $S_{k\ell}$, such that $|A| = \ell$ and $d_A \leq N^{\frac{1}{k}}$. Similarly let ζ_2 be the collection of all subsets B of $S_{k\ell}$, such that $|B| = \ell$ and $d_B > N^{\frac{1}{k}}$. Since any $C \subseteq S_{k\ell}$ having ℓ elements belongs to exactly one of ζ_1 or ζ_2, we have

$$|\zeta_1| + |\zeta_2| = \binom{k\ell}{\ell}. \tag{1}$$

Since $\prod_j d_{A_j} = N$, $\exists A_j$ such that $d_{A_j} \le N^{\frac{1}{k}}$, and so we have for every permutation π,

$$|\zeta_2 \cap \xi_\pi| \le (k-1)|\zeta_1 \cap \xi_\pi|,$$

which can be written as

$$\sum_{B\in\zeta_2} \delta_\pi(B) \le (k-1)\sum_{A\in\zeta_1} \delta_\pi(A).$$

Now summing over all π, we get

$$\sum_{B\in\zeta_2}\sum_{\pi} \delta_\pi(B) \le (k-1)\sum_{A\in\zeta_1}\sum_{\pi} \delta_\pi(A).$$

Since for any C with exactly ℓ elements, $\sum_\pi \delta_\pi(C)$ being k times $\ell!(k\ell-\ell)!$ is independent of C, the above inequality leads to

$$|\zeta_2| \le (k-1)|\zeta_1|. \tag{2}$$

From (1) and (2), we obtain

$$k|\zeta_1| \ge \binom{k\ell}{\ell},$$

which completes the proof of the proposition.

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REFERENCES

1. K. Alladi, P. Erdös and J. D. Vaaler, Multiplicative functions and small divisors, II, *J. Number Theory* **31** (1989), 183–190.

2. Z. S. Baranyai, On the factorization of the complete uniform hyper graph, in *Colloquia Mathematica Societatis Janos Bolyai*, **10** (Keszthely, Hungary, 1973), 91–102.

3. S. Srinivasan, On an arithmetical inequality, *Glasgow Math. J.* **36** (1994), 81–86.