Can. J. Math., Vol. XXXII, No. 3, 1980, pp. 695–702

AN INCLUSION RELATION FOR ABEL, BOREL, AND LAMBERT SUMMABILITY

W. GAWRONSKI, H. SIEBERT AND R. TRAUTNER

1. In this paper a new type of inclusion theorem concerning Abel, Borel and Lambert summability is established. To state our results we need some definitions and notations. With a formal series $\sum_{k=0}^{\infty} a_k$, $a_k \in \mathbf{C}$, and its partial sums s_n we associate the series

$$(1.1)\quad A(v) := \sum_{k=0}^{\infty} a_k v^{-k},$$

$$(1.2)\quad L(v) := a_0 - (v-1)\sum_{m=1}^{\infty} A'(v^m)v^m,$$

$$(1.3)\quad B(x) := e^{-x}\sum_{n=0}^{\infty} \frac{s_n}{n!}x^n.$$

Then $\sum_{k=0}^{\infty} a_k$ is said to be summable to the value s

(a) by Abel's method, if (1.1) is convergent for $|v| > 1$ and $\lim_{v\to 1+} A(v) = s$,
(b) by Lambert's method, if (1.2) is convergent for $|v| > 1$ and $\lim_{v\to 1+} L(v) = s$,
(c) by Borel's method, if (1.3) is convergent for all $x \in \mathbf{R}$ and $\lim_{x\to+\infty} B(x) = s$.

For the domains of summability we write (A), (L) and (B) respectively.

Hardy and Littlewood [**2**] showed the inclusion $(L) \subset (A)$. Actually they inferred this relation from the convergence of $\sum_1^\infty |N(n)|n^{-1}$, where

$$N(x) := \sum_{n\leqq x} \mu(n)n^{-1},$$

$\mu(n)$ being Möbius' function. Hoischen [**3**] proved that $(L) \subset (A)$ in fact is equivalent to the convergence of $\sum_1^\infty |N(n)|n^{-1}$, and this is somewhat deeper than the prime number theorem. (In number theory two statements are called "equivalent" when either of them can be deduced from the other without using deeper results from the analytic theory of numbers.) Furthermore, Hoischen [**4**] proved the equivalence of the

Received September 1, 1978 and in revised form July 30, 1979. This research was supported in part by the National Research Council of Canada.

Riemann hypothesis with certain saturation properties between Lambert's and Abel's method. On the other hand we have $(B) \not\subseteq (A)$ and $(B) \not\subseteq (L)$, since (B) contains series with $\overline{\lim}_{k\to\infty} |a_k|^{1/k} > 1$, which certainly cannot be in (A) or (L). However, if $\sum_0^\infty a_k$ is summable to s by Borel's method and $\overline{\lim}_{k\to\infty} |a_k|^{1/k} < \infty$, then $A(v)$ has an analytic continuation into $\operatorname{Re} v > 1$ and $\lim_{v\to 1+} A(v) = s$. Therefore it makes sense to ask whether $(A) \cap (B) \subseteq (L)$ or not. This question is due to B. L. R. Shawyer (London, Ontario) and will be answered by the following

THEOREM. $(A) \cap (B) \subseteq (L)$.

The detailed proof is established in the following three sections. First we give an outline of the proof. Suppose throughout that

$$(1.4) \qquad \sum_0^\infty a_k \in (A) \cap (B)$$

$$(1.5) \qquad a_0 = a_1 = 0$$

$$(1.6) \qquad \lim_{x\to\infty} B(x) = 0.$$

Assumptions (1.5) and (1.6) do not mean a loss of generality, since A, B, and L are regular methods. Furthermore, we consider the weight function

$$w(x) := \min(x^2, 1), \quad x > 0,$$

and the Banach spaces

$$U_1 := \{f(x) \mid f(x) \text{ continuous for } x > 0,$$
$$\|f\| = \sup_{x>0} |f(x)/w(x)| < \infty\},$$
$$U_2 := \{g(v) \mid g(v) \text{ continuous for } v \in (1, 2],$$
$$\|g\| = \sup_{1<v\leqq 2} |g(v)| < \infty\}.$$

If (1.4) and (1.5) hold, then $B(x) \in U_1$ and if $\sum_0^\infty a_k \in (L)$, then $L(v) \in U_2$.

We shall show the existence of an integral kernel $K_{L,B}(v, x)$ such that

$$(1.7) \qquad g(v) = \int_0^\infty K_{L,B}(v, x) f(x) dx, \quad 1 < v \leqq 2,$$

defines a linear map from U_1 into the space of continuous functions on $(1, 2]$ and satisfying in addition

$$(1.8) \qquad L(v) = \int_0^\infty K_{L,B}(v, x) B(x) dx.$$

The critical part of the proof consists in showing that (1.7) is even a bounded map from U_1 into U_2, that is (see Lemma 4)

$$(1.9) \qquad \|K_{L,B}\| = \sup_{1<v\leqq 2} \int_0^\infty |K_{L,B}(v, x)| w(x) dx < \infty.$$

Finally we show that

(1.10) $\lim_{v\to 1+} K_{L,B}(v, x) = 0$

holds uniformly on any interval $[x_1, x_2]$, $0 < x_1 < x_2 < \infty$, and infer from (1.8), (1.9) and (1.10) by standard techniques (following the proof of the well-known Silverman-Toeplitz theorem) that (1.4), (1.5) and (1.6) imply

(1.11) $\lim_{v\to 1+} L(v) = 0$.

2. In this section we establish transformation formulae between $A(v)$, $L(v)$ and $B(x)$. Throughout this paper we assume that $|B(x)| \leqq M$ (M constant) and $\lim_{k\to\infty} |a_k|^{1/k} \leqq 1$ which are necessary conditions for $\sum_0^\infty a_k$ to be in (B) and (A) respectively.

LEMMA 1. *If* $x \in \mathbf{R}$ *and* $R > 1$, *then*

(2.1) $$B(x) = \oint_{|v|=R} K_{B,A}(x, v)A(v)dv$$

with

(2.2) $$K_{B,A}(x, v) = e^{-x(1-v)}/2\pi i(v - 1).$$

Proof. From (1.1) we get

$$\frac{A(v)}{v-1} = \sum_0^\infty \frac{s_n}{v^{n+1}}$$

and therefore

$$s_n = \frac{1}{2\pi i}\oint_{|v|=R} \frac{A(v)}{v-1} v^n dv.$$

This implies

$$\begin{aligned} B(x) &= e^{-x}\sum_{n=0}^\infty \frac{s_n x^n}{n!} = e^{-x}\sum_{n=0}^\infty \frac{x^n}{n!}\frac{1}{2\pi i}\oint_{|v|=R}\frac{A(v)}{v-1}v^n dv \\ &= \oint_{|v|=R}\frac{A(v)}{2\pi i(v-1)} e^{-x}\sum_{n=0}^\infty \frac{(xv)^n}{n!} dv, \end{aligned}$$

which completes the proof.

LEMMA 2. *If* $a_0 = 0$, *then we have for* $\operatorname{Re} v > 1$

(2.3) $$A(v) = \int_0^\infty K_{A,B}(v, x)B(x)dx$$

with

(2.4) $$K_{A,B}(v, x) = (v - 1)e^{-x(v-1)}.$$

Proof. We first compute

$$\int_0^\infty K_{A,B}(v, x)K_{B,A}(x, t)dx = (v-1)\int_0^\infty e^{-x(v-1)} \frac{e^{-x(1-t)}}{2\pi i(t-1)}dx$$
$$= \frac{1}{2\pi i}\frac{(v-1)}{(t-1)(v-t)}$$

valid for $\operatorname{Re}(v-t) > 0$. Hence we get for $\operatorname{Re} v > R > 1$ from Lemma 1

$$\int_0^\infty K_{A,B}(v, x)B(x)dx = \int_0^\infty K_{A,B}(v, x)\oint_{|t|=R} K_{B,A}(x, t)A(t)dt\,dx$$
$$= \frac{1}{2\pi i}\oint_{|t|=R} A(t)\frac{(v-1)}{(t-1)(v-t)}dt = A(v).$$

In the last step we used $a_0 = 0$. Since $R > 1$ was arbitrary, the proof is complete.

LEMMA 3. *Under* (1.5) *we have for* $1 < v \leqq 2$

$$L(v) = \int_0^\infty K_{L,B}(v, x)B(x)dx \tag{2.5}$$

with

$$K_{L,B}(v, x) = -(v-1)\sum_{m=1}^\infty e^{-x(v^m-1)}(1 - x(v^m - 1))v^m. \tag{2.6}$$

Proof. Differentiating (2.3) we obtain

$$A'(v) = \int_0^\infty \frac{\partial}{\partial v}K_{A,B}(v, x)B(x)dx$$
$$= \int_0^\infty e^{-x(v-1)}(1 - x(v-1))B(x)dx$$

and further from (1.2) we get

$$L(v) = -(v-1)\sum_{m=1}^\infty A'(v^m)v^m$$
$$= -(v-1)\sum_{m=1}^\infty \int_0^\infty e^{-x(v^m-1)}(1 - x(v^m-1))v^m B(x)dx$$

leading to Lemma 3 by interchanging sum and integral, which can be justified as follows. We have for $x > 0$, $1 < v \leqq 2$,

$$\frac{1}{v-1}|K_{L\,B}(v, x)| \leqq \sum_{m=1}^\infty e^{-x(v^m-1)}(1 + x(v^m-1))v^m$$
$$= \sum_{m=1}^\infty e^{-x(v^m-1)}(1 + (v^m-1)(x+1)$$
$$+ (v^m-1)^2x).$$

https://doi.org/10.4153/CJM-1980-054-5 Published online by Cambridge University Press

Using $y^\nu e^{-y} \leqq Ke^{-y/2}$, $y > 0$, $(\nu = 0, 1, 2)$ and $(v^m - 1) \geqq (v - 1)$ we obtain for $x > 0$, $1 < v \leqq 2$

$$\frac{1}{(v-1)} |K_{L,B}(v, x)| \leqq K\left(1 + \frac{1+x}{x} + \frac{1}{x}\right) \sum_{m=1}^{\infty} e^{-x(v^m-1)/2}$$
$$\leqq 2K \frac{1+x}{x} \frac{e^{-x(v-1)/2}}{1 - e^{-x(v-1)/2}}$$

and hence

$$\text{(2.7)} \qquad |K_{L,B}(v, x)| \leqq 4K\left(\frac{1+x}{x}\right)^2 e^{-x(v-1)/2} \qquad 1 < v \leqq 2, x > 0,$$

since $e^{-t}(1 - e^{-t})^{-1} \leqq e^{-t}(1 + t)t^{-1}$ for $t > 0$.

In view of (1.5) there exists a constant M' such that $|B(x)| \leqq M'(x/(1 + x))^2$, $x > 0$, which implies that

$$\int_0^\infty |K_{L,B}(v, x)B(x)|dx \leqq 4KM' \int_0^\infty \left(\frac{1+x}{x}\right)^2 \left(\frac{x}{1+x}\right)^2 e^{-x(v-1)/2} dx$$

is finite. Now Fubini's theorem justifies the interchange in question.

3. This section is devoted to the essential part of the proof of our theorem. We show that

$$\text{(1.7)} \qquad g(v) = \int_0^\infty K_{L,B}(v, x)f(x)dx$$

defines a bounded map from the Banach space U_1 into U_2. Therefore we have to prove

LEMMA 4. *The norm of $K_{L,B}$ is finite, that is*

$$\text{(1.9)} \qquad ||K_{L,B}|| = \sup_{1<v\leqq 2} \int_0^\infty |K_{L,B}(v, x)|w(x)dx < \infty.$$

Proof. From (2.7) we get

$$\text{(3.1)} \qquad \int_0^1 |K_{L,B}(v, x)|w(x)dx \leqq 4K \int_0^1 \left(\frac{1+x}{x}\right)^2 e^{-x(v-1)/2} x^2 dx$$

which is bounded for $v \in (1, 2]$. To estimate the remaining part of the integral we write a primitive function $I(v, x)$ of $K_{L,B}(v, x)$ as a Laplace–Stieltjes integral. We have

$$\text{(3.2)} \qquad I(v, x) = -x(v-1) \sum_{m=1}^{\infty} e^{-x(v^m-1)} v^m = -x \int_0^\infty e^{-xt} dF(v, t)$$
$$= -x^2 \int_0^\infty e^{-xt} F(v, t)dt,$$

where

$$\text{(3.3)} \qquad F(v, t) := (v-1) \sum_{1 \leqq k \leqq \log(t+1)/\log v} v^k$$

https://doi.org/10.4153/CJM-1980-054-5 Published online by Cambridge University Press

is a step function having jumps of height $(v-1)v^k$ at $t = v^k - 1$, $k \in \mathbf{N}$. Obviously we have

$$(3.4) \qquad F(v,t) = 0 \quad \text{for } 0 \leqq t < v-1,$$

while

$$(3.5) \qquad |F(v,t) - t| \leqq (v-1)(1+t) \quad \text{for } t \geqq v-1.$$

To prove (3.5) suppose that $v^m - 1 \leqq t < v^{m+1} - 1$ for some $m \in \mathbf{N}$. In this interval $F(v,t) = F(v, v^m - 1) = v(v^m - 1)$ holds and hence

$$F(v,t) - t \leqq v(v^m - 1) - (v^m - 1) = (v-1)(v^m - 1) \leqq (v-1)t$$

and

$$F(v,t) - t \geqq v(v^m - 1) - (v^{m+1} - 1) = 1 - v.$$

Thus we get

$$|F(v,t) - t| \leqq \max((v-1),(v-1)t) \leqq (v-1)(1+t)$$

which proves (3.5). From (3.2) and (3.4) we obtain

$$(3.6) \qquad K_{L,B}(v,x) = -2x\int_0^\infty e^{-xt}F(v,t)dt + x^2\int_0^\infty e^{-xt}tF(v,t)dt = -x^2\int_{v-1}^\infty e^{-xt}(2G(v,t) - tF(v,t))dt$$

where $G(v,t)$ satisfies

$$\frac{\partial}{\partial t}G(v,t) = F(v,t) \quad \text{and} \quad G(v, v-1) = 0.$$

From (3.5) we infer $(t \geqq v-1)$

$$|tF(v,t) - t^2| \leqq (v-1)(t^2 + t) \quad \text{and}$$
$$|2G(v,t) - t^2| \leqq (v-1)(t^2 + 2t)$$

giving

$$(3.7) \qquad |2\,G(v,t) - tF(v,t)| \leqq 2(v-1)(t^2 + 2t),\ t \geqq v-1.$$

Thus, by (3.6), we have

$$\int_1^\infty |K_{L,B}(v,x)|w(x)dx \leqq \int_1^\infty x^2\int_{v-1}^\infty e^{-xt}2(v-1)(t^2+2t)dt\,dx$$
$$= 2(v-1)\int_{v-1}^\infty t^2\int_1^\infty e^{-xt}x^2dxdt + 4(v-1)\int_{v-1}^\infty t\int_1^\infty e^{-xt}x^2\,dx\,dt$$
$$< 2(v-1)\int_{v-1}^\infty t^2\int_1^\infty e^{-xt}x^3\,dx\,dt + 8(v-1)\int_{v-1}^\infty \frac{dt}{t^2}$$
$$< (12(v-1) + 8(v-1))\int_{v-1}^\infty \frac{dt}{t^2} = 20$$

which gives (1.9).

4. In this section we finish the proof of our theorem by showing (1.11). This will be done by proving (1.10) which may be regarded as an analogue of Toeplitz's column limit condition. But having (3.6) and (3.7) the following lemma is established immediately. (We are indebted to David Borwein, who made this observation for shortening its proof.)

LEMMA 5. *If* $0 < x_1 < x_2 < \infty$, *then*

(1.10) $\lim_{v\to 1+} K_{L,B}(v, x) = 0$

holds uniformly on $[x_1, x_2]$.

Suppose that $\sum_0^\infty a_k$ is in $(A) \cap (B)$. Further we assume without loss of generality that

(1.5) $a_0 = a_1 = 0$

and

(1.6) $\lim_{x\to\infty} B(x) = 0$

hold. Then we have to show that $\lim_{v\to 1+} L(v) = 0$.

Suppose that $\epsilon > 0$ and that $x_1 < x_2$ are positive numbers which will be chosen suitably depending on ϵ. We write

$$L(v) = \int_0^{x_1} K_{L,B}(v, x)B(x)dx + \int_{x_1}^{x_2} K_{L,B}(v, x)B(x)dx + \int_{x_2}^{\infty} K_{L,B}(v, x)B(x)dx$$
$$= I + II + III, \text{ say.}$$

By (1.5), we have $|B(x)| \leqq x^2M'$ (M' fixed, $x > 0$) which implies that (use (2.7))

$$|I| \leqq M'\int_0^{x_1} x^2|K_{L,B}(v, x)|dx < \frac{\epsilon}{3},$$

if $x_1 = x_1(\epsilon)$ is sufficiently small. Further we conclude from Lemma 4 and (1.6) that

$$|III| \leqq \sup_{x\geqq x_2} |B(x)|\int_1^{\infty} |K_{L,B}(v, x)|dx < \frac{\epsilon}{3},$$

if $x_2 = x_2(\epsilon)$ is sufficiently large. Finally, by Lemma 5, we get

$$|II| \leqq \sup_{x_1\leqq x\leqq x_2} |B(x)|\int_{x_1}^{x_2} |K_{L,B}(v, x)|dx < \frac{\epsilon}{3},$$

if $|v - 1|$ is sufficiently small.

Now the proof of the theorem is complete.

https://doi.org/10.4153/CJM-1980-054-5 Published online by Cambridge University Press

5. As we have seen the basic step in the preceding proof consists in the norm estimate for the integral kernel $K_{L,B}$. An alternative proof of the finiteness of $\|K_{L,B}\|$ can be given by considering the shifted sequence $\{\hat{a}_k\}_0^\infty$, where $\hat{a}_k: = a_{k-1}$, $k \in \mathbf{N}$, $\hat{a}_0: = 0$, instead of $\{a_k\}_0^\infty$ and by using an index shifting property of Borel's method [**1**].

Finally we should remark that our methods are sufficient to show the inclusion relation

$$(A^*) \cap (B) \subseteq (L^*)$$

where A^* and L^* are generalized Abel's and Lambert's methods respectively which are defined as follows:

A series $\sum_0^\infty a_k$ is said to be summable to the value s

(a) by the method A^*, if (1.1) defines a holomorphic function on $\{v \in \mathbf{C} |\ |v| > \rho\} \cup (1, \infty)$, $\rho > 0$, and $\lim_{v \to 1+} A(v) = s$,

(b) by the method L^*, if (1.2) defines a holomorphic function on $\{v \in \mathbf{C} |\ |v| > \rho\} \cup (1, \infty)$, $\rho > 0$, and $\lim_{v \to 1+} L(v) = s$.

References

1. D. Gaier, *Zur Frage der Indexverschiebung beim Borel-Verfahren*, Math. Zeitschr. Bd. *58* (1953), 453–455.

2. G. H. Hardy and J. E. Littlewood, *On a Tauberian theorem for Lambert's series and some fundamental theorems in the analytic theory of numbers*, Proc. London Math. Soc. *19* (1919), 21–29.

3. L. Hoischen, *An inclusion theorem for Abel and Lambert summability*, Journal London Math. Soc. *42* (1967), 591–594.

4. ——— *Über die Wirkfelder verallgemeinerter Abelscher Limitierungsverfahren*, Acta Math. Acad. Sci. Hung. *20* (1969), 149–157.

Universität Ulm,
Oberer Eselsberg, Germany