An algebraic theory of hypergroups

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The category of groups forms a full subcategory of the category
of hypergroups. This larger category also contains other group
theoretic objects, such as the conjugacy class hypergroup and
character hypergroup of a finite group.

Definitions of hypergroups and hypergroup morphisms are given and
related to double algebras (which are simultaneously algebras and
cogebras, but not Hopf algebras in general). Quotient and orbit
hypergroups are defined. Coefficients are allowed in a more or
less arbitrary field.

These concepts provide a new language in which groups and their
character tables can be fruitfully discussed.

Introduction

It is classical that the characters of a compact abelian group form a
discrete abelian group. In recent years, harmonic analysts have endowed
the dual of even a compact nonabelian group with an algebraic
structure in order to exploit analogies with the abelian case. Thus the
hypergroup has been born, and with its sister the locally compact
topological hypergroup toils daily in the harmonic-analytic vineyard.

Despite their algebraic nature, however, an appropriate algebraic
treatment of discrete hypergroups has not appeared until now, although the
literature is already of significant size, as perusal of the bibliography
of [6] shows (see especially [8], an analytic treatment, and [7] for an
algebraic notion of hypergroup with weaker structure).

In presenting what we feel is such a treatment, there are two

additional motives. In [1], Brauer introduced the idea of studying
character tables *in abstracto*, by considering them as square arrays of
complex numbers satisfying certain postulates. It was shown in [2] that
such character tables, though not always having an underlying group, always
have an underlying hypergroup. Hence the hypergroup suggests itself as a
tool for group theorists: this is our first motive. Our second motive is
to illustrate the satisfying way in which hypergroup theory extends group
theory.

Accordingly, we give herein definitions of hypergroups and their
morphisms, and their relation to double algebras (§1). Quotient hyper-
groups (§2), and finally orbit hypergroups (§3) are studied. This
selection of topics is made with algebraic readers in mind. Much of the
treatment allows coefficients in a more or less general ring.

A duality theorem for finite abelian hypergroups over a splitting
field is presented in a companion paper [3], which logically depends only
on §1 of the present paper.

1.   Hypergroups and hypergroup algebras

1.0.

Let R be a fixed commutative ring with identity 1 . Let C be a
subset of R . A *hypergroup* $(H, n, d, 1, -)$ *with coefficients in* C is
a set H together with three maps $n : H^3 \to C$, $d : H \to C\backslash\{0\}$,
$- : H \to H$, called *multiplicity*, *degree*, and *conjugation*, respectively,
together with a distinguished element $1 \in H$, satisfying the following
axioms:

(H1)   (Associativity) for $a, b \in H$, $n(a, b, x) \neq 0$ for only
finitely many $x \in H$, and

$$\sum_{x \in H} n(a, b, x)n(x, c, f) = \sum_{y \in H} n(b, c, y)n(a, y, f) ;$$

(H2)   (Identity) for $a, b \in H$, $n(1, a, b) = n(a, 1, b) = \delta_{ab}$;

(H3)   (Reversibility) for $a, b \in H$, $n(a, \overline{b}, 1) \neq 0$ if and
only if $a = b$; and for all $a \in H$,
$n(a, \overline{a}, 1) = n(\overline{a}, a, 1)$;

(H4) (Conjugation) $n(\overline{a},\ \overline{b},\ \overline{c}) = n(b,\ a,\ c)$; $d(\overline{a}) = d(a)$;

(H5) (Degree) $d(a)d(b) = \sum\limits_{x \in H} n(a,\ b,\ x)d(x)$;

(H6) (Division) $n(a,\ \overline{a},\ 1)$ divides $d(a)^2$ in R , for each
$a \in H$, and the quotient belongs to C .

Elementary examples of hypergroups are the *character hypergroup* $\hat{G}$ of
a compact group G , and *conjugacy class* hypergroup $\overline{G}$ of a group G
whose conjugacy classes are finite. These are *abelian* hypergroups in that
they satisfy the identity

(1) $n(a,\ b,\ c) = n(b,\ a,\ c)$,

and their coefficients are nonnegative integers. Every finite group is
also a hypergroup.

It follows from (H1), (H3), and (H4) that the identity

(2) $n(a,\ b,\ c)r(c) = n(\overline{a},\ c,\ b)r(b) = n(c,\ \overline{b},\ a)r(a)$

holds, where $r(a) = n(a,\ \overline{a},\ 1) = n(\overline{a},\ a,\ 1)$. This explains the word
"reversible". Notice that if G is a finite group, then for $\hat{G}$, $r \equiv 1$,
while for $\overline{G}$, $r(x) = d(x)$ for $x \in \overline{G}$. Such hypergroups are called
standard and *classlike*, respectively.

1.1. REMARKS ON THE AXIOMS

(i) If $d(a) = 1$ for all a , and if the coefficients are real and
nonnegative, then our notion of a hypergroup coincides with that of a
discrete hypergroup in Spector [4], [5]. We call a hypergroup *grouplike*
if $d(a) = 1$ for all $a \in H$. All hypergroups are equivalent in a certain
sense with a grouplike hypergroup, as demonstrated in Proposition 1 below.

(ii) If H is finite and has positive real coefficients then d is
unique. This follows from the orthogonality relations [3, Section 8].

(iii) Every abelian hypergroup has an underlying "canonical hyper-
group" structure (see [7]) when we define the hyperproduct by

(3) $a \times b = \{x \in H \mid n(a,\ b,\ x) \neq 0\}$.

However, there exists a canonical hypergroup with no corresponding hyper-
group structure, if we require the coefficients to be integers.

(iv) It follows from the axioms that $\overline{\overline{a}} = a$ for $a \in H$ and that $\overline{a}$

is unique.

1.2. NORM OF A HYPERGROUP

This is defined for a finite hypergroup by

$$(4) \qquad\qquad \|H\| = \sum_{a \in H} d(a)^2 r(a)^{-1}$$

which by (H6) is an element of the coefficient ring. It is analogous to the order of a group, and coincides with this concept if H is a group.

1.3. GROUP OF SCALARS

Let H be a hypergroup with positive integral coefficients, and let $a \in H$. Then adapting [2, Lemma 2.4] we see that the conditions

(i) $d(a) = 1$,

(ii) $n(a, \overline{a}, b) = \delta_{1b}$ for $b \in H$, and

(iii) for each $b \in H$ there exists a unique $c \in H$ such that
        $n(a, b, x) = \delta_{cx}$ for all $x \in H$,

are all equivalent $\big($note that by (H6), $d(a) = 1$ implies $r(a) = 1 \big)$. The set of such elements forms a group if we put $a.b = c$ when (iii) holds. This group is called the *group of scalars* of H.

Note that the three conditions certainly are not equivalent if the ceofficients are not assumed to be integral and positive.

1.4. HYPERGROUP ALGEBRAS

Every group may be regarded as a standard classlike hypergroup with positive integral coefficients (and conversely, by 1.3). Just as a group gives rise to its group Hopf algebra, a hypergroup gives rise to a "double algebra", which we now define. It is of great theoretical importance and implies the duality theorem of [3].

A *double algebra* over a field k is a k module A with the following structure (see [9] for terminology on cogebras):

(A1) A is a k-algebra with unit $u : k \to A$;

(A2) A is a k-cogebra with counit $d : A \to k$;

(A3) the grouplikes in the cogebra A span A (hence are a

basis);

(A4) u is a cogebra map;

(A5) d is an algebra map;

(A6)   there is a mapping $S : A \to A$ which is a cogebra map, and

an algebra map of A into its *opposite* algebra A^0 (it

follows that S maps grouplikes to grouplikes and

$S(1) = 1$);

(A7)   let $\theta_1 : A \to k$ be the unique k-linear map such that

$\theta_1(x)$ is the coefficient of   1   when x is written in

terms of grouplikes. Then for grouplikes $a, b \in A$,

$\theta_1(bS(a)) = \theta_1(aS(b)) \neq 0$ if and only if $a = b$ (it

follows that $S^2 = 1_A$ and that S is unique).

PROPOSITION 1. *Let H be a hypergroup with coefficients in a field*
k . *Let kH be the free k-module with basis H . Define multiplication*
on kH by $(a, b) \to \sum\limits_{c \in H} n(a, b, c)c$, let $u(1) = 1 \in H$, and let Δ, d, S
be the unique k-linear extensions of the maps

$$\Delta : a \to d(a)^{-1}a \otimes a \quad (a \in H) ,$$

(5) $$d : a \to d(a) \quad (a \in H) ,$$

$$S : a \to \overline{a} \quad (a \in H) .$$

Then kH is a double algebra, and $\{d(a)^{-1}a \mid a \in H\}$ is its set of group-
likes.

Conversely, let A be a double algebra, and let $H_0(A)$ be the set of
grouplikes in A . Then there is a unique hypergroup structure on $H_0(A)$
for which A is isomorphic to $k(H_0(A))$ both as algebra and cogebra.

The proof is a trivial consequence of the axioms.   □

1.5. MORPHISMS

A *morphism* $A \xrightarrow{\phi} B$ between two double algebras over the same field

k is a k-linear map which is an algebra map, a cogebra map, and such that
$S \circ \phi = \phi \circ S$. A morphism necessarily preserves grouplikes. A *morphism*
$H \xrightarrow{\phi} J$ between two hypergroups H and J with coefficients in k is a
morphism $kH \xrightarrow{\phi} kJ$ of the corresponding double algebras. By *isomorphism*
we mean a morphism possessing an inverse morphism.

If $R \leq k$ is a subring of k then ϕ is a *R-morphism* if
$\phi(RH) \subseteq RJ$ (that is if $\phi(H) \subseteq RJ$). A *weak R-morphism* is an algebra map
$\phi : RH \to RJ$ preserving the counit and S .

Note that if $\phi : H \to J$ is an R-morphism of hypergroups, then $\phi(a)$
is an R-multiple of an element of J , for each $a \in H$.

PROPOSITION 2. *(i) Let $A = kH$ be a double algebra. Let $H_2(A)$*
be the set of classlike *elements of A , that is, those of the form*
$d(a)r(a)^{-1}a$ $(a \in H)$. *Then $H_2(A)$ is independent of the original hyper-*
group H , and under its induced hypergroup structure $H_2(A)$ is a class-
like hypergroup.

(ii) Let $A = kH$ be a double algebra, where H has positive real
coefficients. Let $H_1(A)$ denote the set of standard elements of A ,
namely those of the form $r(a)^{\frac{1}{2}}a$ $(a \in H)$, where $r(a)^{\frac{1}{2}}$ denotes the
positive square root. Then $H_1(A)$ is independent of the original hyper-
group H , and under its induced hypergroup structure $H_1(A)$ is a standard
hypergroup.

Proof. It is easy to see that $H_1(A)$ and $H_2(A)$ may be defined in
terms of the grouplike hypergroup $H_0(A)$. The rest follows.   □

COROLLARY. *If H is a hypergroup with coefficients in a field k ,*
then there is a classlike hypergroup with coefficients in k which is
k-isomorphic with H . If k contains the field $\mathbb{R}$ and H has positive
real coefficients, then there is also a standard hypergroup with these
properties.   □

We stress here that a morphism $\phi : H \to J$ of hypergroups - even an
isomorphism - is not necessarily a set-theoretic mapping of H into J ,

save as indicated in the following proposition.

PROPOSITION 3. *Let* $\phi : A \rightarrow B$ *be a morphism of double algebras. Then* ϕ *maps grouplikes to grouplikes, and classlikes to classlikes. If* $x \in A$ *is standard, then there exists a standard* $y \in B$ *such that* $\phi(x) = d(x)d(y)^{-1}(y)$. *If* ϕ *is* injective, *and* $x \in A$ *is standard, then* $\phi(x)$ *is standard.*   □

1.6. EXAMPLES OF MORPHISMS ·

(i) $\{1\}$ is a hypergroup in a unique way and so $k \cong k\{1\}$ is a double algebra (in fact, a Hopf algebra). If H is any hypergroup then the unit and counit $d : kH \rightarrow k$ and $u : k \rightarrow kH$ are morphisms.

(ii) Let $\phi : G_1 \rightarrow G_2$ be a continuous group homomorphism of compact groups. For $i = 1, 2$, $\hat{G}_i$ can be identified with an algebra of functions on G_i . Put $\hat{\phi}(f) = f \circ \phi$. Then $\hat{\phi}$ is a weak $\mathbb{Z}^+$-morphism from $\hat{G}_2$ to $\hat{G}_1$, and is a $\mathbb{Z}^+$-morphism precisely when ϕ is surjective.

This exhibits $\hat{\ }$ as a functor from the opposite category of the category of compact groups and continuous group homomorphisms to the category of standard abelian hypergroups with positive integral coefficients and weak $\mathbb{Z}^+$-morphisms.

1.7. SUBHYPERGROUPS

Let H be a hypergroup. A subset J of H is a *subhypergroup* if $a, b \in J$ and $c \notin J$ imply $n(a, b, c) = 0$ and $S(J) \subseteq J$. The subset J then inherits a hypergroup structure from H in the obvious way, and kJ is a subalgebra of kH which is simultaneously a subcogebra.

Conversely, if $B \subseteq kH$ is a subalgebra and subcogebra of kH , then it is easily seen that if $x \in B$ and $x = \sum_{a \in H} \alpha(a)a$ then for all $a \in H$ either $\alpha(a) = 0$ or $a \in B$. Thus B is the linear span of some subset J of H , which must be a subhypergroup since B is a subalgebra.

If G is a group, then $J \subseteq G$ is a subhypergroup if and only if it is a subgroup.

It is certainly clear that if $\phi : H \rightarrow K$ is a morphism, then there is a subhypergroup J of K with inclusion morphism i , and a morphism

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$\psi : H \to J$ such that
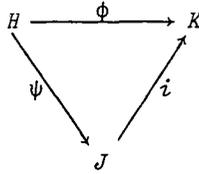
commutes, and $\psi(kH) = kJ$.

1.8. DOUBLE ALGEBRAS WHICH ARE BIGEBRAS

The following is now easy to prove.

THEOREM 1. *Let H be a hypergroup with positive integral coefficients. Let $A = \mathbb{Q}H$ be the rational double algebra of H (see Proposition 1). Then the following are equivalent:*

(i) *A is a Hopf algebra and S is its antipode;*

(ii) *A is a Hopf algebra;*

(iii) *A is a bigebra;*

(iv) *Δ is an algebra map;*

(v) *M is a cogebra map;*

(vi) *$S * I = I * S = ud$ where $*$ denotes convolution in*
$\hom_{\mathbb{Q}}(A, A)$ *;*

(vii) *elements of H are invertible in A ;*

(viii) *$H_0(A)$ is a group.* $\square$

2.   Quotient hypergroups

2.1. QUOTIENTS

Let A be a double algebra. Then a *double ideal* in A is a k submodule V of A which is

(1)   an ideal,

(2)   a coideal (see [9, p. 18ff]), and

(3)   invariant under S .

The kernel of the counit d is the largest coideal, and since it is

also an ideal invariant under S it is the largest double ideal. We plan to establish a correspondence between double ideals in kH of a certain type and completely normal subhypergroups of H .

PROPOSITION 4. *Let kH be a double algebra, let $V \subseteq kH$ be a double ideal, and let $\pi : kH \to kH/V = E$ be the natural k-linear map.*

(a) There is a unique structure of algebra and cogebra on E such that π is a morphism of both algebras and cogebras, and a unique mapping $S : E \to E$ with $S \circ \pi = \pi \circ S$. With these structures, E satisfies all axioms for a double algebra except (A7).

(b) If $f : kH \to D$ is any map into a k-module D which has the above three structures and if f respects these structures then $\ker f$ is a double ideal.

(c) If $V \subseteq \ker f$ then there is a unique k-module morphism f^{*} such that the diagram
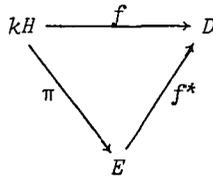
commutes.

The proof is trivial. $\quad\square$

PROPOSITION 5. *Let H be a hypergroup with positive real coefficients contained in the field k . Let V be a double ideal in the double algebra kH . Let $\pi : kH \to kH/V = E$ be the quotient map. Let J be the set*

$$J = \{a \in H \mid \pi(a) = d(a).1\} .$$

Thus J is a subhypergroup of H . Also, E is a double algebra, that is satisfies axiom (A7), *if and only if J satisfies the following condition:*

(a) for all $a, b \in H$,

(*) $$\sum_{z \in J} n(a,\ b,\ z)d(z) = \sum_{z \in J} n(b,\ a,\ z)d(z) .$$

If J is finite, then (a) is equivalent to:

(b) for all $a \in H$,

$$\xi_J a = a \xi_J \ ,$$

where $\xi_J = \sum_{z \in J} d(z) r(z)^{-1} z \in kJ$.

Proof. Obviously $\overline{J} \subseteq J$. Suppose $n(a, b, c) \neq 0$ with $a, b \in J$.
Then we have $\pi(ab) = \pi(a)\pi(b)$, which is a multiple of $1 \in E$, while
$\pi(ab) = \sum_{c' \in H} n(a, b, c')\pi(c')$, so that $n(a, b, c) \neq 0$ entails that $\pi(c)$
is a multiple of 1 . Thus J is a subhypergroup of H .

It is a simple exercise to show that if J is finite then (a) and (b)
are equivalent.

Now, as the grouplikes in E are the elements $d(x)^{-1}\pi(x)$, where x
is a grouplike in kH , (A7) for E is equivalent to axiom (H3) for
elements $\pi(a)$, namely, to the assertions that for $a, b \in H$,
$n(\pi(a), \pi(b), 1) = n(\pi(b), \pi(a), 1)$ and that this expression takes the
value 0 if and only if $\pi(a) \neq \overline{\pi(b)}$. The first assertion is equivalent
to (*). The proof will be complete if we prove that the second assertion
always holds.

Now observe that $n(\pi(a), \pi(b), 1) = \sum_{z \in J} n(a, b, z) d(z) = r(b)\alpha$,
where α is the coefficient of b in $\overline{a}\xi_J$. Also, $\pi(\xi_J) = \|J\| . 1$, so
that $\pi(\overline{a}\xi_J) = \|J\|\overline{\pi(a)}$. If $n(\pi(a), \pi(b), 1) \neq 0$ then $\alpha \neq 0$, and hence
by the linear independence of grouplikes, $\pi(b) = \overline{\pi(a)}$. Conversely
$n(\pi(a), \overline{\pi(a)}, 1) = n(\pi(a), \pi(\overline{a}), 1)$
$$= \sum_{z \in J} n(a, \overline{a}, z) d(z) \geq n(a, \overline{a}, 1) > 0 \ . \qquad \square$$

We shall call a subhypergroup *prenormal*, if it satisfies condition (a)
of Proposition 5. If V is a double ideal in kH , the subhypergroup
defined in Proposition 5 is said to be *associated* with V . The ideal V
is generated as a k-module by the elements $d(a')a - d(a)a'$ where
$a, a' \in H$ and $d(a)\pi(a') = d(a')\pi(a)$.

PROPOSITION 6. *Let k, H, V, π , and J be as in Proposition 5, and*

suppose that J is prenormal. Then the following are equivalent for a, $b \in H$:

    (i) $d(a)\pi(b) = d(b)\pi(a)$;

    (ii) $n\big(\pi(a), \overline{\pi(b)}, 1\big) \neq 0$;

    (iii)   there exists $z \in J$ such that $n(b, z, a) \neq 0$;

   (iv)   there exists $z \in J$ such that $n(a, \overline{b}, z) \neq 0$;

and if J is finite, these are equivalent to

    (v)   the coefficient of b in $a\xi_J$ is nonzero;

   (vi)   the coefficient of a in $b\xi_J$ is nonzero. $\square$

The conditions (iii) and (iv) are plainly equivalent whatever the prenormal subhypergroup J of H . Let us write $a \sim b$ when these conditions hold. It is elementary to show that $\sim$ is an equivalence relation. A $\sim$-class we call a *coset modulo* J .

Proposition 5 can be stated in the following form.

PROPOSITION 5'. *If H is a hypergroup with positive real coefficients in a field k and if $kH \xrightarrow{\alpha} kL$ is a hypergroup morphism surjective as a k-linear map (that is, as a cogebra map) then the largest subcogebra $kJ \overset{\iota}{\rightarrowtail} kH$ for which the diagram*

$$kJ \xrightarrow{\quad \iota \quad} kH \xrightarrow{\quad \alpha \quad} kL$$

(†)

$$k\{1\}$$
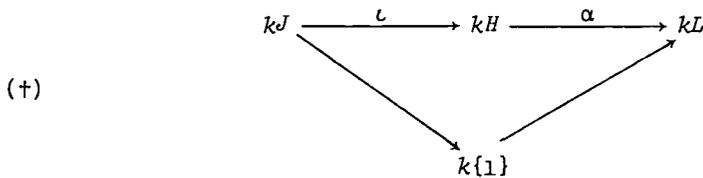
commutes is actually a subalgebra as well, so that J is a (prenormal) subhypergroup and further, any commutative diagram

$$kJ \longrightarrow kH \xrightarrow{\alpha'} kL$$

$$K\{1\}$$
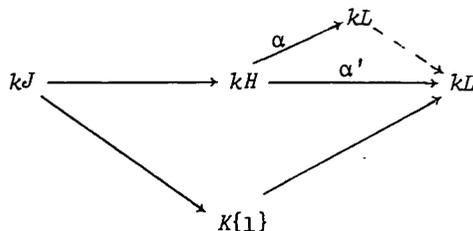
where α *and* α' *are morphisms can be completed commutatively as shown. In particular,* all morphisms have kernels.   □

The dual problem is the following. If $kJ \xrightarrow{\iota} kH$ is a prenormal subhypergroup, what conditions on J ensure that it is a kernel, that is, that there is a morphism with domain kH such that (†) commutes? We call the prenormal subhypergroup J *normal* if this is the case. The equivalence of this to the following conditions is routine to check $\bigl((3)\text{-}(6)$ assuming J is finite, (7), (8) assuming h abelian$\bigr)$:

(1)   for all $a, b, c \in H$, the element

$$\sum_{c'\sim c} n(a,\, b,\, c')d(c')d(a)^{-1}d(b)^{-1}$$

of k depends only on the cosets modulo J to which a and b belong;

(2)   the k-linear span V of the set $\{d(a)^{-1}a\text{-}d(b)^{-1}b \mid a \sim b\}$ is an ideal in kH ;

(3)   for all $a \in H$, $d(a)^{-1}a\xi_J = d(\xi_J)d(\xi_{aJ})^{-1}\xi_{aJ}$, where

$$\xi_{aJ} = \sum_{a'\sim a} d(a')r(a')^{-1}a' \; ;$$

(4) V is the annihilator of ξ_J ;

(5) $a \sim b \Longleftrightarrow d(a)^{-1}a\xi_J = d(b)^{-1}b\xi_J$ $\bigl($note that as $d(\xi_J)^{-1}\xi_J$ is idempotent, $a \longmapsto d\bigl(\xi_J^{-1}\bigr)a\xi_J$ is always a ring homomorphism ;

(6)   the set $\bigl\{d(a)^{-1}a\xi_J \mid a \in H\bigr\}$ is k-linearly independent;

(7) $J = A^{\perp} = \{a \in H \mid \forall \rho \in A, \rho(a) = d(a)\}$ for some set A of algebra maps $kH \to k$;

(8) $J = J^{\perp\perp}$.

The following is an extension of the well-known property of abelian groups.

PROPOSITION 7. *Let H be an abelian hypergroup with positive real coefficients in a splitting field k whose grouplike dual H_0^* also has*

positive coefficients (refer to [3, paragraph 14, proposition]). *Then every subhypergroup of H is normal.*

Proof. It is proved in [3] that the category A of abelian hyper-groups (equivalently, of commutative double algebras) with real coefficients and morphisms is *self dual*, that is to say that $D : H \to H_0^*$ (equivalently, $kH \to (kH)^*$) and defined in the obvious way on morphisms is a functor from this category to its opposite category, and the isomorphisms $\#_H : H \to H^{**}$ constitute an equivalence of the identity

functor on A with D^2 .

The hypergroups with positive real coefficients contained in a fixed splitting field $k \subseteq \mathbb{C}$, and whose duals have positive real coefficients form a full subcategory A_k^+ of A , and $DA_k^+ = \left(A_k^+\right)^{op}$.

Let us call a morphism $H \xrightarrow{\alpha} L$ of hypergroups surjective or injective according to whether it is surjective or injective on the under-lying sets of grouplikes (equivalently, on the k modules kH and kL).

If A is a semisimple commutative algebra over a splitting field k and if $A_1 \subseteq A$ is a k-subalgebra, then every algebra map $\rho_1 : A_1 \to k$

extends to an algebra map $\rho : A \to k$. Consequently, a morphism $H \xrightarrow{\alpha} L$ in A_k^+ is injective if and only if its dual $L_0^* \xrightarrow{\alpha^*} H_0^*$ is surjective. Combining these facts with Proposition 5' we obtain Proposition 7.   □

The hypotheses of Proposition 7 are satisfied by $\overline{G}$ and $\hat{G}$ if G is a finite group. The category A_k^+ is also closed under subhypergroups and morphic images.

The cokernel arising from a subhypergroup $J \subseteq H$ $\left(\text{if } H \in A_k^+ \right)$ we will denote by H/J , and call the *quotient hypergroup* of H by J . Note that $(H/J)^*$ is isomorphic with the subhypergroup $J^{\perp}$ of H^* .

2.2. COEFFICIENTS OF A QUOTIENT HYPERGROUP

Let H be a hypergroup with positive integral coefficients, and let J be a normal *finite* subhypergroup of H . Plainly, each coset of J is then finite. Let V be the double ideal with which J is associated, and

π the quotient map $\mathbb{Q}H \to \mathbb{Q}H/V = E$.

For $a \in H$, define $a_J = d(a_J)d(a)^{-1}\pi(a)$, where

$$d(a_J) = \text{g.c.d.}\{d(x) \mid x \in H, \ x \sim a\} .$$

Let H/J be the hypergroup $\{a_J \mid a \in H\}$, whose double algebra is E . Put

$$\|a_J\| = \sum_{a' \sim a} d(a')^2 r(a')^{-1} .$$

PROPOSITION 8. *The following statements are true:*

(i) $n(a_J, b_J, c_J)$

$$= d(a_J)d(b_J)d(a)^{-1}d(b)^{-1}d(c_J)^{-1} \sum_{c' \sim c} n(a, b, c')d(c') ;$$

(ii) $r(a_J) = n(\overline{a}_J, a_J, 1) = \|J\|d(a_J)^2\|a_J\|^{-1}$;

(iii) $\|H\| = \|J\| \|H/J\|$;

(iv) $n(a_J, b_J, c_J) \in \mathbb{Z}^+$, $d(a_J) \in \mathbb{Z}^+$; if $\|J\|$ is a divisor of $\|a_J\|$ for all $a_J \in H/J$, then H/J has its coefficients in $\mathbb{Z}^+$.

Proof. Statement *(i)* is trivial. Multiplying it by $\dfrac{d(a)^2}{r(a)} \dfrac{d(b)^2}{r(b)}$
and summing over the cosets of a and b we obtain the symmetric expression

$n(a_J, b_J, c_J)$

$$= d(a_J)d(b_J)\|a_J\|^{-1}\|b_J\|^{-1}d(c_J)^{-1} \sum_{a' \sim a} \sum_{b' \sim b} \sum_{c' \sim c} \frac{n(a',b',c')d(c')d(a')d(b')}{r(a')r(b')}$$

whence

$$n(a_J, b_J, c_J).d(c_J)^2\|c_J\|^{-1} = n(c_J, \overline{b}_J, a_J)d(a_J)^2\|a_J\|^{-1} ,$$

and putting $b = \overline{a}$ and $c = 1$, and using $\|1_J\| = \|J\|$, we obtain *(ii)*.

Relation *(iii)* is obtained directly from *(ii)*, and there remains the

proof of *(iv)*. Now it is plain that $n(a_J, b_J, c_J) \geq 0$ for all a, b, $c \in H$, and that $d(a_J) \in \{1, 2, 3, \ldots\}$ by its definition.

Let $a \in H$, and let $\{a_1, \ldots, a_r\}$ be the coset of a. Then by the definition of $d(a_J)$, there are integers $k_1, \ldots, k_r$ such that

$$d(a_J) = \sum_{i=1}^{r} k_i d(a_i) .$$

Therefore

$$\pi\left\{\sum_{i=1}^{r} k_i a_i\right\} = \left[\sum_{i=1}^{r} k_i d(a_i)\right] d(a_J)^{-1} a_J = a_J ,$$

showing that a_J belongs to the ring $\pi(\mathbb{Z}H)$. It follows that $n(a_J, b_J, c_J)$ is an integer·for all a, b, $c \in H$.   □

2.3. $\|J\|$ NEED NOT DIVIDE $\|H\|$

Consider the hypergroup

$$H = \langle 1, a, b : d(a) = 1, d(b) = 5, a^2 = 1, ab = ba = b, b^2 = 5+5a+3b \rangle .$$

It is classlike abelian with integral coefficients, with norm $\|H\| = 7$. The subhypergroup $J = \{1, a\}$ has norm $\|J\| = 2$. The quotient H/J is given by

$$H/J = \left\langle 1, b_J : d(b_J) = 5, b_J^2 = 10+3b_J \right\rangle ,$$

of course, $\|H/J\| = 7/2$, so that H/J does not have coefficients in $\mathbb{Z}$.

However, if H is an abelian hypergroup with integral coefficients, and some form of the dual of H also has integral coefficients (for example, H_0^*, H_1^*, or H_2^*) then every subhypergroup $J \subseteq H$ has norm dividing $\|H\|$. The reason is, of course, that $\|H/J\| = \|(H/J)^*\| = \|J^\perp\|$, where $J^\perp$, the annihilator of J in H^*, is a subhypergroup of H^*. By hypothesis its norm is an integer.

3. Orbit hypergroups

Throughout this section, hypergroups are assumed for convenience to

have positive integral coefficients, although this is essential only in
those paragraphs which deal with quotient hypergroups.

3.1.

An automorphism of a hypergroup H is an isomorphism of H onto H
(see 1.5). The automorphisms of H form a group $\text{aut}(H)$. If H and H'
are isomorphic, then so are $\text{aut}(H)$ and $\text{aut}(H')$.

3.2.

PROPOSITION 9. *Let H^* be any dual of the abelian hypergroup H.*
Then the mapping $\phi \rightarrow (\phi^*)^{-1}$ is a group isomorphism of $\text{aut}(H)$ with
$\text{aut}(H^*)$ *$\left((\phi^*)^{-1}\right.$ has the effect of carrying the element $\rho \in H^*$ to the*
element $\rho \circ \phi^{-1}$).   □

3.3.

Let G be a group. A *G-hypergroup* H is a hypergroup upon which G
acts on the left by means of automorphisms. We say H is *trivial* if the
action of G on H is trivial. If H and H' are G-hypergroups, then
a *G-morphism* $\phi : H \rightarrow H'$ is a morphism of hypergroups rendering the
diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\phi} & H' \\
g \downarrow & & \downarrow g \\
H & \xrightarrow{\phi} & H'
\end{array}
$$
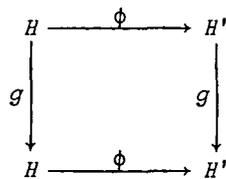
commutative for each $g \in G$. Clearly, if H is a G-hypergroup, then G
acts naturally on RH for any ring $R \supseteq \mathbb{Z}$ as a group of algebra and
cogebra morphisms, each of which respects the operator S.

We may also define a weak $(\mathbb{Z}^+, G)$-morphism to be a weak $\mathbb{Z}^+$-morphism
rendering (3.1) commutative.

3.4.

In the following, $\#X$ denotes the cardinality of the set X.

PROPOSITION 10. *Let H be a G-hypergroup, and suppose that the*
orbits of H under G are finite. Let $G \backslash H$ denote the set of orbit sums

in $\mathbb{Z}H$ *of the form*

$$[a] = \sum_{a' \in G.a} a' \quad (a \in H) \; .$$

Then $\{[a] : a \in H\}$ *is linearly independent in* $\mathbb{Q}H$ *and under the ring multiplication in* $\mathbb{Z}H$ *we have*

$$(3.2) \qquad [a][b] = \sum_{[c] \in G \backslash H} \left(\sum_{a' \in G.a} \sum_{b' \in G.b} n(a', \; b', \; c) \right) [c]$$

and also

$$(3.3) \qquad \qquad d([a]) = \#[a] d(a) \; .$$

Then $G \backslash H$ *is a hypergroup with coefficients in* $\mathbb{Z}^+$ *, and*

$$(3.4) \qquad \qquad r([a]) = \#[a] r(a) \; ,$$

$$(3.5) \qquad \qquad \|G \backslash H\| = \|H\| \quad if \quad H \quad is \ finite.$$

Proof. Relation (3.2) is obtained directly, (3.3) is obvious. Now [1] occurs in $[a][b]$ in the decomposition (3.2) if and only if for some $g_1, g_2 \in G$, we have $n(g_1 a, \; g_2 b, \; 1) \neq 0$; that is $g_1 a = g_2 \overline{b}$. This means that $[a] = [\overline{b}]$, and so $G \backslash H$ is a hypergroup. Finally, (3.4) is computed directly from the relation $r([a]) = n([a], \; [\overline{a}], \; 1)$.   □

3.5.

We shall call $G \backslash H$ the *orbit hypergroup* of H under G . Clearly, for any ring $R \supseteq \mathbb{Z}$, $R(G \backslash H)$ is the set of fixed points of the double algebra RH under the action of G .

If $\phi : H \to H'$ is a G-morphism $\left(\text{respectively weak } (\mathbb{Z}^+, G)\text{-morphism}\right)$ then $\phi(G \backslash H)$ is fixed elementwise by the action of G on H' , and it is easy to see that we thus obtain a morphism $\left(\text{respectively weak } (\mathbb{Z}^+, G)\text{-}\right.$ morphism$\left.\right)$

$$G \backslash \phi \; : \; G \backslash H \to G \backslash H' \; .$$

THEOREM 2. *The mappings* $H \mapsto G \backslash H$, $\phi \mapsto G \backslash \phi$ *constitute a functor from the category of orbit-finite* G-*hypergroups and* G-*morphisms* $\left(\textit{respectively weak } (\mathbb{Z}^+, G)\textit{-morphisms}\right)$ *to the category of trivial* G-*hypergroups and hypergroup morphisms (respectively weak* $\mathbb{Z}^+$-*morphisms).*   □

3.6.

The following is immediate from Proposition 10.

PROPOSITION 11. *If H is classlike with positive integral coefficients, then so is $G \backslash H$.*   □

3.7. ORBITS IN STAGES

Let H be a G -hypergroup, and let $\psi : G_1 \to G$ be a group homomorphism. Then putting $g_1 . x = \psi(g_1) . x$ $(x \in H, g_1 \in G_1)$ makes H into a G_1 -hypergroup. If $\psi(G_1)$ is *normal* in G , then putting $g . [a] = [ga]$ $(g \in G, a \in H)$ makes $G_1 \backslash H$ into a G -hypergroup, on which $\psi(G_1)$ acts trivially. Thus $G_1 \backslash H$ may be regarded as a $(G / \psi(G_1))$ -hypergroup, and we have a natural isomorphism between $G \backslash H$, $(G/G_1) \backslash (G_1 \backslash H)$, and $G \backslash (G_1 \backslash H)$.

3.8. REMARKS

(i) Let G be a group, and let it act on itself by conjugation. Then $G \backslash G$ is by definition the conjugacy class hypergroup $\overline{G}$. Relation (3.5) recovers the class equation for G .

(ii) If N is a normal subgroup of G_1 and $\iota : N \to G$ is the inclusion mapping, then $G \backslash \iota : G \backslash N \to G \backslash G$ maps $G \backslash N$ isomorphically onto a subhypergroup of $\overline{G}$, namely, to the hypergroup of all G -classes which are contained in N .

(iii) $G \backslash N$ and $G \backslash \overline{N}$ are isomorphic by (i) and 3.7.

3.9. THE DUAL OF AN ORBIT HYPERGROUP

Let   be a finite G -hypergroup with coefficients in a splitting field k , of characteristic not dividing $\#G$. Then $k(G \backslash H^*)$ is the subalgebra of kH^* consisting of those linear functionals on kH that are fixed under the action of G given by

$$(g . \rho)(x) = \rho(g^{-1} . x) \ , \quad x \in kH \ , \quad \rho \in kH^* \ ;$$

see Proposition 9 above. If $\rho \in k(G \backslash H^*)$, then ρ has a restriction to $k(G \backslash H) \in kH$ which we denote $\kappa(\rho)$.

THEOREM 3. *The mapping* κ *is a hypergroup isomorphism of* $k(G\backslash H^*)$ *onto* $k(G\backslash H)^*$.

Proof. We first show that κ is surjective. Let $\eta \in k(G\backslash H)^*$. Then η is a linear functional on $k(G\backslash H) \subseteq kH$, and so has an extension η_* to all of kH . Put $\rho = (\#G)^{-1} \sum_{g\in G} g.\eta_*$. Then $\rho \in kH^*$ and is plainly fixed by G , so $\rho \in k(G\backslash H^*)$. If $x \in k(G\backslash H)$, then $x \in kH$ and G fixes x . Hence

$$\rho(x) = (\#G)^{-1} \sum_{g\in G} g.\eta_*(x) = \eta_*\left[(\#G)^{-1} \sum_{g\in G} g^{-1}.x\right] = \eta_*(x) = \eta(x) \ .$$

Thus $\kappa(\rho) = \eta$.

Now we show κ is injective. Suppose $\rho \in k(G\backslash H^*)$ and $\kappa(\rho) = 0$; that is, that $\rho(x) = 0$ whenever $x \in kH$ is fixed by G . Let $x \in kH$. Then

$$\rho(y) = \left[(\#G)^{-1} \sum_{g\in G} g.\rho\right](y) = \rho\left[(\#G)^{-1} \sum_{g\in G} g^{-1}.y\right] = 0 \ ,$$

since the argument is fixed by G . Hence $\rho = 0$.

As an algebra, $k(G\backslash H^*)$ is a subalgebra of kH^* . Accordingly, if ρ_1, $\rho_2 \in k(G\backslash H^*)$ and $x_0 \in kH$ is group like, then $(\rho_1\rho_2)(x_0) = \rho_1(x)\rho_2(x_0)$. Now a typical grouplike in $k(G\backslash H)$ has the form $x = (\#G)^{-1} \sum_{g\in G} g.x_0$, where x_0 is grouplike in kH . Thus we easily see that

$$\kappa(\rho_1)\kappa(\rho_2)(x) = \kappa(\rho_1)(x)\kappa(\rho_2)(x) = \rho_1(x_0)\rho_2(x_0) = (\rho_1\rho_2)(x_0) = \kappa(\rho_1\rho_2)(x) \ .$$

This shows that κ is an algebra map.

It is plain that κ respects the conjugation on $k(G\backslash H^*)$. We have now but to show that κ is a cogebra map, that is, if $\gamma \in G\backslash H^*$ is grouplike, then $\kappa(\gamma)$ is grouplike in $(G\backslash H)^*$.

Now $\gamma \in k(G\backslash H^*)$ being grouplike signifies that for some grouplike $\tau \in kH^*$, $\gamma = (\#G)^{-1} \sum_{g\in G} g.\tau$. Now $\kappa(\tau)$ is the restriction of τ to

$k(G\backslash H)$, and $\kappa(\gamma) = (\#G)^{-1} \sum_{g \in G} \kappa(g.\tau)$. The statement that $\tau \in kH^*$ is

grouplike entails that $\tau : kH \rightarrow k$ is an algebra map. Consequently, if
$x, y \in k(G\backslash H)$, since $xy \in k(G\backslash H)$ is fixed by g , we obtain

$$\kappa(\gamma)(xy) = (\#G)^{-1} \sum_{g \in G} (g\ \tau)(xy)$$

$$= (\#G)^{-1} \sum_{g \in G} \tau\big(g^{-1}(x,\ y)\big)$$

$$= (\#G)^{-1} \sum_{g \in G} \tau(x)\tau(y)$$

$$= \tau(x)\tau(y)\ ,$$

while similarly $\kappa(\gamma)(x) = \tau(x)$ and $\kappa(\gamma)(y) = \tau(y)$. Thus
$\kappa(\gamma) : k(G\backslash H) \rightarrow k$ is an algebra map, so is grouplike in $k(G\backslash H)^*$. This
completes the proof.   □

3.10.   ORBIT HYPERGROUPS AND QUOTIENT HYPERGROUPS

Let H be an orbit finite G-hypergroup with positive integral
coefficients. Let $J \leq H$ be a normal G-invariant subhypergroup of H .
Then plainly the action of G respects cosets, and hence the double ideal
V with which J is associated is G-invariant. Therefore H/J becomes a
G-hypergroup under the quotient action of G on $\mathbb{Q}H/V$, and the quotient
map $\pi : \mathbb{Q}H \rightarrow \mathbb{Q}H/V$ is a G-morphism. That is to say $\pi : H \rightarrow H/J$ is a
G-morphism. The following proposition is now easily proved.

PROPOSITION 12. *$G\backslash J$ is a normal subhypergroup of $G\backslash H$, and the
hypergroups*

$$G\backslash(H/J)\quad and\quad (G\backslash H)/(G\backslash J)$$

are isomorphic.   □

The following related result is a consequence of Clifford's Theorem on
restriction of characters to a normal subgroup, and is an analogue of [7,
Theorem 2.7].

PROPOSITION 13. *Let G be a finite group, N a normal subgroup of
G , $\hat{G}$ the character hypergroup of G , $\hat{N}$ the character group of N .
Let G act on N , hence on $\hat{N}$, by conjugation. Let J be the set of
characters $\chi \in \hat{G}$ which are constant on cosets of N . Then J is a
normal subhypergroup of $\hat{G}$.*

Let $\mathrm{res}(\hat{G}, N)$ *be the set of restrictions to* N *of characters of* G *, with pointwise multiplication. Then* $\mathrm{res}(\hat{G}, N)$ *is a hypergroup, and indeed the three hypergroups*

$$\mathrm{res}(\hat{G}, N), \ G\backslash\hat{N}, \ \hat{G}/J$$

are isomorphic.   □

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