This prompts a query. If positive integers k, l, M satisfy $4k^2 - Ml^2 = 1$, so that M and l are necessarily odd, then $M \equiv 3 \pmod 4$, and the irrationality of $\sqrt{4k^2 - 1}$ entails that of $l\sqrt{M}$ and hence $\sqrt{M}$. For many values of $M \equiv 3 \pmod 4$, the fundamental solution of the Pell equation $x^2 - My^2 = 1$ has x even and y odd, which supplies the $k = \frac{1}{2}x$ and $l = y$ required. But what explains the exceptional values of such M, the first few examples of which are 39, 55, 95 and 111?

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Infinite sums of reciprocal quadratics

Summing series such as $\displaystyle\sum_1^\infty \frac{1}{(n + p)(n + q)}$ with $0 \leqslant p < q$ integers is a familiar topic for many students. Partial fractions rewrite the terms of the series as

$$\frac{1}{q - p} \sum_1^\infty \left(\frac{1}{n + p} - \frac{1}{n + q}\right)$$

which telescopes to give

$$\sum_1^\infty \frac{1}{(n + p)(n + q)} = \frac{1}{q - p} \sum_1^{q - p} \frac{1}{n + p}.$$

After this, a natural question to ask is ‘Can we evaluate the sum

$$\sum_1^\infty \frac{1}{n^2 + An + B}$$

with integer coefficients A, B when the quadratic $n^2 + An + B$ does not factorise?’

A clue that something more sophisticated is needed comes from the case $p = q$ above where we need to invoke the sum

$$\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$$

to deduce that

$$\sum_1^\infty \frac{1}{(n + p)^2} = \frac{\pi^2}{6} - \sum_1^p \frac{1}{n^2}.$$

It is convenient to split cases according to the parity of A.

- *Case* 1: $A = 2a$ is even

  Then $n^2 + An + B = (n + a)^2 - \Delta$, where $4\Delta = A^2 - 4B$ is the discriminant of $n^2 + An + B$, assumed not to be the square of a non-negative integer. We also assume that $\Delta \neq 0$, since the case $\Delta = 0$ has already been dealt with.

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The infinite product representation of $\sin x$ gives

$$\frac{\sin \pi x}{\pi x} = \prod_1^\infty \left(1 - \frac{x^2}{n^2}\right)$$

and logarithmic differentiation then produces the series for $\cot \pi x$,

$$\pi \cot \pi x - \frac{1}{x} = \sum_1^\infty \frac{-2x}{n^2 - x^2}.$$

Substituting $x = \sqrt{\Delta}$ then gives

$$\frac{\pi \cot \pi\sqrt{\Delta}}{-2\sqrt{\Delta}} + \frac{1}{2\Delta} = \sum_1^\infty \frac{1}{n^2 - \Delta}.$$

If $a \geqslant 0$, our required sum is given by

$$\sum_1^\infty \frac{1}{(n+a)^2 - \Delta} = \sum_1^\infty \frac{1}{n^2 - \Delta} - \sum_1^a \frac{1}{n^2 - \Delta} = -\frac{\pi \cot \pi\sqrt{\Delta}}{2\sqrt{\Delta}} + \frac{1}{2\Delta} - \sum_1^a \frac{1}{n^2 - \Delta}.$$

If $a < 0$, the sum is

$$\sum_1^\infty \frac{1}{(n+a)^2 - \Delta} = \sum_1^\infty \frac{1}{n^2 - \Delta} + \sum_0^{-a-1} \frac{1}{n^2 - \Delta} = -\frac{\pi \cot \pi\sqrt{\Delta}}{2\sqrt{\Delta}} + \frac{1}{2\Delta} - \sum_0^{-a-1} \frac{1}{n^2 - \Delta}.$$

If $\Delta < 0$, the cot terms evaluate as

$$-\frac{\pi \cot \pi\sqrt{-\Delta}i}{2\sqrt{-\Delta}i} = \frac{\pi \coth \pi\sqrt{-\Delta}}{2\sqrt{-\Delta}}.$$

- *Case* 2: $A = 2a - 1$ is odd

Then $n^2 + An + B = \left(n + a - \frac{1}{2}\right)^2 - \Delta$, where

$$4\Delta = A^2 - 4B = 4a^2 - 4a - 4B + 1$$

is the discriminant of $n^2 + An + B$, assumed not to be the square of a non-negative integer.

This time we start with the infinite product for $\cos x$ in the form

$$\cos \pi x = \prod_1^\infty \left(1 - \frac{x^2}{(n - \frac{1}{2})^2}\right).$$

Logarithmic differentiation gives the series for $\tan x$,

$$\pi \tan \pi x = \sum_1^\infty \frac{2x}{(n - \frac{1}{2})^2 - x^2}.$$

Substituting $x = \sqrt{\Delta}$ then gives

$$\frac{\pi \tan \pi\sqrt{\Delta}}{2\sqrt{\Delta}} = \sum_1^\infty \frac{1}{(n - \frac{1}{2})^2 - \Delta}.$$

If $\alpha \geqslant 0$, our required sum is given by

$$\sum_1^\infty \frac{1}{(n + a - \frac{1}{2})^2 - \Delta} = \sum_1^\infty \frac{1}{(n - \frac{1}{2})^2 - \Delta} - \sum_1^a \frac{1}{(n - \frac{1}{2})^2 - \Delta}$$
$$= \frac{\pi \tan \pi\sqrt{\Delta}}{2\sqrt{\Delta}} - \sum_1^a \frac{1}{(n - \frac{1}{2})^2 - \Delta}.$$

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If $a < 0$, the sum is.

$$\sum_1^\infty \frac{1}{(n + a - \frac{1}{2})^2 - \Delta} = \sum_1^\infty \frac{1}{(n - \frac{1}{2})^2 - \Delta} + \sum_{a+1}^0 \frac{1}{(n - \frac{1}{2})^2 - \Delta}$$
$$= \frac{\pi \tan \pi\sqrt{\Delta}}{2\sqrt{\Delta}} + \sum_{a+1}^0 \frac{1}{(n - \frac{1}{2})^2 - \Delta}.$$

If $\Delta < 0$, the tan terms evaluate as

$$\frac{\pi \tan \pi\sqrt{-\Delta}i}{2\sqrt{-\Delta}i} = \frac{\pi \tanh \pi\sqrt{-\Delta}}{2\sqrt{-\Delta}}.$$

We illustrate these formulae with two numerical examples; the series $\sum_1^\infty \frac{1}{n^2 + n + 41}$ was also evaluated by this method in [1].

- $n^2 + 6n + 1$

  Here, $\Delta = \frac{1}{4}(6^2 - 4) = 8, a = 3$. So, by the $a \geqslant 0$ sum from Case 1,

$$\sum_1^\infty \frac{1}{n^2 + 6n + 1} = -\frac{\pi \cot \pi\sqrt{8}}{2\sqrt{8}} + \frac{1}{16} - \left(\frac{1}{1^2 - 8} + \frac{1}{2^2 - 8} + \frac{1}{3^2 - 8}\right)$$
$$= -\frac{\pi \cot \pi\sqrt{8}}{2\sqrt{8}} - \frac{61}{112}.$$

- $n^2 - 5n + 7$

  Here, $\Delta = \frac{1}{4}\left((-5)^2 - 4 \times 7\right) = -\frac{3}{4}$, $a = -2$. So, by the $a < 0$. $\Delta < 0$ sum from Case 2,

$$\sum_1^\infty \frac{1}{n^2 - 5n + 7} = \frac{\pi \tanh \frac{1}{2}(\pi\sqrt{3})}{\sqrt{3}} + \left(\frac{1}{(-\frac{3}{2})^2 - (-\frac{3}{4})} + \frac{1}{(-\frac{1}{2})^2 - (-\frac{3}{4})}\right)$$
$$= \frac{\pi \tanh \frac{1}{2}(\pi\sqrt{3})}{\sqrt{3}} + \frac{4}{3}.$$

It is also well-worth emphasising and generalising an observation made in [1]. Using the mid-ordinate approximation to the sum, in which the area, $\frac{1}{n^2 - n + B}$, of the rectangle with base $n - \frac{1}{2} \leqslant x < n + \frac{1}{2}$ and height $\frac{1}{n^2 - n + B}$ is approximated by $\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{x^2 - x + B}dx$, we see that for positive integers B,

$$\sum_1^\infty \frac{1}{n^2 - n + B} \approx \int_{\frac{1}{2}}^\infty \frac{1}{x^2 - x + B}dx = \left[\frac{1}{\sqrt{B - \frac{1}{4}}} \tan^{-1} \frac{x - \frac{1}{2}}{\sqrt{B - \frac{1}{4}}}\right]_{\frac{1}{2}}^\infty = \frac{\pi}{\sqrt{4B - 1}}.$$

By Case 2 with $a = 0$, the exact value of the sum is

$$\sum_1^\infty \frac{1}{n^2 - n + B} = \frac{\pi}{\sqrt{4B - 1}} \tanh \pi\frac{\sqrt{4B - 1}}{2}.$$

So the error in using the integral approximation is

$$\frac{\pi}{\sqrt{4B - 1}}\left[1 - \tanh \pi\frac{\sqrt{4B - 1}}{2}\right] < \frac{2\pi}{\sqrt{4B - 1}}e^{-\pi\sqrt{4B - 1}},$$

since $1 - \tanh x = \dfrac{2}{e^{2x} + 1} < 2e^{-2x}$. This means that the approximation can be amazingly accurate: for $B = 10$, the latter bound is 3.0×10^{-9} while for $B = 100$ it is 1.8×10^{-28}.

I claim no great originality for these results but thought that the short proofs via the infinite products for sin and cos might be of interest. It seems quite hard to locate these sums in these specific forms in the literature, although a similar method is hinted at for an exercise in [2]. This is almost certainly because there is an alternative method (contour integration) that gives a uniform method for all cases of the doubly infinite sum $\displaystyle\sum_{-\infty}^{\infty} \frac{1}{n^2 + An + B}$ for real numbers A, B. Integration of the function $g(z) = \dfrac{\pi \cot \pi z}{z^2 + Az + B}$ over large square contours gives $\displaystyle\sum_{-\infty}^{\infty} \frac{1}{n^2 + An + B}$ in terms of the residues of $g(z)$ at the roots of $z^2 + Az + B = 0$, [3]. Writing $n^2 + An + b = (n + a)^2 - \Delta$ as before and evaluating the residues gives

$$\sum_{-\infty}^{\infty} \frac{1}{(n + a)^2 - \Delta} = -\frac{\pi}{2\sqrt{\Delta}} \left[\cot \pi \left(a + \sqrt{\Delta}\right) + \cot \pi \left(-a + \sqrt{\Delta}\right)\right]$$
$$= \frac{\pi}{\sqrt{\Delta}} \cdot \frac{\sin 2\pi\sqrt{\Delta}}{\cos 2\pi\sqrt{\Delta} - \cos 2a\pi}. \qquad (*)$$

if $\Delta > 0$, with $\sin 2\pi\sqrt{\Delta}$, $\cos 2\pi\sqrt{\Delta}$ being replaced by their hyperbolic counterparts if $\Delta < 0$.

In the case that A, B are integers the sum we originally considered, $\displaystyle\sum_{1}^{\infty} \frac{1}{n^2 + An + B}$, can be extracted from $\displaystyle\sum_{-\infty}^{\infty} \frac{1}{n^2 + An + B}$ using the observation that, if $f(n) = n^2 + An + B$, then $f(-n) = n^2 - An + B = f(n - A)$.

Using this, when $A \geqslant 0$,

$$\sum_{-\infty}^{\infty} \frac{1}{f(n)} = \sum_{1}^{\infty} \frac{1}{f(n)} + \sum_{0}^{\infty} \frac{1}{f(n - A)} = 2\sum_{1}^{\infty} \frac{1}{f(n)} + \sum_{0}^{A} \frac{1}{f(n - A)},$$

with a similar splitting, $\displaystyle 2\sum_{1}^{\infty} \frac{1}{f(n)} - \sum_{1}^{-A-1} \frac{1}{f(n)}$, if $A < 0$.

We can reconcile (*) with our earlier formulae by noting that in Case 1, the right-hand side of (*) is

$$\frac{\pi}{\sqrt{\Delta}} \cdot \frac{\sin 2\pi\sqrt{\Delta}}{\cos 2\pi\sqrt{\Delta} - 1} = -\frac{\pi \cot \pi}{\sqrt{\Delta}},$$

while in Case 2 it is

$$\frac{\pi}{\sqrt{\Delta}} \cdot \frac{\sin 2\pi\sqrt{\Delta}}{\cos 2\pi\sqrt{\Delta} + 1} = \frac{\pi \tan \pi}{\sqrt{\Delta}}.$$

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Nick Lord : An appreciation

Nick Lord, who died after a short illness on 1st March 2024, was both a fine mathematician and a first-rate teacher. These qualities are not always combined, but Nick believed that there was no point in doing mathematics if you could not communicate your ideas enthusiastically to other people, and, in particular, to school pupils. His belief was reinforced by an excellent series of Teaching Notes in *The Mathematical Gazette*, whose objective was to suggest ways to bring depth and enthusiasm into the ordinary mathematics syllabus.

Part of this involved exploring connections between different aspects of the subject. If you look at his three contributions in the Teaching Notes to the March 2024 issue of the *Gazette*, you will see how he achieves this. The first takes the form of a imaginary dialogue between the author and a "Devil's Advocate" as to the reasons why you might be interested in high-order approximations to irrational numbers such as $\sqrt{2}$. The second is a cautionary tale about polar coordinates and the need to be careful about ambiguous endpoints of a curve. The third, inspired by plenary talk at MA Conference, discusses problem-solving strategies and the unexpected occurrence of the golden ratio in an unlikely context. The fecundity of ideas explored in these three short pieces is absolutely typical of the way Nick thought mathematics should be communicated.

Nick also contributed many longer pieces to the *Gazette* which went beyond the level of school mathematics, but these are also remarkable for his insistence in communicating ideas elegantly. Their subjects range from abstract algebra to analysis and from geometrical inequalities to number theory. He will be particularly remembered for his 20 years in charge of the Problem Corner, during which time he supervised the selection of some 250 tantalising challenges for readers of the journal to tackle. These covered topics from combinatorics, conics and constructions to polynomials, probability and packing. His comments on the solutions were both rigorous and imaginative, and he was always keen to emphasise that a published solution is never the final word – there is always something new and unexpected to develop.