Journal of the History of Economic Thought,
Volume 45, Number 3, September 2023

LETTER TO THE EDITORS: SOLVING VINCENT CARRET'S PUZZLE: A REBUTTAL OF CARRET'S FALLACIES AND ERRORS

BY
JEAN-MARC GINOUX
AND
FRANCK JOVANOVIC

In his article published in JHET *in 2022, Vincent Carret (2022a) criticizes our work. In footnote 19, pages 630–631, he claims that our result "is based on a mistaken interpretation of the paragraph at the bottom of p. 191 of Frisch (1933)." He then states that we "take to mean that the coefficient of each cycle in the general sum of solutions is arbitrary, while … these coefficients [depended] on initial conditions and the parameters of the system." The present rejoinder aims at rebutting Carret's allegation of mistaken interpretation in our work. We demonstrate that his statements are based on a misunderstanding of Frisch's econometric model and approach. Then, we show that Carret's results are not supported by the demonstration he claims to have made, and that he misrepresents our arguments.*

I. CARRET'S HYPOTHESES HAVE NO ECONOMIC JUSTIFICATION

In his criticism of our work, Vincent Carret (2022a) states that we made a mistaken interpretation of Ragnar Frisch's paragraph in which he expressed a condition that must be respected in order to obtain the general solution. Frisch called it a "proviso." According to the *Cambridge Dictionary*, a proviso is "a statement in an agreement, saying that a particular thing must happen before another can." In modern terms, Frisch's proviso is a closure relation, according to which the sum of the coefficients k_j, which are the weight of each cycle, must be equal to unity. With this closure relation, Frisch normalized the weight of each cycle.

Jean-Marc Ginoux: Centre de Physique Théorique, Université de Toulon. Email: ginoux@univ-tln.fr. Franck Jovanovic: School of Business Administration, TELUQ University. Email: franck.jovanovic@teluq.ca. We wish to acknowledge the financial support of this research provided by TELUQ University (Fonds d'aide à la recherche).

ISSN 1053-8372 print; ISSN 1469-9656 online/23/03000503-510 © The Author(s), 2023. Published by Cambridge University Press on behalf of the History of Economics Society.
doi:10.1017/S1053837223000068

https://doi.org/10.1017/S1053837223000068 Published online by Cambridge University Press

We recently demonstrated that this closure relation must be considered in Frisch's work; a point that Lionello Punzo (2022) confirmed in a symposium published in *History of Economic Ideas* following our demonstration that Stefano Zambelli's assertion is based on a mathematical error, and consequently does not hold (Ginoux and Jovanovic 2022a). Punzo (2022) also validated our result by claiming that Zambelli failed to prove that "the rocking horse does not rock." Consequently, Zambelli's assertion is pure speculation, without any mathematical or numerical demonstration. Taking Zambelli's work as a starting point, Carret reproduced the same errors. Moreover, since the value of our coefficients respect Frisch's closure relation, they necessarily respect Frisch's initial conditions and cannot be arbitrary (Ginoux and Jovanovic 2022b, pp. 180–181), a fact that Carret fails to understand, as shown by his criticism in his *JHET* article.

We would like to address here another fundamental problem that Carret avoids answering in his criticism of our work: Why did he ignore this closing relationship in his publications? Indeed, he has never explained his choice to ignore this condition that must be respected by definition. This question is crucial since Frisch used this condition for some economic reasons, as we will clarify here.

In his model, Carret does not normalize the weight of each cycle, as Frisch did; on the contrary, Carret assumes that all coefficients k_j have the same value, namely 1, and then that $k_1 = k_2 = k_3 = \ldots = 1$. Does Carret's hypothesis make sense from an economic viewpoint? With his model, Frisch reproduced two cycles observed in the literature at his time, i.e., the primary cycle of 8.57 years and the secondary cycle of 3.50 years (the tertiary cycle of 2.20 years is a prediction made by Frisch). The closure relation implies that Frisch considered that these different cycles (such as Kitchin and Juglar cycles), which do not have the same origin according to the economic theories, do not impact the economic activity with the same amplitude. Carret, however, states that these different cycles impact the economic activity with the same amplitude. Unfortunately, there is no economic justification for Carret's hypothesis, as explained, for instance, by Joseph Schumpeter (1939) and more recently by Muriel Dal Pont Legrand and Harald Hagemann (2007, p. 12).

This is not the only fundamental problem that Carret avoids answering. What business cycle did Carret try to replicate with his model? Because Carret (2022a) ignored Frisch's closure relation, which ensures that Frisch's system oscillates, he was obliged to change the value of the parameters to obtain oscillations (Ginoux and Jovanovic 2022b). The problem here is that Carret dismisses the consequences of his choice too easily. Indeed, Carret (2022a, p. 632) obtained "a primary cycle with a period of about 6.5 years, a secondary cycle with a period of about 3.2 years … all values rather close to those in Frisch's article." In Carret's view, such differences do not represent an issue: "do[es] not think that it necessarily is [a problem]" (2022a, p. 633).

Of course this is a problem. Contrary to what Carret claims, the period of his primary cycle (6.5 years) is not "rather close to" the cycle observed in the literature (an 8.5-year cycle) and which was reproduced by Frisch. An 8.5-year cycle is very different from a 6.5-year cycle: over twenty years, we will have two cycles, and therefore two economic recessions in one case, and three in the other. That is very different. Moreover, a 6.5-year cycle does not correspond to any known cycle in history of economics (Juglar cycles are eight to ten years).

Then, Carret (2022a, p. 633) admits that "[t]here is, however, one caveat, compared with Frisch's original article: in order to obtain apparent cycles at the aggregate level, we

https://doi.org/10.1017/S1053837223000068 Published online by Cambridge University Press

Table 1. Comparison between the Parameters Used

Carret (2022a)	λ = 0.3	r = 1	s = 2	m = 1	μ = 15	ε = 6	c = 0.165**
Frisch (1933)	λ = 0.05	r = 2	s = 1	m = 0.5	μ = 10	ε = 6	c = 0.165
Ginoux and Jovanovic (2023)	λ = 0.05	r = 2	s = 1	m = 0.5	μ = 10	ε = 6	c = 0.165

**: Carret did not provide the value he used for c in his demonstrations. However, we were able to calculate it based on the value he used for a_*. By using Frisch's characteristic equations (1933, p. 184, eq. 12 and 13) and Carret's parameters, we found $a_* = 0.1833$. This value is consistent with what we observe on Carret's fig. 1 (2022a, p. 632). Then, since Frisch (1933, p. 188, eq. 17) stated $c = \lambda a_*(r + sm)$, we have $c = 0.3*0.1833*(1+2*1) = 0.165$.

had to decrease the damping of the system. In fact, the return to equilibrium is much longer than in Frisch's original article."

Indeed, Carret's "damping exponent" for the first cycle is 8.5 times lower than Frisch's.[1] This means that Carret's general solution takes 8.5 times more time to damp, as seen in Carret's figures 1 and 2 (see the horizontal axis, which extends for a century). What economic significance should be made of an econometric model that takes 100 years to return to the stationary level? Did we observe such behavior in Frisch's time? Not at all.

To make his general solution oscillate for a while, Carret tuned the values of Frisch's parameters, and left out elements of Frisch's demonstration when it did not suit him. For instance, his parameter λ is six times higher than Frisch's in order to increase the importance of the *transient regime* until reaching the stationary level (horizontal asymptote at 0.18 in Carret's figures 1 and 2), which obviously does not oscillate around it. In fact, Carret worked on a solution of Frisch's model that is different from the original one. As we can see in Table 1, all of Carret's parameters are different from Frisch's, except ε and c.

In his work, Carret does not make it possible to reproduce the observed cycles that Frisch sought to reproduce. Therefore, he ignores the problem that Frisch was faced with. Moreover, because his model does not have the same economic behavior as Frisch's, his results cannot be directly compared with those of Frisch. Carret has not explained why he ignored Frisch's closure condition (or closure relation). By ignoring its role, Carret's results have no merit in either mathematics or economics.

II. CARRET IGNORED A WELL-KNOWN THEOREM

In footnote 19, to criticize our work, Carret refers to another article (2022b) in which he claimed to have demonstrated our mistake. Specifically, Carret (2022b, p. 168) states that Frisch's paragraph on page 191 refers to the resolution of a system with a homogenous part and a non-homogeneous part. He clarifies by stating that "[a] combination of a homogeneous solution [i.e., Frisch's eq. 18] and a particular

[1] See also Ginoux and Jovanovic (2022b, p. 179, table 1).

https://doi.org/10.1017/S1053837223000068 Published online by Cambridge University Press

solution [i.e., a_*, b_* and c_*] will thus solve the system," and "the functions in [Frisch's eq.] (16) solved [the non-homogeneous system] because they were composed of the sum of the particular solutions a_*, b_* and c_* and the trends which solved respectively the nonhomogeneous and the homogeneous parts of the system."

Then, to explain Frisch's proviso, Carret claims that

> [w]hat Frisch was saying was that, when adding two of the solutions …, for instance x_0 [$x_0 = a_* + a_0 e^{\rho_0 t}$] and g_1 [$g_1 = a_* + A_1 e^{-\beta_1 t}\sin(\phi_1 + \alpha_1 t)$], we should be careful to end up with only one particular solution a_*.
>
> Suppose we do this addition; using arbitrary coefficients r_0 and r_1, we obtain
>
> $$r_0 x_0 = r_1 x_1 = a_*(r_0 + r_1) + r_0 a_0 e^{\rho_0 t} + r_1 A_1 e^{-\beta_1 t}\sin(\phi_1 + \alpha_1 t).$$
>
> It clearly appears that the coefficients r_0 and r_1 must sum to 1 so that we obtain only one particular solution a_* solving the nonhomogeneous part of the system.

Obviously, Carret is referring indirectly to the Superposition Theorem.

Unfortunately, Carret has misunderstood this theorem and applied it poorly. Indeed, this theorem is "an *existence and a uniqueness theorem*" (Tenenbaum and Pollard 1985, p. 208). It states that we must look:

– first, for a *particular solution* to the non-homogeneous equation for $c \neq 0$;
– second, for a *general solution* to the homogeneous equation for $c = 0$, which is given by a linear combination of linearly independent solutions;
– and third, for the general solution of the non-homogeneous equation, which will be equal to the sum of the *particular* and *general* solutions (Tenenbaum and Pollard 1985, p. 208; Warusfel 1966, p. 138).

Given that a_* is a *particular solution*, and that $a_0 e^{\rho_0 t}$ and $A_1 e^{-\beta_1 t}\sin(\phi_1 + \alpha_1 t)$ are *general solutions*, Carret clearly did not respect the theorem. Indeed, he made a linear combination of a mixture of both *particular* and *general* solutions that led him to count the *particular solution* twice, which we never did. The superposition theorem requires precisely that the *particular solution* be counted only once.

In fact, Carret (2022a, p. 631, eq. 5) should have written the *general solution* of Frisch's system as:

$$x(t) = \underbrace{a_*}_{(P)} + \underbrace{x_0^H(t) + x_C^H(t)}_{(H)} = \underbrace{a_* + a_0 e^{-\beta_0 t}}_{(T)} + \underbrace{\sum_{j=1}^{\infty} k_j A_j e^{-\beta_j t}\sin\left(\phi_j + \alpha_j t\right)}_{(C)} \tag{1}$$

Where (P) represents the particular solution, (H) the homogeneous solution, (T) the trend, and (C) the cyclic components;[2] a_* is the *particular solution* while $x_0^H(t) = a_0 e^{\rho_0 t} = a_0 e^{-\beta_0 t}$ and $x_C^H(t) = \sum_{j=1}^{\infty} k_j A_j e^{-\beta_j t}\sin\left(\phi_j + \alpha_j t\right)$ are the *general solutions*

[2] Note that Frisch (1933, p. 188, eq. 16) could write the trend like this because the characteristic exponent of $a_0 e^{\rho_0 t}$ has no imaginary part.

https://doi.org/10.1017/S1053837223000068 Published online by Cambridge University Press

of the homogeneous equation. This latter is a linear combination of Frisch's *cyclical components.*

By claiming that we have misinterpreted Frisch's paragraph, Carret attempts to draw readers into a false debate. Careful readers will have noted that in his criticism, Carret (2022b, p. 168) did not discuss the fact that "the sum of the coefficients [must be] equal to unity" in order to add the constant terms a_*, b_*, and c_* "to (18) in order to get a correct solution." We suspect that Carret did not want to discuss the coefficients k_j, because this would lead him to question the superposition theorem and consequently the existence of a general solution in Frisch's model; especially since the superposition theorem applies even when using the Laplace transform. Therefore, it is no coincidence that Carret criticizes our work in this paragraph: as soon as we respect Frisch's closure relation, which follows from the superposition theorem, Carret's whole demonstration collapses and his "results" on Frisch's model would have to be abandoned.

III. SOLVING CARRET'S PUZZLE

Now let us solve Carret's puzzle about the error he purports to have demonstrated in Frisch's work. As Carret has rightly mentioned in all of his publications, Frisch gave different initial conditions for the trends and the cycles. Carret claims that this was a serious error: "Frisch erred, because he gave different initial conditions for the trend and for the cycles, even though they should depend on the same initial development" (2022b, p. 165).

This problem is crucial for Carret, since he has built his argumentation on this element and presents it as one of his major contributions: "With the hindsight of a more complete theory of mixed differential–difference equations, we can show analytically by using the Laplace transform that all components, whether cyclical or oscillatory, will depend on the same initial conditions" (Assous and Carret 2022, p. 67).

In fact, there is no error in Frisch's work. In the general case, Carret is correct: we should have the same initial condition for the trend and the cyclical components. But Frisch managed to find a particular case that works. Punzo (2022, p. 174) also pointed this out, leading him to claim that it is "almost an honor" for Frisch to "find one such constellation," i.e., the three cyclical components and their periods. Carret has missed this in all of his publications, including his *JHET* article (Carret 2022a, pp. 634, 637), overlooking the originality of Frisch's work. Indeed, this "honor" results from Frisch's calibration with his econometric model (Ginoux and Jovanovic 2022b, p. 179).

This result demonstrates that Carret attempts to make the reader believe that he has shown something that is not in Frisch's writings, but this is not true.

IV. CARRET'S DEMONSTRATION IS INCOMPLETE AND UNVERIFIABLE

Carret's criticism hides a major methodological difference between our work and his: Carret does not give the information needed to reproduce his work; we do. Therefore, anyone can verify our conclusions, but no one can verify Carret's work. For proof, in footnote 19 of his *JHET* article (2022a, p. 630), Carret criticizes our work by invoking his "solution based on the Laplace transform" and its inverse. However, he does not

https://doi.org/10.1017/S1053837223000068 Published online by Cambridge University Press

provide the demonstration of his analytical solution. He explains that "[the] full derivation of this solution is published in Assous and Carret (2022)."

Unfortunately, Michaël Assous and Carret (2022) did not provide all the calculations of the inverse Laplace transform they used. Carret's work is therefore unverifiable. This is embarrassing, because the Laplace transform and its inverse underpin all Carret's analysis and arguments. It is on this basis that he could claim that Frisch "erred," or made "an error," or that he "can give a more elegant answer than Frisch" (Carret 2022a, p. 630). Moreover, according to Carret, the Laplace transform and its inverse are "modern mathematical tools that [Frisch] did not know" (2022a, p. 624). This is an astonishing claim to make, given that the Laplace transform was introduced in 1737, that the first use of its modern formulation dates back to 1910, and that in "the 1920s and 1930s it was seen as a topic of front-line research" (Deakin 1992, p. 265).

This is not our only issue with Carret's "solution based on the Laplace transform." As Roy Allen (1959, pp. 155–156) explained, the Laplace transform is a "trick" of mathematicians. One of the main problems with this trick is that when we use the Laplace transform and its inverse, we automatically introduce new constants (i.e., new initial conditions). Thus, "when the solution is obtained, it has the initial conditions 'built in,'" and "*n* arbitrary constants, to be 'fitted' or evaluated with great labor from the initial conditions" (Allen 1959, p. 159). In other words, to solve a system similar to Frisch's with a Laplace transform and its inverse, we have to generate at least one or two new initial conditions that are arbitrary by construction.

It is surprising, even shocking, that Carret keeps silent about this problem in his work, while he constantly criticizes Frisch on the value of his initial conditions. More vexatious is the fact that in none of his publications does Carret provide the value for his initial condition(s), including the new ones he introduced with the Laplace transform. Specifically, since he did not provide the expression of $x(t)$ for $0 \leq t \leq \varepsilon$, we cannot compute the values of his A_i and k_i used in his equation $A_i = 2|k_i|$. It is legitimate to ask why he does not bother to provide the initial conditions he used in his work. By not providing his initial condition(s) and all of his calculations for his inverse Laplace transform, Carret ensures that no one can reproduce and thus verify his work, as is expected in a scientific work.

V. CONCLUSION

Carret pretends that "[we] have suggested another approach [than his] to exhibit fluctuations in Frisch's propagation mechanism" (2022a, p. 630n19). This statement is fallacious. Unlike Carret, we worked within Frisch's framework, and strictly followed Frisch's demonstration step by step without introducing any new mathematical tools or economic reasoning, as Carret does. In so doing, we proved that Frisch's model fluctuates with its original values (Ginoux and Jovanovic 2023, 2022a, 2022b). Indeed, while Frisch did not provide the value of the coefficients k_j (because he did not give an explicit general solution of his model), we provide a general solution by choosing a set of coefficients that respect Frisch's closure relation and for which his propagation model fluctuates.[3]

[3] Note that our article (Ginoux and Jovanovic 2023) contains a typographical error corrected by an erratum (Ginoux and Jovanovic 2022c).

https://doi.org/10.1017/S1053837223000068 Published online by Cambridge University Press

Therefore, in the light of our recently published demonstration, Carret's attempt to modify the Frisch parameters to show that his model can oscillate seems pointless, since Frisch's model fluctuates with its original parameters.

By putting forward economic arguments that have the appearance of being based on mathematical analysis, Carret claims several things about our work, as well as Frisch's, that are simply false. Moreover, his criticisms are based on fallacious arguments that will mislead economists who are not familiar enough with the mathematics used in his work. By ignoring Frisch's closure relation, Carret replicated Zambelli's error and continues to spread the same baseless arguments (Carret 2022a, pp. 630–632). By changing Frisch's parameters in his publications, Carret introduced additional new puzzles and additional economic errors, ignoring the relevance of the econometric approach defended by Frisch. Moreover, Carret asserted things that are false and did not give all the information needed to verify his results.

COMPETING INTERESTS

The author declares no competing interests exist.

REFERENCES

Allen, Roy George Douglas. 1959. *Mathematical Economics*. Second edition. London and Basingstoke: ELBS and The Macmillan Press.

Assous, Michaël, and Vincent Carret. 2022. *Modeling Economic Instability. A History of Early Macroeconomics*. Springer Studies in the History of Economic Thought. Cham: Springer.

Carret, Vincent. 2022a. "Fluctuations and Growth in Ragnar Frisch's Rocking Horse Model." *Journal of the History of Economic Thought* 44 (4): 622–641.

———. 2022b. "A Comment on Ginoux and Jovanovic, 'The Rocking Horse Model Does Rock.'" *History of Economic Ideas* 30 (2): 165–171.

Dal Pont Legrand, Muriel, and Harald Hagemann. 2007. "Business Cycles in Juglar and Schumpeter." *History of Economic Thought* 49 (1): 1–18.

Deakin, Michael A. B. 1992. "The Ascendancy of the Laplace Transform and How It Came About." *Archive for History of Exact Sciences* 44 (3): 265–286.

Frisch, Ragnar. 1933. "Propagation Problems and Impulse Problems in Dynamic Economics." In *Economic Essays in Honour of Gustav Cassel*. London: George Allen and Unwin Ltd., pp. 171–205.

Ginoux, Jean-Marc, and Franck Jovanovic. 2022a. "The 'Rocking Horse Model Is Rocking': Solving Zambelli's Puzzle." *History of Economic Ideas* 30 (2): 139–158.

———. 2022b. "The 'Rocking Horse Model Is Rocking': A Rejoinder." *History of Economic Ideas* 30 (2): 175–182.

———. 2022c. "Correction to Frisch's Propagation-Impulse Model: A Comprehensive Mathematical Analysis." *Foundations of Science*. https://doi.org/10.1007/s10699-022-09895-5. Accessed April 5, 2023.

———. 2023. "Frisch's Propagation-Impulse Model: A Comprehensive Mathematical Analysis." *Foundations of Science* 28: 57–84. https://doi.org/10.1007/s10699-021-09827-9. Accessed April 28, 2023.

Punzo, Lionello. 2022. "A Comment on Ginoux and Jovanovic, 'The Rocking Horse Model Does Rock.'" *History of Economic Ideas* 30 (2): 173–174.

https://doi.org/10.1017/S1053837223000068 Published online by Cambridge University Press

Schumpeter, Joseph Alois. 1939. *Business Cycles: A Theoretical, Historical and Statistical Analysis of the Capitalist Process*. New York: McGraw-Hill.

Tenenbaum, Morris, and Harry Pollard. 1985. *Ordinary Differential Equations*. New York: Dover Publications.

Warusfel, André. 1966. *Dictionnaire raisonné de mathématiques*. Paris: Seuil.

https://doi.org/10.1017/S1053837223000068 Published online by Cambridge University Press