SOME REMARKS ON RAMSAY'S THEOREM

P. Erdős

A special case of a well known theorem of Ramsay [3] states that an infinite graph either contains an infinite complete subgraph or it contains an infinite independent set; in other words there exists an infinite subset of its vertices so that either every two of them are joined by an edge or no two of them are joined by an edge. Thus if we have a graph whose vertices are the integers, and which has no infinite complete sub-graph, it certainly has an infinite independent set. The question can now be asked if there exists an independent set whose vertices $n_1 < n_2 < \dots$ do not tend to infinity too fast. It is quite clear however, that no such theorem can hold. To see this let $1 = m_1 < m_2 < \dots$ be a sequence of integers tending to infinity sufficiently fast. Two integers u and v are joined in our graph G if and only if for some i, $m_i \le u < v < m_{i+1}$. Clearly G contains no infinite complete subgraph (in fact it contains no infinite connected subgraph), but every infinite independent sequence $n_1 < n_2 < \dots$ satisfies $n_i \ge m_i$ for all i.

On the other hand the following simple remark is perhaps not entirely without interest.

Let G be a graph whose vertices are the integers and which contains no triangle. Then there exists an infinite independent sequence $n_1 < n_2 < \dots$ so that

$$(1) \qquad n_k < (1 + o(1)) \frac{k^2}{2}$$

holds for infinitely many k.

Before we prove (1) we remark that it can not be strengthened to a statement like: $n_k \le f(k)$ holds for all k,

https://doi.org/10.4153/CMB-1964-059-6 Published online by Cambridge University Press

where $f(x)$ is an arbitrary function (increasing as fast as we please). To see this, let to each i correspond an interval (x_i, y_i) where

$$(2) \qquad f(i) < x_i, \; f(x_i) < y_i, \; y_i < x_{i+1} .$$

The intervals (x_i, y_i) clearly do not overlap. Let further,

$$(3) \qquad (a_k, b_k), \; f(a_k) = b_k, a_{k+1} > y_{b_k}, \quad k = 1, 2, \ldots$$

be a sequence of non-overlapping intervals. The vertices i and j are joined if and only if for some k

$$(4) \qquad a_k < i < b_k, \; x_i < j < y_i .$$

(2), (3) and (4) imply that G contains no triangle; in fact it contains no path of length three, and no independent sequence satisfies $n_k < f(k)$ for all k.

Now we prove (1). Put $A(x) = \sum_{n_i \leq x} 1$. (1) then means that there is an independent sequence for which

$$(5) \qquad A(x) > (1 + o(1)) \sqrt{2x}$$

holds for infinitely many x. To prove (5) denote by $S(i)$ the set of vertices joined to i, and by $N_x(i)$ the number of elements of $S(i)$ which are $\leq x$. Clearly $S(i)$ is an independent set (for otherwise G contains a triangle). Thus, if for some i and infinitely many x

$$N_x(i) > x^{3/4} ,$$

then (5) is clearly satisfied. Thus we can assume that for all i there is an $x_o(i)$ so that for $x < x_o(i)$

$$(6) \qquad N_x(i) \leq x^{3/4} .$$

https://doi.org/10.4153/CMB-1964-059-6 Published online by Cambridge University Press

Let now $m_1 < m_2 < \ldots$ satisfy

$$(7) \qquad m_{k+1} > 2^{m_k}, \quad m_{k+1} > \max_{j \le m_k} x_o(j) .$$

Denote by G_k the graph spanned by those vertices $m_k < j \le m_{k+1}$ which are not joined to any $i \le m_k$. By (6) and (7) G_k has at least

$$m_{k+1} - m_k\, m_{k+1}^{3/4} = (1+o(1))m_{k+1}$$

vertices. By a well known theorem [1] G_k contains an independent set having at least $(1+o(1))\sqrt{2m_{k+1}}$ elements and, since clearly any two vertices of G_{k_1} and G_{k_2} are independent, we obtain an independent sequence which satisfies (5) for $x = m_k$, $k \to \infty$; hence (1) is proved.

Using the methods of [2] we can show that there is a graph G which contains no triangle and every independent sequence $n_1 < n_2 < \ldots$ satisfies for every $\epsilon > 0$ and $k > k_o(\epsilon)$

$$n_k > k^{2-\epsilon} .$$

Thus (1) can not be improved very much.

By the same method we could easily prove that if G does not contain a complete k-gon then it has an independent sequence $n_1 < n_2 < \ldots$ satisfying

$$A(x) > c_k x^{1/k-1}$$

for infinitely many x. The general theorem of Ramsay states that if in an infinite set S we have a system of k-tuples so that there is no infinite $S_1 \subset S$ all whose k-tuples are in our system,

https://doi.org/10.4153/CMB-1964-059-6 Published online by Cambridge University Press

then there is an infinite independent S_2, in other words an infinite $S_2 \subset S$ which does not contain any of our k-tuples.

Here the following result holds, the simple proof of which we leave to the reader. If a system of k-tuples of integers is given, no two of which have a common (k-1)-tuple, then there is an independent sequence $n_1 < n_2 < \ldots$ satisfying $n_t \leq t^{k-1}$ for all t; on the other hand if we only require that every (k+1)-tuple contains at most two k-tuples no similar theorem can hold. Results of the type (1) might hold, but I have not investigated this.

REFERENCES

1. P. Erdős and G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2(1935), 463-470.

2. P. Erdős, Graph theory and probability II, Can. J. Math., 13 (1961), 346-352.

3. F. P. Ramsay, On a problem of formal logic, Proc. London Math. Soc., 30 (1929), 264-286.

McGill University

https://doi.org/10.4153/CMB-1964-059-6 Published online by Cambridge University Press