SYMMETRIC COORDINATE SPACES AND SYMMETRIC BASES

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1. Introduction. In this paper properties of symmetric coordinate spaces and symmetric bases are investigated. Since a space which possesses a basis is essentially a space of sequences (**12**, p. 207), the interrelation of these two concepts naturally suggests itself.

Section 2 is a summary of the terminology and methods employed, which fall into four categories: (1) set theoretical properties of coordinate spaces such as symmetry and dual spaces; (2) the notion of FK and BK space (**12**, p. 202; **13**); (3) the theory of the Schauder basis in F-space applied to the case when $\mathfrak{E}$ (see § 2) is a basis for a coordinate space; (4) the concept of a sequential norm, which the author introduced in (**7**) to illustrate the underlying unity of the first three ideas.

In § 3 we examine a class of spaces which might be regarded as archetypal perfect symmetric spaces. We note that these spaces were introduced from a somewhat different viewpoint by W. L. C. Sargent in (**8**) and further studied by her in (**9**).

Some properties of a perfect symmetric space are discussed in § 4. The chief result here is Theorem 4.5, which states that every perfect symmetric BK space can be given an equivalent norm having a certain form.

In the final section we apply our work to a new proof of a result of Singer (**11**) concerning symmetric bases.

2. Preliminary observations. A coordinate space is a linear space, X, of scalar (real or complex) sequences with addition and scalar multiplication defined coordinatewise. We designate by e^i the ith coordinate vector, i.e. the sequence with 1 in the ith place and 0's elsewhere. The set $\{e^i: i = 1, 2, \ldots\}$ is denoted by $\mathfrak{E}$. Unless we specify otherwise we assume that $\mathfrak{E} \subseteq X$. The sequence whose ith term is t_i is written (t_i) or merely t.

The following concepts are of long standing (**6**; **5**, p. 427).

Definition 2.1. For X a coordinate space:

(a) X^α, the *α-dual* of X, is

$$\left\{y: \sum_{i=1}^{\infty} |x_i y_i| < \infty \text{ for each } x \in X\right\}.$$

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(b) X^β, the *β-dual* of X, is

$$\left\{y:\sum_{i=1}^{\infty} x_i y_i \text{ converges for each } x \in X\right\}.$$

(c) X is *perfect* if $X^{\alpha\alpha} = X$.

(d) X is *balanced* (or normal) if $x \in X$ implies that $(a_i x_i) \in X$ for each $a \in m$, the space of all bounded sequences.

(e) X is *symmetric* if $x \in X$ implies that $x^\pi \in X$ for each permutation π on the non-negative integers where $(x_i)^\pi = (x_{\pi(i)})$.

The results summarized in the following proposition are widely known **(3; 5; 6)**.

PROPOSITION 2.2. (a) *X^α is balanced and perfect for every coordinate space X.*

(b) *If X is symmetric, X^α is symmetric.*

(c) *If X is perfect and symmetric, then $X = R^\infty$, the space of all finite sequences, or $X = s$, the space of all sequences or $l^1 \subseteq X \subseteq m$.*

Definition 2.3. The *symmetric dual*, X^σ, of a coordinate space X is

$$\left\{y:\sum_{i=1}^{\infty} |y_i x_{\pi(i)}| < \infty \text{ for each } x \in X \text{ and each permutation } \pi \text{ on the positive integers}\right\}.$$

PROPOSITION 2.4. *If X is symmetric, $X^\sigma = X^\alpha$.*

Proof. For π a permutation on the positive integers, let $X_\pi = \{x^\pi: x \in X\}$. Then $X^\sigma = \bigcap\{X_\pi{}^\alpha: \text{all } \pi\}$, but if X is symmetric, $X_\pi = X$ for each π, so $X^\sigma = X^\alpha$.

Definition 2.5. A coordinate space X is an FK space if X is an F-space (complete linear metric space) and the linear functionals defined by $f_i(x) = x_i$ are continuous.

An FK space which is a Banach space is called a BK space **(12**, § 11.3**)**.

Definition 2.6. A *sequential norm* is a function, N, from s into R^* which satisfies the following conditions:

(1) N is an extended norm, i.e.,

(a) $N(x + y) \leqslant N(x) + N(y)$,

(b) $N(ax) = |a|N(x)$ for each scalar a,

(c) $N(x) \geqslant 0$, $N(x) = 0$ if and only if $x = 0$.

(2) $N(e_i) < \infty$ for each i.

(3) $N(x) = \sup_n N(P_n x)$, where

$$P_n x = \sum_{i=1}^{n} x_i e_i.$$

If in addition N satisfies

(4) $0 < \inf_n N(e_n) \leqslant \sup_n N(e_n) < \infty$,

N is a *proper sequential norm.*

For a coordinate space X on which a topology has been fixed we shall write X^0 for the closed linear span of $\mathfrak{E}$ in X.

The concept of a proper sequential norm (p.s.n.) was introduced and studied in **(7)**. If N is a p.s.n., the set S_N of all x for which $N(x) < \infty$ is a BK space with norm N, and $\mathfrak{E}$ is a basis for $S_N{}^0$ **(7**, Theorem 2.2**)**. Conversely, if $\mathfrak{X} = \{x_1, x_2, \ldots\}$ is a basic sequence in a Banach space which is bounded in norm away from 0 and ∞, we can find a p.s.n. N such that $\mathfrak{X}$ is equivalent **(1, 2)** to the basic sequence $\mathfrak{E}$ in S_N; namely, let

$$N(t) = \sup_n \left\| \sum_{i=1}^{n} t_i x_i \right\| .$$

Definition 2.7. The *conjugate* p.s.n. of a p.s.n. N is the function from s into R^* given by

$$N'(y) = \sup \left\{ \sup_n \left| \sum_{i=1}^{n} x_i y_i \right| : N(x) \leqslant 1 \right\} .$$

By **(7**, Theorem 3.2**)**, N' is a p.s.n. and the conjugate space of $S_N{}^0$, $(S_N{}^0)^*$ is isometric to $S_{N'}$ under the correspondence of f in $(S_N{}^0)^*$ to $(f(e^i))$ in $S_{N'}$, and

$$f(x) = \sum_{i=1}^{\infty} x_i f(e^i) \qquad \text{for each } x \text{ in } S_N{}^0.$$

Definition 2.8. A p.s.n., N, is balanced

$$\text{if } N(x) = \sup \{N(a_i, x_i): |a_i| \leqslant 1\} \qquad \text{for each } x \text{ in } s.$$

Definition 2.9. A p.s.n., N, is *symmetric* if $N(x) = N(x^\pi)$ for each permutation π on the positive integers.

For N a p.s.n. we have $S_{N'} = (S_N{}^0)^\beta$ **(7**, Corollary 3.3**)** and if N is a balanced p.s.n., then $S_{N'} = (S_N{}^0)^\alpha$ **(7**, proof of Theorem 4.5**)**.

PROPOSITION 2.10. (a) *If* $\{N_\alpha\}$ *is a family of p.s.n.'s and there is a* $K > 0$ *such that* $\sup_\alpha N_\alpha(e_i) < K$ *for each* i, *then* $\sup_\alpha N_\alpha$ *is a p.s.n.*

(b) *If each* N_α *is balanced (symmetric) and* $\sup_\alpha N_\alpha$ *is a p.s.n., then* $\sup_\alpha N_\alpha$ *is balanced (symmetric).*

Proof. (a) Let $N = \sup_\alpha N_\alpha$ By hypotheses $\sup_i N(e_i) \leqslant K < \infty$ and since N is a sup, $\inf_i N(e_i) > 0$. We shall verify condition (3) of Definition 2.6. The proof of the norm condition is analogous.

$$\begin{aligned} N(x) &= \sup_\alpha N_\alpha(x) = \sup_\alpha \sup_n N_\alpha(P_n x) \\ &= \sup_n \sup_\alpha N_\alpha(P_n x) = \sup_n N(P_n x). \end{aligned}$$

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(b) The proof that N is balanced (symmetric) if each N_α is balanced (symmetric) is also obtained by permuting sup operators.

3. The first and second symmetric duals of a sequence.

Definition 3.1. The *symmetric dual* of a sequence x is

$$x^\sigma = \left\{ y : \sum_{i=1}^{\infty} |x_{\pi(i)}\, y_i| < \infty \text{ for each } \pi \right\}.$$

PROPOSITION 3.2. (a) *$y \in x^\sigma$ if and only if $x \in y^\sigma$.*
(b) *x is unbounded if and only if $x^\sigma = R^\infty$.*
(c) *$x \in R^\infty$ if and only if $x^\sigma = s$.*
(d) *$x \in l^1 \sim R^\infty$ if and only if $x^\sigma = m$.*
(e) *If $x \in m \sim c_0$, $x^\sigma = l^1$.*

Proof. (a) Obvious.

(b) First note that x^σ is perfect and symmetric so that by 2.2(c) x^σ is either s, R^∞, or $l^1 \subseteq x^\sigma \subseteq m$. For every x, $R^\infty \subseteq x^\sigma$. If x is not bounded, then there is a y in l^1 such that $\sum_{c=1}^{\infty} |x_i\, y_i|$ does not converge since $(l^1)^\alpha = m$. This implies that $x^\sigma \subset l^1$ properly, so $x^\sigma = R^\infty$. If $x \in m$, $x^\sigma \supseteq m^\sigma = l^1$, so if $x^\sigma = R^\infty$, x is unbounded.

(c) If $x \in R^\infty$, then $x^\sigma = s$ by (a) and (b). If $x^\sigma = s$, then $x \in y^\sigma$ for each $y \in s$ so by (a) $x \in R^\infty$.

(d) If $x \in l^1 \sim R^\infty$, we have $x^\sigma \subset s$ properly, but $x^\sigma \supseteq (l^1)^\sigma = m$. Thus $x^\sigma = m$. If $x^\sigma = m$, then $\sum_{i=1}^{\infty} |x_i|$ converges since $(1, 1, \ldots)$ is in m.

(e) If $x \in m \sim c_0$, there is a subsequence x' of x such that

$$\inf_n |x'_n| = \epsilon > 0.$$

Then $x^\sigma \subseteq x'^\sigma \subseteq l^1$ and $x^\sigma \supseteq m^\sigma = l^1$.

In the following we shall derive the converse of (e). In view of Proposition 3.2 we shall restrict our attention to the σ-dual of a sequence x which is in c_0 but not in l^1.

Definition 3.3. The *reduced form* of a sequence $x \in c_0$ is the sequence $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots)$ in which $\hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots$ exhaust the non-zero values assumed by $|x_1|, |x_2|, |x_3|, \ldots$ allowing repeated values and $\hat{x}_1 \geqslant \hat{x}_2 \geqslant \hat{x}_3 \geqslant \ldots$.

A sequence x is in reduced form if $x = \hat{x}$.

If $x \in c_0$, then $x^\sigma = \hat{x}^\sigma$ so that whenever we consider the space x^σ we may assume x is in reduced form.

THEOREM 3.4. *Given a sequence x in $c_0 \sim l^1$ define*

$$Q(y) = \sup_\pi \sum_{i=1}^{\infty} |x_{\pi(i)}\, y_i|;$$

then Q is a proper, balanced, symmetric, sequential norm, $S_Q = S_Q{}^0 = x^\sigma$, *and*

$$(1) \qquad Q(y) = \sum_{i=1}^{\infty} \hat{x}_i \hat{y}_i \quad \text{for } y \in c_0, \\ = \infty \quad \text{for } y \notin c_0.$$

Proof. By hypothesis $Q(y) = \sup_\pi N_\pi(y)$, where

$$N_\pi(y) = \sum_{i=1}^{\infty} |x_{\pi(i)} y_i|.$$

Note that N_π is an extended seminorm for each π and has the property that $N_\pi(y) = \sup_n N_\pi(P_n y)$. Thus Q is an extended seminorm and has the property that $Q(y) = \sup_n Q(P_n y)$. It is obvious that Q is a norm and that $Q(e_i) = \sup_n |x_n|$ for each i. Thus Q is a proper sequential norm.

Next we shall verify the equality (1). If $y \notin c_0$, $y^\sigma \subset l^1$ so that $x \notin y^\sigma$, which implies that $Q(y) = \infty$. If $y \in c_0$, then for each n there is a rearrangement $\hat{y}_{\phi(1)}, \hat{y}_{\phi(2)}, \ldots, \hat{y}_{\phi(n)}$ of $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n$ such that $\hat{y}_{\phi(i)} \geqslant |y_{\pi(i)}|$ for $i \leqslant n$, for π a given permutation. Since $\hat{x}_1 \geqslant \hat{x}_2 \geqslant \ldots \geqslant \hat{x}_n$ and $\hat{y}_1 \geqslant \hat{y}_2 \geqslant \ldots \geqslant \hat{y}_n$,

$$\sum_{i=1}^{n} \hat{x}_i \hat{y}_i \geqslant \sum_{i=1}^{n} |x_i \hat{y}_{\phi(i)}| \geqslant \sum_{i=1}^{n} |x_i y_{\pi(i)}|.$$

Therefore,

$$\sum_{i=1}^{\infty} \hat{x}_i \hat{y}_i \geqslant Q(y).$$

On the other hand, for each n there are permutations π and ϕ such that $|y_{\pi(i)}| = \hat{y}_i$ and $|x_{\phi(i)}| = \hat{x}_i$ for $i \leqslant n$. Given $\epsilon > 0$, let n be such that

$$\sum_{i=1}^{n} \hat{x}_i \hat{y}_i + \epsilon > \sum_{i=1}^{\infty} \hat{x}_i \hat{y}_i,$$

and let π and ϕ correspond to this n. Then

$$Q(y) \geqslant \sum_{i=1}^{\infty} |x_i y_{\phi^{-1}\pi(i)}| \geqslant \sum_{i=1}^{n} |x_{\phi(i)} y_{\pi(i)}| \\ \geqslant \sum_{i=1}^{\infty} \hat{x}_i \hat{y}_i + \epsilon.$$

Therefore

$$Q(y) \geqslant \sum_{i=1}^{\infty} \hat{x}_i \hat{y}_i.$$

The validity of (1) implies that Q is balanced and symmetric, and that $S_Q = \hat{x}^\sigma = x^\sigma$. In order to show that $S_Q = S_Q{}^0$ we first prove the following lemma.

LEMMA 3.5. *If Q is a symmetric balanced sequential norm, S_Q and $S_Q{}^0$ are symmetric spaces.*

Proof. Since Q is symmetric, it is obvious that S_Q is symmetric. Since

$$Q\left(\sum_{i=1}^{n} t_i e_i\right) = Q\left(\sum_{i=1}^{n} t_i e_{\pi^{-1}(i)}\right)$$

for each n and each π, $e_{\pi^{-1}(i)}$ is a basis for $S_Q{}^0$ equivalent to $\mathfrak{E}$ so that $\sum_{i=1}^{\infty} t_i e_i$ converges implies that $\sum_{i=1}^{\infty} t_i e_{\pi^{-1}(i)}$ converges. By the biorthogonality of the coefficient functionals

$$\sum_{i=1}^{\infty} t_i e_{\pi^{-1}(i)} = \sum_{i=1}^{\infty} t_{\pi(i)} e_i,$$

so that if $t \in S_Q{}^0$, so is t^π. Therefore, $S_Q{}^0$ is symmetric.

In view of Lemma 3.5 it suffices to prove that for each $y \in S_Q$, $\hat{y} \in S_Q{}^0$. By the definition of $S_Q{}^0$, $\hat{y} \in S_Q{}^0$ if and only if

$$\lim_n Q(\hat{y} - P_n \hat{y}) = \lim_n \sum_{i=n}^{\infty} \hat{x}_i \hat{y}_{n+1} = 0.$$

We conclude the proof by showing that

$$\lim_n \sum_{i=1}^{\infty} \hat{x}_i \hat{y}_{n+i} = 0$$

if $y \in x^\sigma$. Given $\epsilon > 0$, let N_1 be such that

$$\sum_{i=N_1+1}^{\infty} \hat{x}_i \hat{y}_i < \epsilon/2.$$

Then

$$\sum_{i=N_1+1}^{\infty} \hat{x}_i \hat{y}_{n+i} < \sum_{i=N_1+1}^{\infty} \hat{x}_i \hat{y}_i < \epsilon/2 \qquad \text{for each } n.$$

Since $\lim_n \hat{y}_n = 0$, let N_2 be such that $n \geqslant N_2$ implies that

$$y_n < \epsilon \Big/ \left(2\sum_{i=1}^{N_1} \hat{x}_i\right).$$

If $n \geqslant N_2$

$$\sum_{i=1}^{\infty} \hat{x}_i \hat{y}_{n+i} = \sum_{i=1}^{N_1} \hat{x}_i \hat{y}_{n+i} + \sum_{i=N_1+1}^{\infty} \hat{x}_i \hat{y}_{n+i}$$
$$\leqslant \left(\epsilon \Big/ 2\sum_{i=1}^{N_1} \hat{x}_i\right)\left(\sum_{i=1}^{N_1} \hat{x}_i\right) + \epsilon/2 = \epsilon.$$

Given $x \in c_0 \sim l^1$ we denote by Q_x the sequential norm defined in the previous theorem. If no ambiguity results, we shall simply write Q for Q_x.

THEOREM 3.6. *For* $x \in c_0 \sim l^1$

$$Q'_x(y) = \sup_n \left(\sum_{i=1}^{n} \hat{y}_i\right) \Big/ \left(\sum_{i=1}^{n} \hat{x}_i\right).$$

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Proof. By definition

$$Q'(y) = \sup\left\{\sup_n \sum_{i=1}^{n} a_i y_i : Q(a) \leqslant 1\right\}$$
$$= \sup\left\{\sup_n \sum_{i=1}^{n} a_i y_i : \sum_{i=1}^{\infty} \hat{a}_i \hat{x}_i \leqslant 1\right\}$$
$$= \sup\left\{\sum_{i=1}^{\infty} \hat{a}_i \hat{y}_i : \sum_{i=1}^{\infty} \hat{a}_i \hat{x}_i \leqslant 1\right\}.$$

The last equality holds since

$$\left|\sum_{i=1}^{n} a_i y_i\right| \leqslant \sum_{i=1}^{n} \hat{a}_i \hat{y}_i \leqslant \sum_{i=1}^{\infty} \hat{a}_i \hat{y}_i.$$

If m is so large that $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n$ are included in $\{|y_1|, |y_2|, \ldots, |y_m|\}$, let π be any permutation such that $\hat{y}_i = |y_{\pi(i)}|\ i \leqslant n$. Let $b_i = (\operatorname{sgn} y_i)\hat{a}_{\pi^{-1}(i)}$. Then $\hat{b} = \hat{a}$ and

$$\sum_{i=1}^{m} b_i y_i \geqslant \sum_{\pi(i)\leqslant m} b_{\pi(i)} y_{\pi(i)}$$
$$\geqslant \sum_{i\leqslant n} b_{\pi(i)} y_{\pi(i)}$$
$$= \sum_{i=1}^{n} \hat{a}_i |y_{\pi(i)}| = \sum_{i=1}^{n} \hat{a}_i \hat{y}_i.$$

Now

$$\sum_{n=1}^{n} \hat{a}_i \hat{y}_i = \sum_{i=1}^{n-1} (\hat{a}_i - \hat{a}_{i+1})\left(\sum_{j=1}^{i} \hat{y}_j\right) + \hat{a}_n \sum_{j=1}^{n} \hat{y}_j$$
$$= \left[\sum_{i=1}^{n-1} (\hat{a}_i - \hat{a}_{i+1})\right]\left[\sum_{j=1}^{i} \hat{y}_j \Big/ \sum_{j=1}^{i} \hat{x}_j\right]\sum_{j=1}^{i} \hat{x}_j + \hat{a}_n\left[\sum_{j=1}^{n} \hat{y}_j \Big/ \sum_{j=1}^{n} \hat{x}_j\right]\sum_{j=1}^{n} \hat{x}_j$$
$$\leqslant \sup_n\left(\sum_{i=1}^{n} \hat{y}_i \Big/ \sum_{i=1}^{n} \hat{x}_i\right)\left[\sum_{i=1}^{n-1} (\hat{a}_i - \hat{a}_{i+1}) \sum_{j=1}^{i} \hat{x}_j + \hat{a}_n \sum_{j=1}^{n} \hat{x}_j\right]$$
$$= \sup_n\left(\sum_{i=1}^{n} \hat{y}_i \Big/ \sum_{i=1}^{n} \hat{x}_i\right)\sum_{i=1}^{n} \hat{a}_i \hat{x}_i$$
$$\leqslant \sup_n \sum_{i=1}^{n} \hat{y}_i \Big/ \sum_{i=1}^{n} \hat{x}_i$$

if

$$Q(a) = \sum_{i=1}^{\infty} \hat{a}_i \hat{x}_i \leqslant 1.$$

On the other hand, if n is such that

$$\sum_{i=1}^{n} \hat{y}_i \Big/ \sum_{i=1}^{n} \hat{x}_i > \sup_n \sum_{i=1}^{n} \hat{y}_i \Big/ \sum_{i=1}^{n} \hat{x}_i - \epsilon,$$

then let

$$b_i = 1 \Big/ \sum_{i=1}^{n} \hat{x}_i$$

so that

$$\sum_{i=1}^{n} \hat{x}_i b_i = 1$$

while

$$\sum_{i=1}^{n} \hat{b}_i \hat{y}_i > \sup_n \sum_{i=1}^{n} \hat{y}_i \Big/ \sum_{i=1}^{n} \hat{x}_i.$$

In view of Theorems 5 and 7 we conclude that $x^\sigma = S_Q$ is the space $n\phi$ and $x^{\sigma\sigma} = S_Q$ is the space $m\phi$ studied by W. L. C. Sargent in **(8)** and **(9)**, where

$$\phi_n = \sum_{i=1}^{n} \hat{x}_i.$$

The norm given by Sargent for $n\phi$ coincides with Q, but the norm she gave for $m\phi$ does not necessarily coincide with Q'.

The following proposition is Lemma 10 of **(8)** with a different proof.

PROPOSITION 3.7. *$y^\sigma \supseteq x^\sigma$ if and only if*

$$\sup_n \sum_{i=1}^{n} \hat{y}_i \Big/ \sum_{i=1}^{n} \hat{x}_i < \infty.$$

Proof. If

$$\sup_n \sum_{i=1}^{n} \hat{y}_i \Big/ \sum_{i=1}^{n} \hat{x}_i < \infty,$$

then $y \in S_{Q'} = x^{\sigma\sigma}$. Thus $y^\sigma \supseteq x^{\sigma\sigma\sigma} = x^\sigma$.

If $y^\sigma \supseteq x^\sigma$, then $y \in y^{\sigma\sigma} \subseteq S_{Q'}$ so that

$$\sup_n \sum_{i=1}^{n} \hat{y}_i \Big/ \sum_{i=1}^{n} \hat{x}_i = Q'(y) < \infty.$$

PROPOSITION 3.8. *$x_0^{\sigma\sigma} \neq x^{\sigma\sigma}$. (By $x_0^{\sigma\sigma}$ we mean the closed linear span of $\mathfrak{E}$ in $x^{\sigma\sigma}$.)*

Proof. Define f on $x^{\sigma\sigma}$ by

$$f(y) = \lim_n \sum_{i=1}^{n} y_i \Big/ \sum_{i=1}^{n} \hat{x}_i, \qquad n = 1, 2, \ldots.$$

Then $||f|| \leqslant 1$ and $f(e_1) = 0$ for each i. Since $f(\hat{x}) \notin x_0^{\sigma\sigma}$ although $\hat{x} \in x^{\sigma\sigma}$.

In view of the previous statement and Theorem 5.5 of **(7)** (see also **(4,** Lemmas 1 and 2**))** we arrive at the following.

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PROPOSITION 3.9. (a) *There is a closed subspace of* $x^{\sigma\sigma}$ *topologically isomorphic to m.* (b) *There is a closed subspace of* $x_0^{\sigma\sigma}$ *topologically isomorphic to* c_0. (c) *There is a closed subspace of* x^σ *topologically isomorhphc to* l^1. (d) *The spaces* x^σ, $x^{\sigma\sigma}$, *and* $x_0^{\sigma\sigma}$ *are not reflexive.*

In connection with the above proposition see **(9**, Theorems 8 and 9**)**.

PROPOSITION 3.10. (*converse of* 3(e)). *If* $x^\sigma = l^1$, $x \in m \sim c_0$.

Proof. If $x^\sigma = l^1$, $x \in x^{\sigma\sigma} = m$. If $x \in c_0$,

$$\inf_n \sum_{i=1}^{n} \hat{x}_i \Big/ n = \lim_n \frac{1}{n}\sum_{i=1}^{n} \hat{x}_i = 0$$

so that $x^\sigma = (1, 1, 1, \ldots)^\sigma = l^1$ by Proposition 3.7.

4. Symmetric coordinate spaces. The following proposition is a generalization of **(8**, Lemma 12d**)**.

PROPOSITION 4.1. *If X is any perfect symmetric space, then*

$$X = \bigcup\{x^{\sigma\sigma}: x \in X\} = \bigcup\{x^{\sigma\sigma}: X^\sigma \supseteq X^\sigma\}$$
$$= \bigcap\{x^\sigma: x \in X^\sigma\}.$$

Proof. If $x \in X$, then $x^\sigma \supseteq X^\sigma$, and if $x^\sigma \supseteq X^\sigma$, then $x^{\sigma\sigma} \subseteq X^{\sigma\sigma} = X$ so that $x \in X$ if and only if $x^{\sigma\sigma} \subseteq X$ and $x \in X$ if and only if $x^\sigma \supseteq X^\sigma$. This yields the first two equalities.

THEOREM 4.2. *If X is a perfect symmetric* BK *space, there is a balanced, symmetric sequential norm N of the form*

$$N(x) = \sup_\alpha \sum_{i=1}^{\infty} \hat{x}_i \hat{y}_i^{(\alpha)}$$

for which $S_N = X$.

Proof. By 4.1, $X = \bigcap\{y^\sigma: y \in X^\sigma\}$. For each $y \in X^\sigma$, $y^\sigma \supseteq X$ so there is an $m_y > 0$ such that

$$m_y\, Q_y(x) \leqslant ||x||, \qquad x \in X,$$

where $||\ ||$ is the norm on X **(12**, p. 203**)**. Note that

$$m_y\, Q_y(x) = \sum_{i=1}^{\infty} \widehat{m_y\, y_i}\, \hat{x}_i.$$

Let

$$N = \sup\{m_y\, Q_y: y \in X^\sigma\}.$$

Then N is a balanced symmetric p.s.n. by 2.10 and $N(x) \leqslant ||x||$ for $x \in X$ so that $S_N \supseteq X$. In order to apply 2.10 we observe that

$$m_y\, Q_y(e_i) = m_y\, Q_y(e_1) \leqslant ||e_1|| \qquad \text{for each } i.$$

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On the other hand, $S_N \subseteq x^\sigma$ for each $x \in X^\sigma$ so that $S_N \subseteq X$. Finally

$$N(x) = \sup\left\{ \sum_{i=1}^{\infty} \widehat{m_y\, y_i}\, \hat{x}_i : y \in X^\sigma \right\}.$$

5. Applications to symmetric bases. Recall the definition by Singer **(10)** that a basis $\{x_n\}$ of a Banach space X is *symmetric* if

$$(\mathrm{SB}_1) \qquad \sup_{\pi \in P} \sup_{\substack{|\delta_i| \leqslant 1 \\ 1 \leqslant n < \infty}} \left\| \sum_{i=1}^{n} \delta_i f_i(x) x_{\pi(i)} \right\| < \infty$$

for all $x \in X$ where P denotes the set of all permutations on the positive integers and $\{f_n\}$ is the sequence of continuous linear functionals biorthogonal to $\{x_n\}$.

PROPOSITION 5.1. *The basis $\mathfrak{X} = x_n$ of a Banach space X is symmetric if and only if $S_{\mathfrak{X}} = S_N{}^0$ for N a balanced, symmetric p.s.n., where*

$$S_{\mathfrak{X}} = \left\{ t : \sum_{i=1}^{\infty} t_i x_i \text{ converges in } X \right\}.$$

Proof. If $\mathfrak{X}$ is symmetric, define

$$N(t) = \sup_{\pi \in P} \sup_{\substack{|\delta_i| \leqslant 1 \\ 1 \leqslant n < \infty}} \left\| \sum_{i=1}^{n} \delta_i\, t x_{\sigma(i)} \right\|.$$

Then N is a balanced, symmetric p.s.n. and by (SB_1) $S_{\mathfrak{X}} \subseteq S_N$. However, for $t \in S_{\mathfrak{X}}$, $N(t) \geqslant \|t\|$, so $S_{\mathfrak{X}}$ is a closed subspace of S_N **(12**, p. 203**)**, which implies that $S_{\mathfrak{X}} = S_N{}^0$.

If N is a balanced, symmetric p.s.n. and $S_N{}^0 = S$, then the norm $\|\ \|$ defined on X by

$$\left\| \sum_{i=1}^{\infty} t_i x_i \right\| = N(t)$$

yields the original topology on X and has the property indicated in (SB_1).

In **(11)**, Singer proved that the following is equivalent to (SB_1):

(SB_3) Every permutation $\{x_{\pi(n)}\}$ of the basis $\{x_n\}$ is a basis of X equivalent to the basis $\{x_n\}$.

We shall offer an alternative proof of the equivalence of (SB_1) and (SB_3).

THEOREM 5.2. (SB_1) *is equivalent to* (SB_3).

Proof. $(\mathrm{SB}_1) \Rightarrow (SB_3)$. Let $\mathfrak{X} = \{x_n\}$ be a basis for a Banach space X which satisfies (SB_1). Define a new norm $\|\ \|'$ on X by

$$\|x\|' = \sup_{\pi \in P} \sup_{\substack{|\delta_i| \leqslant 1 \\ 1 \leqslant n < \infty}} \left\| \sum_{i=1}^{n} \delta_i f_i(x) x_{\pi(i)} \right\|.$$

Then $\|\ \|'$ is equivalent to $\|\ \|$ and

$$\left\|\sum_{i=1}^{n} t_i x_i\right\|' = \left\|\sum_{i=1}^{n} t_i x_{\pi(i)}\right\|'$$

for every n and every permutation π. Therefore, $\{x_n\}$ is an equivalent basis to $\{x_{\pi(n)}\}$ for every permutation π.

$(SB_3) \Rightarrow (SB_1)$. If $\mathfrak{X}$ satisfies (SB_3), then $S_{\mathfrak{X}}$ is symmetric. To see this assume that $t \in S_{\mathfrak{X}}$ and π is any permutation on the positive integers. Then $\sum_{i=1}^{\infty} t_i x_i$ converges so that $\sum_{i=1}^{\infty} t_i x_{\pi^{-1}(i)}$ converges necessarily to $\sum_{i=1}^{\infty} t_{\pi(i)} x_i$ so that $t_\pi \in S_{\mathfrak{X}}$.

Since $\mathfrak{X}$ is an unconditional basis for X, there is a balanced sequential norm such that $S_N{}^0 = S_{\mathfrak{X}}$. In fact, define $N(t)$ to be

$$\sup\left\{\left\|\sum_{i=1}^{\infty} a_i t_i x_i\right\| : |a_i| \leqslant 1\right\}.$$

Since $S_N{}^0$ is symmetric, so is $(S_N{}^0)^{\alpha\alpha} = S_N$. By Theorem 4.5 there is a balanced symmetric p.s.n. M such that $S_M = S_N$. Thus $S_M{}^0 = S_N{}^0 = S_{\mathfrak{X}}$, which implies that $\mathfrak{X}$ satisfies (SB_1) by Proposition 5.1.

Added November 10, 1966. The author wishes to point out that many of the results in § 3 or generalizations thereof appear in the related work of D. J. H. Garling **(14)** of which he was unaware at the time of writing this paper.

References

1. M. G. Arsove, *Similar bases and isomorphisms in Fréchet spaces*, Math. Ann., *135* (1958), 283–293.

2. C. Bassaga and A. Pełczynski, *On bases and unconditional convergence of series in Banach spaces*, Studia Math., *17* (1958), 151–164.

3. R. G. Cooke, *Infinite matrices and sequence spaces* (Macmillan, 1950).

4. R. C. James, *Bases and reflexivity in Banach spaces*, Ann. of Math., *52* (1950), 518–527.

5. G. Köthe, *Topologische lineare Räume*, Vol. 1 (Berlin, 1960).

6. G. Köthe and O. Toeplitz, *Lineare Räume mit unendlicher vielen Koordinaten und Ringe unendlicher Matrizen*, J. für Math., *171* (1934), 193–226.

7. W. Ruckle, *On the construction of sequence spaces that have Schauder bases*, Can. J. Math., *18* (1966), 1281–1293.

8. W. L. C. Sargent, *Some sequence spaces related to the l^p spaces*, J. London Math. Soc., *35* (1960), 161–171.

9. ——— *On sectionally bounded* BK-*spaces*, Math. Z., *83* (1964), 57–66.

10. I. Singer, *On Banach spaces with symmetric basis*, Rev. Math. Pures Appl., *7* (1961), 159–166 (in Russian).

11. ——— *Some characterizations of symmetric bases*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys., *10* (1962), 185–192.

12. A. Wilansky, *Functional analysis* (Blaisdell, New York, 1964).

13. K. Zeller, *Abschnittskonvergenz in* FK *Raumen*, Math. Z., *55* (1952), 55–70.

14. D. J. H. Garling, *On symmetric sequence spaces*, Proc. London Math. Soc., (3), *16* (1966), 85–105.

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