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ON EXISTENCE OF DISTINCT REPRESENTATIVE SETS FOR SUBSETS OF A FINITE SET

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1. Introduction. Let S be a finite set and let $S_1, S_2, \ldots, S_t$ be subsets of S, not necessarily distinct. Does there exist a set of distinct representatives (SDR) for $S_1, S_2, \ldots, S_t$? That is, does there exist a subset $\{a_1, a_2, \ldots, a_t\}$ of S such that $a_i \in S_i$, $1 \leqq i \leqq t$, and $a_i \neq a_j$ if $i \neq j$? The following theorem of Hall [**2**; **6**, p. 48] gives the answer.

THEOREM. *The subsets $S_1, S_2, \ldots, S_t$ have an SDR if and only if for each s, $1 \leqq s \leqq t$, $|S_{i_1} \cup S_{i_2} \cup \ldots \cup S_{i_s}| \geqq s$ for each s-combination $\{i_1, i_2, \ldots, i_s\}$ of the integers $1, 2, \ldots, t$.*

(Here and below, $|A|$ denotes the number of elements in A.)

In this paper we use Hall's theorem and a generalization [**1**, p. 231] of a theorem of Macaulay [**4**, p. 537] to solve a related problem. This time we are interested in representing distinct l-element subsets of a finite set having several different kinds of elements by distinct $(l-k)$-element subsets. More precisely, let H be a set of $k_1 + k_2 + \ldots + k_n = K$ (billiard) balls, k_i of colour i, $1 \leqq i \leqq n$, $k_1 \leqq k_2 \leqq \ldots \leqq k_n$, and let $1 \leqq k \leqq l \leqq K$ be given. For a set $\{A_1, A_2, \ldots, A_t\}$ of distinct l-element subsets of H, does there exist an SDR for $\{A_1, A_2, \ldots, A_t\}$ among the $(l-k)$-element subsets of H? That is, does there exist a set $\{B_1, B_2, \ldots, B_t\}$ of distinct $(l-k)$-element subsets of H such that $B_i \subset A_i$, $1 \leqq i \leqq t$? The answer, not surprisingly, is yes if t does not exceed some integer $M(l, k; k_1, k_2, \ldots, k_n)$ and not necessarily if t does exceed $M(l, k; k_1, k_2, \ldots, k_n)$. When there is no danger of ambiguity, we will abbreviate $M(l, k; k_1, \ldots, k_n)$ by M. The problem corresponding to $k_1 = k_2 = \ldots = k_n = 1$ was formulated and completely solved by Katona [**3**]. In this paper we give an explicit method for producing any M. Also, we shall give closed form expressions for M in some cases. In particular, we shall obtain Katona's results.

2. An explicit method for producing M. We identify the subset of H consisting of j_i balls of colour i, $1 \leqq i \leqq n$, with the n-tuple $(j_1, j_2, \ldots, j_n)$. Let the set $F(k_1, k_2, \ldots, k_n)$ of these $(k_1+1)(k_2+1)\ldots(k_n+1) = \theta$ n-tuples be ordered lexicographically; that is, we define $\mathbf{j} = (j_1, \ldots, j_n) < \mathbf{i} = (i_1, \ldots, i_n)$ if and only if $j_r < i_r$ for the smallest integer r such that $j_r \neq i_r$. If there is no danger of ambiguity, we will abbreviate $F(k_1, \ldots, k_n)$

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by F. We imagine the θ elements of F arrayed in $K + 1$ columns and $R = \theta/(k_n + 1)$ rows by writing them in increasing order from left to right, top to bottom, with $k_n + 1$ elements in each row and with element $\mathbf{j}$ in column $j_1 + \ldots + j_n$. Thus all l-element subsets are in column l, $0 \leqq l \leqq K$. For example, $F(2, 3, 4)$ is arrayed as follows:

Column number	0	1	2	3	4	5	6	7	8	9
	000	001	002	003	004					
		010	011	012	013	014				
			020	021	022	023	024			
				030	031	032	033	034		
		100	101	102	103	104				
			110	111	112	113	114			
				120	121	122	123	124		
					130	131	132	133	134	
			200	201	202	203	204			
				210	211	212	213	214		
					220	221	222	223	224	
						230	231	232	233	234

FIGURE 1

We observe for later use that the $F(k_1, k_2, \ldots, k_n)$-array consists of $(k_1 + 1)$ $F(k_2, \ldots, k_n)$-arrays in which each entry in the ith array is preceded by i, $0 \leqq i \leqq k_1$, and each of these modified $F(k_2, \ldots, k_n)$-arrays is one column to the right of its predecessor. With some abuse of language, we will recall this situation by saying that the array $F(k_1, \ldots, k_n)$ consists of $(k_1 + 1)$ $F(k_2, \ldots, k_n)$-arrays.

In order to state the generalized Macaulay theorem, we define the set-valued operator Γ on F by

$$\Gamma((j_1, \ldots, j_n)) = \{(j_1 - 1, j_2, \ldots, j_n), (j_1, j_2 - 1, \ldots, j_n), \ldots, (j_1, \ldots, j_n - 1)\} \cap F.$$

Thus $\Gamma(\mathbf{j})$ is the set of those sets obtainable by removing one element from the set $\mathbf{j}$; in particular, $\Gamma((0, \ldots, 0))$ is the empty set. We also define $\Gamma^k(\mathbf{j}) = \Gamma(\Gamma^{k-1}(\mathbf{j}))$ for $k \geqq 2$, where Γ^1 is Γ, and $\Gamma^k(A) = \bigcup_{\mathbf{a} \in A} \Gamma^k(\mathbf{a})$ for subsets A of F, $k \geqq 1$.

Now let $\{\mathbf{j}_1, \ldots, \mathbf{j}_s\}$ be any s distinct l-element subsets of F, where $0 \leqq l \leqq K$. $C(\{\mathbf{j}_1, \ldots, \mathbf{j}_s\})$ denotes the first s l-element sets. It is called the *compression* of $\{\mathbf{j}_1, \ldots, \mathbf{j}_s\}$. The generalized Macaulay theorem [**1**] asserts that

$$C\Gamma(\{\mathbf{j}_1, \ldots, \mathbf{j}_s\}) \supset \Gamma C(\{\mathbf{j}_1, \ldots, \mathbf{j}_s\}). \tag{1}$$

Applying Γ to both sides here yields

$$\Gamma C\Gamma(\{\mathbf{j}_1, \ldots, \mathbf{j}_s\}) \supset \Gamma^2 C(\{\mathbf{j}_1, \ldots, \mathbf{j}_s\}), \tag{2}$$

and therefore, applying (1) to the left side of (2),

$$C\Gamma^2(\{\mathbf{j}_1, \ldots, \mathbf{j}_s\}) \supset \Gamma^2 C(\{\mathbf{j}_1, \ldots, \mathbf{j}_s\}). \tag{3}$$

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In general, for arbitrary k we have in this way

$$C\Gamma^k(\{\mathbf{j}_1, \ldots, \mathbf{j}_s) \supset \Gamma^k C(\mathbf{j}_1, \ldots, \mathbf{j}_s\}). \tag{4}$$

We now combine Hall's theorem and the generalized Macaulay theorem to obtain a method for determining $M(l, k; k_1, \ldots, k_n)$.

THEOREM 1. *Let $k_1 \leqq k_2 \leqq \ldots \leqq k_n$ and l, k be positive integers such that $1 \leqq k \leqq l \leqq k_1 + \ldots + k_n = K$. For any integer l' let $F_{l'}$ denote the set of l'-element subsets of F (the empty set if $l' < 0$ or $l' > K$), and let $(F_{l'})_t$ denote the first t elements of $F_{l'}$. Finally, let $M(l, k; k_1, \ldots, k_n) = M$ be the largest integer $\leqq |F_l|$ such that $t \leqq M$ implies $|\Gamma^k((F_l)_t)| - t \geqq 0$. Then if $\{\mathbf{j}_1, \ldots, \mathbf{j}_t\}$ is a set of distinct elements of F_l, there exists a set $\{\mathbf{r}_1, \ldots, \mathbf{r}_t\}$ of distinct elements of F_{l-k} such that $\mathbf{r}_i \subset \mathbf{j}_i$, $1 \leqq i \leqq t$, provided $t \leqq M$, but there does not necessarily exist such a set if $t > M$.*

Example. Suppose that $k_1 = 2$, $k_2 = 3$, $k_3 = 4$, $l = 5$, $k = 3$. Assuming the theorem for the moment, inspection of Figure 1 shows that $M = 5$. Then it is possible to represent any five 5-element sets in $F(2, 3, 4)$ by distinct 2-element sets. For instance, the set of five 5-element sets {014, 113, 212, 221, 230} is represented by the set of 2-element sets {011, 002, 200, 110, 020}. However, the set of six 5-element sets $(F_5)_6 =$ {014, 023, 032, 104, 113, 122} has no set of distinct representatives among the 2-element sets since $|\Gamma^3((F_5)_6)| = |\{002, 011, 020, 101, 110\}| = 5 < 6$. We will see in general that $(F_l)_t$ is a subset least likely to have an SDR.

Proof of Theorem 1. First suppose that $t \leqq M$. Let $\{\mathbf{j}_1, \ldots, \mathbf{j}_t\} \subset F_l$ be given, and consider $\{\Gamma^k(\mathbf{j}_1), \Gamma^k(\mathbf{j}_2), \ldots, \Gamma^k(\mathbf{j}_t)\}$. The question facing us is, does this set of subsets of F_{l-k} contain an SDR for $\{\mathbf{j}_1, \mathbf{j}_2, \ldots, \mathbf{j}_t\}$? According to Hall's theorem, the answer is yes, provided that for each $s \leqq t$ and each s-combination $i_1, \ldots, i_s$ of the integers $1, 2, \ldots, t$, the inequality

$$|\Gamma^k(\mathbf{j}_{i_1}) \cup \ldots \cup \Gamma^k(\mathbf{j}_{i_s})| \geqq s$$

holds. But this does indeed hold since

$$\begin{aligned}|\Gamma^k(\mathbf{j}_{i_1}) \cup \ldots \cup \Gamma^k(\mathbf{j}_{i_s})| &= |\Gamma^k\{\mathbf{j}_{i_1}, \ldots, \mathbf{j}_{i_s}\}| = |C(\Gamma^k\{\mathbf{j}_{i_1}, \ldots, \mathbf{j}_{i_s}\})| \\ &\geqq |\Gamma^k(C(\{\mathbf{j}_{i_1}, \ldots, \mathbf{j}_{i_s}\}))| = |\Gamma^k((F_l)_s)| \geqq s.\end{aligned}$$

(Compressing a set does not alter the number of elements in it; the first inequality follows from (4); the last inequality follows from $s \leqq t \leqq M$.)

Next suppose that $|F_l| \geqq t > M$. Take $\{\mathbf{j}_1, \ldots, \mathbf{j}_{M+1}\}$ such that

$$|\Gamma^k(\{\mathbf{j}_1, \ldots, \mathbf{j}_{M+1}\})| < M + 1.$$

This is possible in view of the definition of M. Hall's theorem now shows that any t-element subset of $|F_l|$ which contains $\{\mathbf{j}_1, \ldots, \mathbf{j}_{M+1}\}$ has no SDR among the $(l - k)$-element subsets. This completes the proof.

Thus the problem is completely solved in the sense that for given l, k, $k_1, \ldots, k_n$ one can form the $F(k_1, \ldots, k_n)$-array and determine

$$M(l, k; k_1, \ldots, k_n)$$

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by inspection. For large k_i and large n this will of course be tedious. The following lemmas can be used to shorten the process in some cases, and sometimes even lead to formulas for M.

LEMMA 1. *With* $T(l) = |F_l|$,

$$T(l) = T(K - l), \qquad l \text{ an integer}, \tag{5}$$

and

$$T(l - 1) \leqq T(l), \qquad 1 \leqq l \leqq [K/2]. \tag{6}$$

Remark. If $k_1 = k_2 = \ldots = k_n = 1$, then $T(l) = \binom{n}{l}$, and (5) and (6) are familiar properties of binomial coefficients.

Proof. We use induction on n. For $n = 1$, $T(l) = 1$ for $0 \leqq l \leqq K$ and the lemma is trivial. Now assuming the lemma holds for arrays of $(n - 1)$-tuples, we consider an array of n-tuples, $F(k_1, \ldots, k_n)$.

In view of the way $F(k_1, \ldots, k_n)$ consists of $(k_1 + 1)$ $F(k_2, \ldots, k_n)$ arrays, it is clear that for $0 \leqq l \leqq K$, $T(l)$ is the sum of the lth column of the $(k_1 + 1)$-rowed array of integers

$$\begin{array}{l}
T^*(0) \quad T^*(1) \quad T^*(2) \ldots T^*(k_1 - 1) \quad T^*(k_1) \quad \ldots T^*(K^*) \\
\qquad\quad T^*(0) \quad T^*(1) \ldots T^*(k_1 - 2) \quad T^*(k_1 - 1) \ldots T^*(K^* - 1) \quad T^*(K^*) \\
\qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
\qquad\qquad T^*(0) \qquad T^*(1) \qquad T^*(2) \qquad \ldots \qquad\qquad T^*(K^*) \\
\qquad\qquad\qquad\quad T^*(0) \qquad T^*(1) \qquad \ldots \qquad\qquad T^*(K^* - 1) T^*(K^*)
\end{array}$$

FIGURE 2

where $K^* = k_2 + \ldots + k_n$ and $T^*(l) = |F_l(k_2, \ldots, k_n)|$. In view of the induction hypothesis for the numbers T^*, and the symmetry of this array, (5) is clear. Also one reads off from this array (by cancelling the element in row r of column $l - 1$ with the element in row $r + 1$ of column l and remembering that $T^*(j) = 0$ if $j < 0$) that

$$T(l) - T(l - 1) = T^*(l) - T^*(l - k_1)$$

for $0 \leqq l \leqq K^*$. From this, (6) now follows: for instance, if K^* and k_1 are both even, the two parts of the induction hypothesis together show that $T^*(l) - T^*(l - k_1) \geqq 0$ for $0 \leqq l \leqq (K^* + k_1)/2$, and $(K^* + k_1)/2 = K/2 = [K/2]$. The other cases follow in a similar fashion.

LEMMA 2. *Let* $1 \leqq k \leqq l \leqq K$ *be given and let* ${}_{l-k}N_l(r) = N(r)$ *be the number of elements in* F_{l-k} *above or in row* r *less the number of elements in* F_l *above or in row* r. *If there exists* $r < R$ *such that* $N(r) < 0$, *then* $N(R) \leqq N(r)$, *where* $R = \theta/(k_n + 1)$ *is the total number of rows in* $F(k_1, \ldots, k_n)$.

Proof. We use induction on n. For $n = 1$, R is 1, and the lemma is vacuously true. For $n = 2$, the lemma follows easily from the parallelogram form of the

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array $F(k_1, k_2)$. Assuming the lemma for $n - 1$, we consider n. Suppose that $N(r) < 0$ and that elements in row r begin with i_0. Let L_i and R_i denote (respectively) the number of elements in columns $l - k$ and l beginning with i, $0 \leqq i \leqq k_1$, and let $L_{i_0}{}^r$, $R_{i_0}{}^r$ denote the number of elements beginning with i_0 above or in row r and in columns $l - k$, and l, respectively. In view of the way $F(k_1, \ldots, k_n)$ consists of $(k_1 + 1)$ $F(k_2, \ldots, k_n)$-arrays, we have

$$(7) \quad L_i = |F_{l-k-i}(k_2, \ldots, k_n)|, \qquad R_i = |F_{l-i}(k_2, \ldots, k_n)|, \qquad 0 \leqq i \leqq k_1.$$

Then

$$(8) \quad N(r) = L_0 + \ldots + L_{i_0-1} + L_{i_0}{}^r - R_0 - \ldots - R_{i_0-1} - R_{i_0}{}^r < 0.$$

Now if $L_{i_0}{}^r - R_{i_0}{}^r < 0$, it follows from the $(k_2, \ldots, k_n)$-instance of the induction hypothesis that $L_{i_0} - R_{i_0} \leqq L_{i_0}{}^r - R_{i_0}{}^r$. But then $L_{i_0} - R_{i_0} < 0$, and it follows from Lemma 1 and (7) that $L_j - R_j \leqq 0$, $i_0 + 1 \leqq j \leqq k_1 + 1$. We then have

$$\begin{aligned} N(R) &= L_0 + \ldots + L_{k_1+1} - R_0 - \ldots - R_{k_1+1} \\ &\leqq L_1 + \ldots + L_{i_0}{}^r + \ldots + L_{k_1+1} - R_1 - \ldots - R_{i_0}{}^r - \ldots - R_{k_1+1} \\ &= N(r) + (L_{i_0+1} - R_{i_0+1}) + \ldots + (L_{k_1+1} - R_{k_1+1}) \\ &\leqq N(r). \end{aligned}$$

If $L_{i_0}{}^r - R_{i_0}{}^r \geqq 0$, then in view of (8), we must have $L_{j_0} - R_{j_0} < 0$ for some $j_0 < i_0$. But then $L_j - R_j \leqq 0$ for all $j > j_0$, and so

$$\begin{aligned} N(R) = (L_0 - R_0) + \ldots + (L_{k_1} - R_{k_1}) &\leqq (L_0 - R_0) + \ldots + (L_{i_0} - R_{i_0}) \\ &\leqq (L_0 - R_0) + \ldots + (L_{i_0}{}^r - R_{i_0}{}^r) = N(r). \end{aligned}$$

This completes the proof of Lemma 2.

The next lemma gives further information about ${}_{l-k}N_l$ as a function of r if columns $(l - k)$ and l are symmetrically located about the middle of the K-columned sub-array $F(k_1 - 1, k_2, \ldots, k_n)$ of $F(k_1, k_2, \ldots, k_n)$; that is, if $(l - k) + l = K - 1$.

LEMMA 3. *Suppose that $2l + 1 - k = K$. Then with ${}_{l-k}N_l(r) = N(r)$,*
(i) *if $n = 1$, then $N(1) = 0$,*
(ii) *if $n = 2$, then $N(r) \geqq 0$ for $1 \leqq r \leqq k_1$ and $N(k_1) = 0$,*
(iii) *if $n \geqq 3$, then $N(r) \geqq 0$ for $1 \leqq r \leqq k_1(k_2 + 1) \ldots (k_{n-1} + 1)$ and $N(k_1(k_2 + 1) \ldots (k_{n-1} + 1)) = 0$.*

Proof. In view of the way the $F(k_1, \ldots, k_n)$-array consists of $(k_1 + 1)$ $F(k_2, \ldots, k_n) = F^*$ arrays and since the first $k_1(k_2 + 1) \ldots (k_{n-1} + 1)$ rows of $F(k_1, \ldots, k_n)$ is the first k_1 of the F^* arrays, it follows that

$$\begin{aligned} (9) \quad N(k_1(k_2 + 1) \ldots (k_{n-1} + 1)) = |F^*_{l-k}| + |F^*_{l-k-1}| + \ldots + |F^*_{l-k-k_1+1}| \\ - [|F^*_l| + |F^*_{l-1}| + \ldots + |F^*_{l-k_1+1}|]. \end{aligned}$$

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By Lemma 1, $|F_j^*| = |F_{k_2+\ldots+k_n-j}^*|$ for all j, and so

$$|F_l^*| + |F_{l-1}^*| + \ldots + |F_{l-k_1+1}^*| = |F_{k_1+k_2+\ldots+k_n-l-k_1}^*| + |F_{k_1+\ldots+k_n-(l-1)-k_1}^*| + \ldots + |F_{k_1+\ldots+k_n-l+k_1-1-k_1}^*|$$
$$= |F_{2l+1-k-l-k_1}^*| + |F_{2l+1-k-(l-1)-k_1}^*| + \ldots + |F_{2l+1-k-l-1}^*|$$
$$= |F_{l-k-k_1+1}^*| + |F_{l-k-k_1+2}^*| + \ldots + |F_{l-k}^*|.$$

Hence (9) shows that $N(k_1(k_2+1)\ldots(k_{n-1}+1)) = 0$. The first

$$k_1(k_2+1)\ldots(k_{n-1}+1)$$

rows of $F(k_1, \ldots, k_n)$ is exactly $F(k_1 - 1, k_2, \ldots, k_n)$; thus applying Lemma 2 to $F(k_1 - 1, k_2, \ldots, k_n)$ or to $F(k_2, \ldots, k_n)$ if k_1 happens to be 1, shows that $N(r) \geqq 0$ for all $r \leqq k_1(k_2+1)\ldots(k_{n-1}+1)$, completing the proof.

We can now give an estimate for M.

THEOREM 2. *If* $2l + 1 - k = K$, *then*

$$C_n^{\,l} \leqq M(l, k; k_1, \ldots, k_n) \leqq |F_l|, \tag{10}$$

where

$$(1 + t + \ldots + t^{k_1-1})(1 + t + \ldots + t^{k_2})\ldots(1 + t + \ldots + t^{k_n}) = \sum_{l=0}^{K-1} C_n^{\,l} t^l \tag{11}$$

and

$$(1 + t + \ldots + t^{k_1})(1 + t + \ldots + t^{k_2})\ldots(1 + t + \ldots + t^{k_n}) = \sum_{l=0}^{K} |F_l| t^l. \tag{12}$$

Proof. In view of the definitions of M and N (see Theorem 1 and Lemma 2), it follows from Lemma 3 that M is greater than or equal to the number of elements in column l and the first $k_1(k_2+1)\ldots(k_{n-1}+1)$ rows of $F(k_1, \ldots, k_n)$. These elements are exactly the compositions [**5**] of l with n parts p_i, $0 \leqq p_1 \leqq k_1 - 1$, $0 \leqq p_i \leqq k_i$, $2 \leqq i \leqq n$. If $C_n^{\,l}$ is the number of these elements, it is clear that $C_n^{\,l}$ satisfies (11). Hence the left side of (10) follows. The right side of (10) is immediate from the definition of M; (12) holds since $|F_l|$ is the number of compositions of l with parts p_i, $0 \leqq p_i \leqq k_i$, $1 \leqq i \leqq n$.

COROLLARY. *Under the assumptions in Theorem* 2, *if the last* n *of* $j_1, j_2, \ldots, j_m$, $i_1 \leqq j_2 \leqq \ldots \leqq j_m$, *are* $k_1, k_2, \ldots, k_n$ *and*

$$M(l, k; k_1, \ldots, k_n) < |F_l(k_1, \ldots, k_n)|,$$

then

$$M(l, k; j_1, \ldots, j_m) = M(l, k; k_1, \ldots, k_n).$$

Proof. The $F(k_1, \ldots, k_n)$-array is the upper left part of the $F(j_1, \ldots, j_m)$-array.

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Example. $12 = C_3^6 \leqq M(6, 2; 3, 4, 4) \leqq |F_6(3, 4, 4)| = 16$. The last $k_2 + 1 = 5$ rows of columns $l - k = 4$, $l = 6$ in $F(3, 4, 4)$ are related as the 5 rows of columns $4 - k_1 = 1$ and $6 - k_1 = 3$ in $F(4, 4)$. From this it follows that $M = 14$, and from the corollary it now follows, for instance, that $M(6, 2; 2, 2, 2, 3, 4, 4) = 14$ also.

3. Formulas for certain $M(l, k; k_1, k_2, \ldots, k_n)$**.** When $n = 1$, formulas for $M(l, k; b)$ are simple: $M(l, k; b) = 1$ for all l, k such that $1 \leqq k \leqq l \leqq b$.

THEOREM 3. *Suppose that b is a positive integer, that $n \geqq 2$, $k_1 = k_2 = \ldots = k_n = b$, $1 \leqq k \leqq l \leqq nb$, and $2l + 1 - k = nb$. Then with $C_n^{\,l} = 0$ if $l < 0$ or $l > (nb - 1)$ and otherwise defined by*

$$(1 + t + \ldots + t^{b-1})(1 + t + \ldots + t^b)^{n-1} = \sum_{l=0}^{nb-1} C_n^{\,l} t^l,$$

the formulas

$$M(l, k; nb) = C_n^l + C_{n-2}^{l-b} + C_{n-4}^{l-2b} + \ldots + \begin{cases} C_3^{l-((n-3)/2)b} + A(b, k) & \text{if } n \text{ is odd}, \\ C_2^{l-((n-2)/2)b} & \text{if } n \text{ is even}. \end{cases}$$

hold, where $M(l, k; nb)$ abbreviates the symbol $M(l, k; b, \ldots, b)$ having n bs, and

$$A(b, k) = \begin{cases} 0 & \text{if } l \geqq 2b, \\ (b - k + 1)/2 & \text{if } l < 2b. \end{cases}$$

Moreover, $M(l, k; nb) < |F_l(nb)|$, where $F_l(nb)$ abbreviates the symbol $F(b, b, \ldots, b)$ having n bs.

Proof. The proof is by induction on even ns and odd ns. Consider $n = 2$ and suppose that $1 \leqq k \leqq l \leqq 2b$ are such that $2l + 1 - k = 2b$. By Lemma 3, ${}_{l-k}N_k(r) \geqq 0$ if $r < b$ and ${}_{l-k}N_k(b) = 0$. From $2l + 1 - k = 2b$ and $k \geqq 1$ it follows that $l - k < l + (1/2) - (k/2) = b \leqq l$. Hence the $(b + 1)$st row of $F(2b)$ has 0- and 1-element in columns $l - k$ and l, respectively. Thus the number of elements in column l or on above row b is $M(l, k; 2b) < |F_l(2b)|$. But these elements (p_1, p_2) are exactly the compositions of l with two parts, $0 \leqq p_1 \leqq (b - 1)$, $0 \leqq p_2 \leqq b$. If C_2^l is the number of these compositions, $0 \leqq l \leqq 2b - 1$, then

$$(1 + t + \ldots + t^{b-1})(1 + t \ldots + t^b) = \sum_{l=0}^{2b-1} C_2^l t^l,$$

and so $M(l, k; 2b) = C_2^l$.

Next consider $n = 3$. Suppose that $1 \leqq k \leqq l \leqq 3b$ are such that $2l + 1 - k = 3b$. By Lemma 3, ${}_{l-k}N_l(r) \geqq 0$ if

$$r < b(b + 1) \qquad \text{and} \qquad {}_{l-k}N_l(b(b + 1)) = 0.$$

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If $l \geqq 2b$, then

$$l - k = l - (2l + 1 - 3b) = 3b - l - 1 \leqq b - 1 < b,$$

from which it follows that there are no elements in column $(l - k)$ after row $b(b + 1)$ while there are elements in column l after row $b(b + 1)$. This in turn shows that $M(l, k; 3b) = M$ is strictly less than $|F_l(3b)|$ and is the number of elements in column l on or above row $b(b + 1)$; i.e. M is the number of compositions (p_1, p_2, p_3) of l with $0 \leqq p_1 \leqq (b - 1)$ and $0 \leqq p_i \leqq b, i = 2, 3$. Thus $M = C_3^l$.

If $l < 2b$, then

$$0 \leqq -l - 1 + 2b = 3b - l - 1 - b = 2l + 1 - k - l - 1 - b$$
$$= l - k - b < l - b < b.$$

From $0 \leqq l - k - b < l - b < b$, the fact that the elements in columns $l - k$ and l of the last $b + 1$ rows of $F(3b)$ are arrayed exactly as the elements in columns $l - k - b$ and $l - b$ of the $b + 1$ rows of $F(2b)$, and the parallelogram form of $F(2b)$, it follows that $M(l, k; 3b)$ is the number of elements in the first $b(b + 1)$ rows of column l of $F(3b)$ plus

$$l - k - b + 1 = (b - k + 1)/2.$$

That is, $M(l, k; 3b) = C_3^l + A(b, k) < |F_l(3b)|$. Thus the induction is anchored for even and odd $n \geqq 2$. The induction can now be completed for both cases simultaneously.

With $n \geqq 4$, assume the theorem for $(n - 2)$-tuples, and consider n-tuples. By hypothesis, $2l + 1 - k = nb$, and so by Lemma 3, ${}_{l-k}N_l(r) \geqq 0$ if $r < b(b + 1)^{n-2}$ and ${}_{l-k}N_l(b(b + 1)^{n-2}) = 0$. In view of the way $F(nb)$ consists of $(b + 1)$ $F((n - 1)b)$-arrays, it follows that the elements in columns $l - k$ and l in rows $b(b + 1)^{n-2} + 1$ through $b(b + 1)^{n-2} + (b + 1)^{n-3}$ of $F(nb)$ are arrayed exactly as the elements in columns $l - k - b$ and $l - b$ in the $(b + 1)^{n-3}$ rows of $F((n - 2)b)$. Since $2(l - b) + 1 - k = (n - 2)b$ (it is at this point that we need $2l + 1 - k = nb$), we are entitled to write

$$M(l - b, k; (n - 2)b) = C_{n-2}^{l-b} + \ldots < |F_{l-b}((n - 2)b)|$$

by the induction hypothesis. Then $M(l, k; nb)$ is the number of elements in the first $b(b + 1)^{n-2}$ rows of $F(nb)$ (that is C_n^l) plus $M(l - b, k; (n - 2)b)$. Thus

$$M(l, k; nb) = C_n^l + C_{n-2}^{l-b} + \ldots < |F_l(nb)|,$$

and the theorem is proved.

In case $b = 1$, $2l + 1 - k$ is always a multiple n of b and $l + 1 \leqq l + 1 + l - k = n$; thus $l < n$. In this case the C_n^l are given by

$$(1 + t)^{n-1} = \sum_{l=0}^{n-1} C_n{}^l t^l,$$

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and so

$$C_n^l = \binom{n-1}{l} = \binom{2l-k}{l}.$$

If n is even, then

$$M(l, k; n1) = C_n^l + C_{n-2}^{l-1} + \ldots + C_2^{l-(n-2)/2}$$
$$= \binom{2l-k}{l} + \binom{2(l-1)-k}{l-1} + \ldots + \binom{2+k-1-k}{(2+k-1)/2}$$

where $\binom{i}{j} = 0$ if $j > i$. Hence

$$M(l, k; n1) = \binom{2l-k}{l} + \binom{2(l-1)-k}{l-1} + \ldots + \binom{k}{k}. \tag{13}$$

If $n \geqq 3$ is odd, then $2l \geqq 2l + 1 - k = n$, and so $l \geqq 3/2$, $l \geqq 2$, and $A(b, k) = 0$. Formula (13) now follows for odd $n \geqq 3$ also. If $n = 1$, (13) holds vacuously. Formula (13) was first given by Katona [**3**, p. 206].

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