1 Differentiable manifolds

We recall some basic concepts of differentiable manifolds, calculus, and classical geometry. This is not intended as an introduction, but to collect some important facts and to establish the notation. Chapter 2 will provide a lightning review of some basic concepts of Lie groups, including (co)adjoint orbits, which will play an important role later. However, most of this book is accessible without sophisticated mathematics, and readers familiar with the basic concepts can skip this section.

Let $\mathcal{M}$ be an n-dimensional *differentiable manifold* defined in terms of local coordinate charts and their transition functions. *Vector fields* V on $\mathcal{M}$ are best viewed as derivations acting on the algebra of smooth functions $\mathcal{C}(\mathcal{M})$, i.e.

$$V[fg] = fV[g] + gV[f], \qquad f, g \in \mathcal{C}(\mathcal{M}). \tag{1.0.1}$$

In terms of local coordinates x^μ, they can be written as $V = V^\mu \frac{\partial}{\partial x^\mu} \equiv V^\mu \partial_\mu$. Vector fields can also be viewed as sections of the *tangent bundle* $T\mathcal{M}$, which is dual to the *cotangent bundle* $T^*\mathcal{M}$, whose sections are one-forms $\alpha = \alpha_\mu dx^\mu \in \Omega^1(\mathcal{M})$.

Differential forms

The vector space of differential forms or k-forms $\Omega^k(\mathcal{M})$ on $\mathcal{M}$ consists of elements of the form

$$\omega = \frac{1}{k!}\omega_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \qquad \in \Omega^k(\mathcal{M}). \tag{1.0.2}$$

Here $\omega_{\mu_1 \ldots \mu_k}(x)$ is totally antisymmetric, and $\wedge$ denotes the antisymmetric wedge product

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu, \tag{1.0.3}$$

which is sometimes suppressed. The Einstein sum convention will be used throughout. This wedge product defines an algebra structure on the space of all differential forms, denoted by $\Omega^*(\mathcal{M})$. The *exterior derivative*

$$d\colon \ \Omega^n(\mathcal{M}) \to \Omega^{n+1}(\mathcal{M}) \tag{1.0.4}$$

is defined by

$$df = (\partial_\mu f)\, dx^\mu, \tag{1.0.5}$$

and the graded Leibniz rule,

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta, \tag{1.0.6}$$

https://doi.org/10.1017/9781009440776.002 Published online by Cambridge University Press

where α is a p-form and β is an arbitrary differential form, and imposing that $ddf = 0$ for $f \in \mathcal{C}(\mathcal{M})$. Then d satisfies more generally

$$d \circ d = 0. \tag{1.0.7}$$

In local coordinates,

$$d\omega = \frac{1}{k!} d\omega_{\mu_1 \ldots \mu_k} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \tag{1.0.8}$$

for $\omega \in \Omega^k(\mathcal{M})$ as in (1.0.2).

The *interior product* or contraction of a vector field $V = V^\mu \partial_\mu$ with a one-form $\alpha = \alpha_\mu dx^\mu$ is defined through the dual evaluation

$$i_V \alpha = \langle V, \alpha \rangle \ = V^\mu \alpha_\mu \qquad \in \mathcal{C}(\mathcal{M}), \tag{1.0.9}$$

which is extended to k-forms through

$$i_V(\alpha \wedge \beta) = (i_V \alpha) \wedge \beta + (-1)^p \alpha \wedge i_V \beta. \tag{1.0.10}$$

Here α is a p-form and β is an arbitrary differential form. In local coordinates, this takes the form

$$i_V\Big(\frac{1}{k!}\alpha_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}\Big) = \frac{1}{(k-1)!} V^\mu \alpha_{\mu \ldots \mu_k} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_k}. \tag{1.0.11}$$

Push-forward and pullback maps

Any smooth map

$$\phi: \quad \mathcal{M} \to \mathcal{N} \tag{1.0.12}$$

between two manifolds $\mathcal{M}$ and $\mathcal{N}$ defines by differentiation a *tangential map* or push-forward

$$d\phi: \quad T_x\mathcal{M} \to T_{\phi(x)}\mathcal{N}, \qquad (d\phi)(V)[f] = V[\phi^* f], \tag{1.0.13}$$

where $(\phi^* f)(y) = f(\phi(y))$ is the pullback of the function f from $\mathcal{N}$ to $\mathcal{M}$. Note that this map is a priori defined only point-wise. If $\phi: \mathcal{M} \to \mathcal{N}$ is a *diffeomorphism*, i.e. a bijective smooth map whose inverse is also smooth, then the push-forward defines a map from vector fields on $\mathcal{M}$ to vector fields on $\mathcal{N}$. If ϕ is not injective, then the push-forward of a vector field is not defined, since vectors on different points in $\mathcal{M}$ can be mapped to the same point in $\mathcal{N}$. This observation will play an important role in the higher-spin theories discussed in Section 5.

By duality, this push-forward map defines a *pullback map* for one-forms

$$\phi^*: \quad T^*_{\phi(x)}\mathcal{N} \to T^*_x\mathcal{M}, \tag{1.0.14}$$

through $\langle V, \phi^*\alpha \rangle = \langle (d\phi)(V), \alpha \rangle$, which extends to a map

$$\phi^*: \ \Omega^*(\mathcal{N}) \to \Omega^*(\mathcal{M}). \tag{1.0.15}$$

https://doi.org/10.1017/9781009440776.002 Published online by Cambridge University Press

In local coordinates x^μ on $\mathcal{N}$ and y^ν on $\mathcal{M}$, these maps reduce to the familiar covariant transformation laws for vectors and covectors. For example,

$$\begin{aligned}\phi^*(dx^\mu) &= \frac{\partial x^\mu}{\partial y^\nu}dy^\nu,\\ (d\phi)\Big(\frac{\partial}{\partial y^\mu}\Big) &= \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu},\end{aligned} \tag{1.0.16}$$

where $x^\mu(y) = \phi^* x^\mu = x^\mu(\phi(y))$ is understood.

Lie derivative

The *Lie derivative* $\mathcal{L}_V$ along a vector field V on $\mathcal{M}$ generalizes the action of V on functions $f \in \Omega^0(\mathcal{M})$ to an action on any forms $\omega \in \Omega^*(\mathcal{M})$. This is again a derivation, which satisfies

$$\begin{aligned}\mathcal{L}_V f &= V[f] = i_V df\\ \mathcal{L}_V(\alpha\wedge\beta) &= (\mathcal{L}_V\alpha)\wedge\beta + \alpha\wedge\mathcal{L}_V\beta\\ \mathcal{L}_V(d\omega) &= d(\mathcal{L}_V\omega), \qquad \omega\in\Omega^*(\mathcal{M}).\end{aligned} \tag{1.0.17}$$

Cartan's magic formula then states that

$$\mathcal{L}_V\omega = (di_V + i_V d)\omega. \tag{1.0.18}$$

This formalism is particularly useful in the context of symplectic manifolds, which will play a central role in this book.

The Lie derivative $\mathcal{L}_V$ can be extended to act also on vector fields and general tensor fields, but this requires a different perspective. The idea is that any vector field V on $\mathcal{M}$ defines a flow

$$\begin{aligned}\phi: \quad \mathbb{R}\times\mathcal{M} &\to \mathcal{M}\\ (t,x) &\mapsto \phi_t(x)\end{aligned} \tag{1.0.19}$$

through

$$\frac{d}{dt}\phi_t(x) = V(x) \quad \in T_x\mathcal{M}. \tag{1.0.20}$$

In other words, ϕ_t is the diffeomorphism that realizes the integral flow along the vector field V. This can be used to drag any tensorial fields along the flow via the differential map

$$d\phi_t: \quad T_x\mathcal{M} \to T_{\phi_t(x)}\mathcal{M} \tag{1.0.21}$$

and similar for $T^*\mathcal{M}$; note that flows are always invertible. Then the Lie derivative is simply the derivative of any tensor fields along this flow, where vectors at different points are subtracted after transporting them to the same point along the flow. This leads to the following explicit formulas:

$$\begin{aligned}\mathcal{L}_V W &= [V,W],\\ \mathcal{L}_V(X\otimes Y) &= \mathcal{L}_V X\otimes Y + X\otimes\mathcal{L}_V Y,\end{aligned} \tag{1.0.22}$$

https://doi.org/10.1017/9781009440776.002 Published online by Cambridge University Press

where $[V, W]$ is the Lie bracket or commutator of the vector fields V, W on $\mathcal{M}$, and X, Y are any tensor fields. In local coordinates x^μ on $\mathcal{M}$, this takes the form

$$(\mathcal{L}_V W)^\mu = V^\rho \partial_\rho W^\mu - W^\rho \partial_\rho V^\mu. \tag{1.0.23}$$

It is important to keep in mind the difference between the Lie derivative $\mathcal{L}_V$ and a connection ∇_V. By definition, a connection (such as the Levi–Civita connection defined in terms of a metric) satisfies $\nabla_{fV} = f\nabla_V$ for any $f \in \mathcal{C}(\mathcal{M})$, and therefore defines a tensor. In contrast, Lie derivations are not tensorial, since $\mathcal{L}_{fV} \neq f\mathcal{L}_V$ in general.

Bundles

In physics, a manifold $\mathcal{M}$ typically carries some extra structure, such as matter fields that live in some vector space over each point of $\mathcal{M}$ or gauge fields that allow to consistently differentiate these. Such structures are typically described by the notion of a bundle over $\mathcal{M}$. The general definition is as follows.

Definition 1.1 A *fiber bundle* is defined in terms of a base manifold $\mathcal{M}$, a total (or bundle) space $\mathcal{B}$ which is also a manifold, and a map $\Pi : \mathcal{B} \to \mathcal{M}$ such that the local structure of $\mathcal{B}$ is that of a product manifold,

$$\mathcal{B} \overset{\text{loc}}{\cong} \mathcal{M} \times \mathcal{F}. \tag{1.0.24}$$

Here $\mathcal{F}$ is called the fiber. All maps are understood to be smooth.

A *section* of a bundle is a map

$$\sigma : \quad \mathcal{M} \to \mathcal{B} \qquad \text{such that} \qquad \Pi \circ \sigma = \mathrm{id}_{\mathcal{M}}. \tag{1.0.25}$$

For example, vector fields on $\mathcal{M}$ can be viewed as sections of the tangent bundle $T\mathcal{M}$, and the space of such sections is denoted as $\Gamma(\mathcal{M})$.

A fiber bundle is called a *vector bundle* if $\mathcal{F}$ is a vector space, and it is called a *principal bundle* if $\mathcal{F}$ is a Lie group (cf. Section 2). Basic examples of vector bundles are the tangent bundle $T\mathcal{M}$ and the cotangent bundle $T^*\mathcal{M}$. In field theory, matter fields are typically described by sections of some vector bundle. Simple examples of principal bundles are $U(1)$ bundles, where the fiber is given by $S^1 \cong U(1)$. We will also encounter other types of bundles such as sphere bundles, where the fiber is a two-sphere $\mathcal{F} \cong S^2$.

1.1 Symplectic manifolds and Poisson structures

To motivate the notion of a symplectic manifold, consider a classical system whose configuration space is an n-dimensional manifold $\mathcal{N}$ with coordinates q^i, $i = 1, \ldots, n$, which in the simplest case is $\mathbb{R}^n$. In Hamiltonian mechanics, it is useful to consider the phase space

$$\mathcal{M} = T^*\mathcal{N} \tag{1.1.1}$$

https://doi.org/10.1017/9781009440776.002 Published online by Cambridge University Press

associated with $\mathcal{N}$. This is by definition the cotangent bundle over $\mathcal{N}$, which consists of all one-forms

$$\alpha = \sum_i \alpha_i(q) dq^i \tag{1.1.2}$$

on $\mathcal{N}$. The cotangent bundle $T^*\mathcal{N}$ captures not only the kinematical information as explained in Section 1.2, but it contains an extra structure, which does not exist on the tangent bundle $T\mathcal{N}$. To see this, we recall the canonical isomorphism $(V^\star)^\star \cong V$, where V is any vector space and $V^\star$ is its dual. Applying this isomorphism to the case of $V = T_q\mathcal{N}$ at some given point $q \in \mathcal{N}$, we obtain canonical maps

$$p_i : \mathcal{M} \to \mathbb{R}, \qquad p_i(dq^j) = \delta_i^j. \tag{1.1.3}$$

Thus, p_i recovers the coefficient functions of a one-form α (1.1.2) expanded in the basis dq^i. In this sense, the p_i are dual to the one-forms dq^i, and they are called canonical momenta. Together, the (q^i, p_j) form the so-called canonical coordinates on $\mathcal{M} = T^*\mathcal{N}$. This means that the p_i are canonically associated with q^i, i.e. there is no need for any extra structure such as a metric.

In particular, there is a canonical one-form on $\mathcal{M} = T^*\mathcal{N}$

$$\theta = \sum_i p_i dq^i = \sum_i \tilde{p}_i d\tilde{q}^i, \tag{1.1.4}$$

which[1] has the same form in any coordinates $\tilde{q}_i$ on $\mathcal{N}$. The exterior derivative of θ defines a canonical two-form ω on $\mathcal{M}$, which is automatically closed,

$$\omega = d\theta = dp_i \wedge dq^i \in \Omega^2(\mathcal{M}), \qquad d\omega = 0. \tag{1.1.5}$$

Recall that the Einstein sum convention is understood, so that $dp_i \wedge dq^i \equiv \sum_i dp_i \wedge dq^i$. Clearly, ω is also nondegenerate, so that ω is a *symplectic form* on $\mathcal{M}$, i.e. a closed nondegenerate two-form. The general definition is as follows.

Definition 1.2 A *symplectic manifold* is a manifold $\mathcal{M}$ equipped with a closed nondegenerate two-form ω.

Note that all symplectic manifolds have even dimension. For example, $\mathcal{M} = T^*\mathcal{N}$ is naturally a symplectic manifold, which is always noncompact; its relation with Hamiltonian mechanics will be recalled in Section 1.2. In this book, we will also encounter other types of symplectic manifolds, including compact symplectic manifolds.

The magic feature of symplectic forms is that they naturally define a Poisson bracket. To see this, consider local coordinates x^a on $\mathcal{M}$, and write ω as

$$\omega = \frac{1}{2}\omega_{ab} dx^a \wedge dx^b. \tag{1.1.6}$$

Recall that ω is closed $d\omega = 0$ if and only if

$$\partial_c\omega_{ab} + \partial_a\omega_{bc} + \partial_b\omega_{ca} = 0 \qquad \text{with} \qquad \omega_{ab} + \omega_{ba} = 0. \tag{1.1.7}$$

[1] In more abstract language, the defining property of θ is that $\alpha^*(\theta) = \alpha$ for any one-form $\alpha = p_i dq^i$ on $\mathcal{N}$. Here α^* is the pullback map associated with α, viewed as section $\alpha : \mathcal{N} \to T^*\mathcal{N} = \mathcal{M}$ of the cotangent bundle.

https://doi.org/10.1017/9781009440776.002 Published online by Cambridge University Press

Since ω_{ab} is nondegenerate, we can consider the inverse tensor field θ^{ab}

$$\theta^{ab}\omega_{bc} = \delta^a_c, \tag{1.1.8}$$

which is also antisymmetric. This defines a bivector field $\theta^{ab}\partial_a \otimes \partial_b$ on $\mathcal{M}$, or equivalently a bracket on the algebra $\mathcal{A} = \mathcal{C}^\infty(\mathcal{M})$ of functions on $\mathcal{M}$ via

$$\{f, g\} := \theta^{ab}\partial_a f \partial_b g. \tag{1.1.9}$$

It is easy to see that this bracket satisfies the following properties:

$$\begin{aligned}
&\{f, g\} = -\{g, f\} && \text{antisymmetry} \\
&\{f, g + \lambda h\} = \{f, g\} + \lambda\{f, h\} && \text{linearity} \\
&\{f, gh\} = g\{f, h\} + h\{f, g\} && \text{Leibniz (product) rule} \\
&\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 && \text{Jacobi identity}
\end{aligned} \tag{1.1.10}$$

for any $f, g, h \in \mathcal{A}$, and $\lambda \in \mathcal{R}$, which constitutes the definition of a Poisson bracket; the Jacobi identity follows from (1.1.7). In terms of local coordinates, the Jacobi identity takes the form

$$\theta^{ad}\partial_d\theta^{bc} + \theta^{bd}\partial_d\theta^{ca} + \theta^{cd}\partial_d\theta^{ab} = 0, \tag{1.1.11}$$

which is a consequence of (1.1.7).

This is a simple but profound result since the Jacobi relation is a nonlinear partial differential equation (PDE) for θ^{ab}. The framework of symplectic forms allows to recast this nonlinear PDE as a linear PDE (1.1.7) for ω_{ab}, which is much easier to handle. Hence, symplectic forms allow to understand and classify nondegenerate Poisson brackets. In particular, the canonical variables q^i and p_j on $\mathcal{M} = T^*\mathcal{N}$ satisfy the following Poisson brackets:

$$\begin{aligned}
\{q^i, p_j\} &= \delta^i_j \\
\{q^i, q^j\} &= 0 = \{p_i, p_j\}.
\end{aligned} \tag{1.1.12}$$

The following important theorem states that this "canonical" form of the Poisson brackets can always be achieved locally for every symplectic manifold.

Theorem 1.3 (Darboux) *For any point x in a symplectic manifold $\mathcal{M}$, there is an open neighborhood $\mathcal{U} \ni x$ and local coordinates $q^i, p_j \colon \mathcal{U} \to \mathbb{R}$ such that*

$$\omega = \sum_i dp_i \wedge dq^i. \tag{1.1.13}$$

In these local coordinates, the Poisson brackets take the form (1.1.12).

We introduce some further concepts and notation. A Poisson manifold is a manifold $\mathcal{M}$ carrying a Poisson structure, i.e. a bracket satisfying the relations (1.1.10). A Poisson structure can be conveniently encoded as a bivector field $\theta^{ab}\partial_a f \partial_b g$, where

$$\theta^{ab} = \{x^a, x^b\} \tag{1.1.14}$$

in local coordinates x^a. The θ^{ab} will be denoted as Poisson tensor.

https://doi.org/10.1017/9781009440776.002 Published online by Cambridge University Press

If the tensor field θ^{ab} is nondegenerate, we can invert it and obtain a symplectic structure. Then Darboux theorem implies that, locally, the Poisson tensor takes the standard form

$$\theta^{ab} = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{1.1.15}$$

in canonical coordinates $x^a = (p_i, q^i)$. If the Poisson tensor is degenerate and hence not invertible, one can still use it to define a *symplectic foliation* into *symplectic leaves*. The general statement is that any (regular, finite-dimensional) Poisson manifold is the disjoint union of its symplectic leaves.[2] For a more detailed discussion, see e.g. [137].

More generally, a (commutative) Poisson algebra $\mathcal{A}$ is an algebra together with a bracket

$$\{.,.\}: \quad \mathcal{A} \times \mathcal{A} \to \mathcal{A} \tag{1.1.16}$$

such that the relations (1.1.10) are satisfied. The most important case is where $\mathcal{A} = \mathcal{C}(\mathcal{M})$ is the algebra of (smooth, typically) functions on some manifold $\mathcal{M}$. Then the Poisson bracket can be identified with a bivector field $\theta^{ab}\partial_a f \partial_b g$.

Another useful result related to Darboux's theorem is Moser's lemma, which is a statement about local deformations of a symplectic structure. The statement is the following.

Lemma 1.4 (Moser) *Let ω_t be a family of symplectic forms depending smoothly on $t \in \mathbb{R}$. Then for any point $p \in \mathcal{M}$, there exists a local neighborhood $\mathcal{U}$ of p and a family of maps $g_t: \mathcal{U} \to \mathcal{U}$ with $g_0 = \mathrm{id}$ and $g_t^*\omega_t = \omega_0$.*

This means that any smooth deformation of a symplectic form can be absorbed locally by some diffeomorphism, which is the essence of Darboux's theorem. The idea of the proof is to consider the infinitesimal version of this relation, given by

$$0 = \frac{d}{dt}(g_t^*\omega_t) = g_t^*\big(\mathcal{L}_{V_t}\omega_t + \frac{d}{dt}\omega_t\big), \tag{1.1.17}$$

where the vector field V_t generates g_t. Since ω_t is closed, this reduces to $0 = i_{V_t}\omega_t + \mu_t$ using Cartan's magic formula (1.0.18), where μ_t is defined locally by $\frac{d}{dt}\omega_t = d\mu_t$ via Poincare's lemma. This can be solved for V_t as ω_t is nondegenerate, which upon integration yields g_t.

1.1.1 Hamiltonian vector fields

For any Poisson manifold, one can define an important class of special vector fields as follows.

Definition 1.5 For any $f \in \mathcal{C}(\mathcal{M})$, the *Hamiltonian vector field* V_f is defined by

$$V_f[g] := \{f, g\} \qquad \forall\, g \in \mathcal{C}(\mathcal{M}). \tag{1.1.18}$$

[2] This is proved using Frobenius' theorem, using the fact that Hamiltonian vector fields are always in involution.

https://doi.org/10.1017/9781009440776.002 Published online by Cambridge University Press

We use here the fact that vector fields are naturally identified with derivations of the algebra $\mathcal{C}(\mathcal{M})$ of smooth functions on a manifold. If the Poisson structure is nondegenerate, then (1.1.18) is equivalent to the well-known relation

$$i_{V_f}\omega = df, \tag{1.1.19}$$

where $i_V\omega$ is the contraction of the vector field V with the two-form ω. Indeed writing the Hamiltonian vector field $V_f = (V_f)^a\partial_a$ in local coordinates, this gives

$$i_{V_f}\omega = (V_f)^a\omega_{ab}\,dx^b \stackrel{!}{=} df = \partial_b f dx^b. \tag{1.1.20}$$

Hence,

$$(V_f)^a = (\omega^{-1})^{ba}\partial_b f = \theta^{ba}\partial_b f = \{f, x^a\} \tag{1.1.21}$$

using (1.1.8), consistent with (1.1.18). The relation between symplectic forms and Poisson brackets can now be stated in a coordinate-free form as follows:

$$\{f, g\} = \omega(V_g, V_f), \tag{1.1.22}$$

which in local coordinates reduces to

$$\theta^{ab}\partial_a f\partial_b g = \omega_{ab}\theta^{ca}\theta^{db}\partial_c f\partial_d g. \tag{1.1.23}$$

In particular, Cartan's magic formula $\mathcal{L}_V\omega = (i_V d + di_V)\omega$ gives immediately Liouville's theorem.

Theorem 1.6 (Liouville)

$$\mathcal{L}_{V_f}\omega = 0 = \mathcal{L}_{V_f}\omega^{\wedge n} \tag{1.1.24}$$

for any Hamiltonian vector field V_f on a symplectic manifold $\mathcal{M}$ of dimension $2n$.

The second statement implies that any symplectic manifold is equipped with a natural volume form

$$\Omega := \frac{1}{n!}\omega^{\wedge n} = \rho_M(x)d^{2n}x, \tag{1.1.25}$$

which is preserved by Hamiltonian vector fields; we will often write $\omega^n \equiv \omega^{\wedge n}$. Here

$$\rho_M(x) := \mathrm{pf}(\omega) = \frac{1}{2^n n!}\varepsilon^{a_1 b_1 \ldots a_n b_n}\omega_{a_1 b_1}\ldots\omega_{a_n b_n} = \sqrt{\det\omega_{ab}} \tag{1.1.26}$$

is the symplectic density on $\mathcal{M}$, which is given by the Pfaffian $\mathrm{pf}(\omega)$ of the antisymmetric matrix ω_{ab}. The relation with the determinant can be seen by writing ω_{ab} in block-diagonal form, noting that both the Pfaffian and the determinant factorize.

The first statement of (1.1.24) also deserves some discussion: it means that the symplectic structure is invariant under Hamiltonian vector fields:

$$\mathcal{L}_{V_f}\omega = 0, \qquad \mathcal{L}_{V_f}\theta^{ab} = 0. \tag{1.1.27}$$

Therefore, the flow defined by any Hamiltonian vector field is a *symplectomorphism*, i.e. a diffeomorphism that leaves ω invariant.

We conclude this brief introduction with some important observations. First, the Jacobi identity implies the following useful identity for any Poisson tensor:

$$0 = \partial_a(\rho_M \theta^{ab}), \tag{1.1.28}$$

where ρ_M is the symplectic volume density (1.1.26). To see this, consider

$$\begin{aligned}\partial_a\theta^{ab} &= -\,\theta^{aa'}\partial_a\theta^{-1}_{a'b'}\theta^{b'b} = \theta^{aa'}\theta^{b'b}(\partial_{a'}\theta^{-1}_{b'a} + \partial_{b'}\theta^{-1}_{aa'})\\ &= -\theta^{aa'}\partial_{a'}\theta^{b'b}\theta^{-1}_{b'a} - \theta^{b'b}\partial_{b'}\theta^{aa'}\theta^{-1}_{aa'}\\ &= -\partial_a\theta^{ab} - 2\theta^{ab}\rho_M^{-1}\partial_a\rho_M,\end{aligned} \tag{1.1.29}$$

noting that $2\rho_M^{-1}\partial_b\rho_M = \partial_b\theta^{aa'}\theta^{-1}_{aa'}$ and using the fact that $\omega_{ab} = \theta^{-1}_{ab}$ is closed in the second step. Then (1.1.28) follows. As an application of this formula, we note that all Hamiltonian vector fields satisfy the following divergence constraint:

$$\partial_a(\rho_M V_f^a) = -\partial_a(\rho_M\theta^{ab}\partial_b f) = 0, \tag{1.1.30}$$

which expresses the fact that V_f preserves $\Omega = \rho_M d^{2n}x$. Another way to see this is by computing (1.1.24) directly:

$$\begin{aligned}0 = \mathcal{L}_V\Omega = d(i_V\Omega) &= \frac{1}{(2n-1)!}d(\rho_M V^b\varepsilon_{ba_2\ldots a_{2n}}dx^{a_2}\ldots dx^{a_{2n}})\\ &= \frac{1}{(2n-1)!}\varepsilon_{ba_2\ldots a_{2n}}\partial_c(\rho_M V^b)dx^c dx^{a_2}\ldots dx^{a_{2n}}\\ &= \partial_b(\rho_M V^b)\Omega\end{aligned} \tag{1.1.31}$$

since $\varepsilon_{ba_2\ldots a_{2n}}\varepsilon^{ca_2\ldots a_{2n}} = (2n-1)!\,\delta^c_b$.

Finally, we note the important identity

$$[V_f, V_g] = V_{\{f,g\}}, \tag{1.1.32}$$

where the left-hand side is the Lie bracket of the Hamiltonian vector fields associated with f and g. This relation follows from the Jacobi identity, upon acting on some test function h:

$$[V_f, V_g][h] = \{f,\{g,h\}\} - \{g,\{f,h\}\} = \{\{f,g\},h\} = V_{\{f,g\}}[h]. \tag{1.1.33}$$

It means that the Hamiltonian vector fields satisfy the same Lie algebra as the Poisson brackets of their generators. In more abstract terms, the map $f \to V_f$ is a Lie algebra homomorphism from the (Poisson) Lie algebra on the space of functions $\mathcal{C}(\mathcal{M})$ on $\mathcal{M}$ to the Lie algebra of vector fields on $\mathcal{M}$. These relations are cornerstones of Hamiltonian mechanics, and they will play an important role in the following.

Exercise 1.1.1 Verify the Jacobi identity (1.1.11) explicitly using $d\omega = 0$ (1.1.7).

1.2 The relation with Hamiltonian mechanics

We have seen that the symplectic structure of the cotangent bundle $T^*\mathcal{N}$ is canonical, i.e. independent of any extra structure such as a Lagrangian. It is nevertheless worthwhile to

https://doi.org/10.1017/9781009440776.002 Published online by Cambridge University Press

recall the relation with Lagrangian and Hamiltonian mechanics. Assume that some physical system is described through some Lagrangian $\mathcal{L}(q^i, v^j)$, which is a function on $T\mathcal{N}$. Here q^i are local coordinates on $\mathcal{N}$, and $v^j = \dot{q}^j$ are the associated tangent space coordinates. We assume that $\mathcal{L}$ is convex in the velocities v^j, which is tantamount to stability. Then the (fiber-wise) map $\phi\colon\ T_q\mathcal{N} \to T_q^*\mathcal{N}$ defined through

$$\phi(v)[w] = \frac{d}{ds}\Big|_0 \mathcal{L}(q, v + sw) \tag{1.2.1}$$

is invertible. Explicitly, v^i is expressed in terms of p_j via

$$p_i = \frac{\partial \mathcal{L}}{\partial v^i}, \tag{1.2.2}$$

where p_i is the canonical momentum (1.1.3). Then the Legendre transformation maps the function $\mathcal{L}$ on $T\mathcal{N}$ to the function H on $T^*\mathcal{N}$ via

$$\begin{aligned} \mathcal{C}(T\mathcal{N}) &\to \mathcal{C}(T^*\mathcal{N}) \\ \mathcal{L}(q^i, v^j) &\mapsto H(q^i, p_j) := p_i \phi_*(v^i) - \phi_*\big(\mathcal{L}(q^i, v^i)\big). \end{aligned} \tag{1.2.3}$$

Here ϕ_* indicates the identification via ϕ by substituting (1.2.2). The Legendre transformation is involutive, i.e. it is its own inverse, and maps convex functions to convex functions. The Euler–Lagrange equations can then be recast in Hamiltonian form,

$$\dot{f} = \{f, H\} = -V_H[f], \tag{1.2.4}$$

for any function f on $T^*\mathcal{N}$, with canonical Poisson structure given by (1.1.5). This formulation leads to powerful tools such as canonical transformations (which by definition preserve the Poisson or symplectic structure) in Hamiltonian mechanics, and it is essential to understand the relation with quantum mechanics.