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ON PERIODIC GROUPS HAVING ALMOST REGULAR 2-ELEMENTS*

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We show that if a periodic residually-finite group G has a 2-element with finite centralizer then G is locally finite.

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1. Introduction

An element a of a group G is called *almost regular* if $C_G(a)$ is finite. In some situations existence of an almost regular element in G implies very strong conclusions about G. For example Šunkov proved in **[14]** that a periodic group G having an almost regular element of order two is locally finite (and even contains a solvable subgroup of finite index). Two other proofs of this result can be found in **[2]** and **[7]**. Locally finite groups having an almost regular element of arbitrary prime order have been studied intensively in the seventies and eighties (see for example **[7]**). Khukhro showed that these groups have a nilpotent subgroup of finite index **[10]**. In general, a very interesting direction in locally finite group theory is to classify in some sense locally finite groups G having a finite subgroup A such that $C_G(A)$ possesses some prescribed property, as for example the property to be a linear group **[8]**.

In this paper we are interested in the following question, which is natural in view of the result of Šunkov.

Given a periodic group G with an almost regular element, under what conditions does it follow that G is locally finite?

According to Ol'shanski, for any positive integer n which has at least one odd divisor there exists an infinite group G having an almost regular element of order n such that all proper subgroups of G are finite **[12]**. Therefore the result of Šunkov cannot be extended to periodic groups having an almost regular element whose order is not a 2-power.

In this paper we consider periodic residually-finite groups having an almost regular element of order 2^n. Note that groups constructed in **[1, 4, 5, 6, 13]** are finitely-generated periodic residually-finite and infinite. We will prove here the following result.

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Theorem A. *Let G be a periodic residually-finite group having an almost regular 2-element. Then G is locally finite.*

The above theorem will be derived from the following closely related result.

Proposition B. *Let G be a periodic residually-nilpotent group admitting a fixed-point-free automorphism of order 2^n. Then G is locally finite.*

2. Proof of Proposition B

Let G be a periodic group. For any element $x \in G$ of odd order let us use $x^{\frac{1}{2}}$ to denote the element $y \in \langle x \rangle$ such that $y^2 = x$. If G is a periodic $2'$-group and ϕ is an involutory automorphism of G then $x(x^{-\phi}x)^{-\frac{1}{2}} \in C_G(\phi)$. Indeed, let $h = x(x^{-\phi}x)^{-\frac{1}{2}}$. Note that

$$(x^{-\phi}x)^{\frac{1}{2}} = x^{-\phi}x(x^{-\phi}x)^{-\frac{1}{2}}.$$

We have

$$h^{\phi} = x^{\phi}(x^{-1}x^{\phi})^{-\frac{1}{2}} = x^{\phi}(x^{-\phi}x)^{\frac{1}{2}} = x^{\phi}x^{-\phi}x(x^{-\phi}x)^{-\frac{1}{2}} = x(x^{-\phi}x)^{-\frac{1}{2}} = h.$$

Lemma 2.1. *Let G be a periodic group acted on by a finite 2-group A. Suppose that $C_G(A)$ contains no 2-element. Then G is a $2'$-group.*

Proof. We will use induction on $|A|$. Suppose first that A is of order two and let ϕ be the involution in A. If G contains elements of even order there exists an involution $\tau \in G$. If $\tau \cdot \tau^{\phi}$ is of even order then the involution from $\langle \tau \cdot \tau^{\phi} \rangle$ is contained in $C_G(A)$, a contradiction. Assume that the order of $\tau \cdot \tau^{\phi}$ is odd. Then $(\tau^{\phi} \cdot \tau)^{\frac{1}{2}} \cdot \tau$ is an involution contained in $C_G(A)$.

Let now $|A| \geq 4$ and ϕ an involution in $Z(A)$. Set $H = C_G(\phi)$. By the preceding paragraph, if G contains elements of even order then so does H. Obviously, H is A-invariant and A induces a group of automorphisms of H whose order is strictly less than that of A. Now the induction hypothesis yields that $C_H(A)$ contains elements of even order. The lemma follows. □

Lemma 2.2. *Let G be a periodic $2'$-group acted on by a finite 2-group A. Suppose that N is a normal A-invariant subgroup of G. Then*

$$C_G(A)N/N = C_{G/N}(A).$$

Proof. Let $|A| = 2^n$. Using induction on n we will show that

$$C_G(A)N/N \geq C_{G/N}(A).$$

Suppose first that $n = 1$ and let ϕ be the involution in A. Let $x \in G$ and $xN \in C_{G/N}(A)$. Then $x^{-\phi}x \in N$. Since

$$x = x(x^{-\phi}x)^{-\frac{1}{2}}(x^{-\phi}x)^{\frac{1}{2}} \text{ and } x(x^{-\phi}x)^{-\frac{1}{2}} \in C_G(\phi),$$

we obtain that $xN \in C_G(A)N/N$.

Let now n be arbitrary and ϕ an involution in $Z(A)$. Set $H = C_G(\phi)$. Let again $x \in G$ and $xN \in C_{G/N}(A)$. Then $(x^{-\phi}x)^{\frac{1}{2}} \in N$. As above we have

$$x = x(x^{-\phi}x)^{-\frac{1}{2}}(x^{-\phi}x)^{\frac{1}{2}} \text{ and } y = x(x^{-\phi}x)^{-\frac{1}{2}} \in H.$$

It is easy to see that

$$y(H \cap N) \in C_{H/H \cap N}(A).$$

Since the automorphism group of H induced by A has order strictly less than $|A|$, we are in a position to apply the induction hypothesis. This yields that $y = ht$, where $h \in C_H(A)$, $t \in H \cap N$. Recall that $x = y(x^{-\phi}x)^{\frac{1}{2}}$. It follows that $xN \in C_G(A)N/N$.

Thus, we showed that $C_{G/N}(A) \leq C_G(A)N/N$. The reverse inclusion is obvious. □

Let now G be any group and p a prime. For $i \geq 1$ set

$$D_i(G) = \prod_{jp^k \geq i} \gamma_j(G)^{p^k},$$

where $\gamma_j(G)$ stands for the j-th term of the lower central series of G and for any subgroup $H \leq G$ the symbol H^{p^k} denotes the subgroup of H generated by the set $\{h^{p^k} \mid h \in H\}$. It follows that $\gamma_i(G) \leq D_i$ for all $i \geq 1$ and that $(D_i(G))_{i \geq 1}$ is a descending series of characteristic subgroups of G. This series is called the Lazard p-series of G.

Lemma 2.3 [**9**, p. 250]. *For any group G and all $i, j \geq 1$ we have*

$$[D_i(G), D_j(G)] \leq D_{i+j}(G).$$

The above lemma shows that the Lazard p-series $(D_i(G))_{i \geq 1}$ is a strongly central series of G in the sense of [**9**].

Given a group G and a prime p we shall associate the Lie algebra $L_p(G)$ to G. Consider the series

$$G = D_1(G) \geq D_2(G) \geq \ldots .$$

For all $i \geq 1$ the factor group $D_i(G)/D_{i+1}(G)$ can be viewed as a vector space over the field with p elements $\mathbb{F}_p$. Let $L(G)$ denote their direct sum,

$$L(G) = \bigoplus_{i=1}^{\infty} D_i(G)/D_{i+1}(G).$$

To avoid overloading the notation, we write D_i for $D_i(G)$. For arbitrary cosets $aD_{i+1} \in D_i/D_{i+1}$ and $bD_{j+1} \in D_j/D_{j+1}$ we define a bracket product

$$[aD_{i+1}; bD_{j+1}] = [a, b]D_{i+j+1}. \tag{2.4}$$

where $[a, b]$ denotes the group commutator $a^{-1}b^{-1}ab$. Lemma 2.3 implies that the product above is well defined in the sense that the right-hand side of 2.4 does not depend on the coset representatives a, b. Extending now the product 2.4 linearly to the whole $L(G)$, we give $L(G)$ the structure of a Lie algebra over the field $\mathbb{F}_p$. The subalgebra of $L(G)$ generated by D_1/D_2 will be denoted by $L_p(G)$.

If G is a finitely-generated residually-finite p-group such that $L_p(G)$ satisfies a non-trivial polynomial identity, then by a deep result of Zelmanov G is finite [**15**, Theorem 1.6].

Proposition B. *Let G be a periodic residually-nilpotent group admitting a fixed-point-free automorphism ϕ of order 2^n. Then G is locally finite.*

Proof. We may assume that G is a finitely-generated p-group. Obviously ϕ induces an automorphism of the Lie algebra $L_p(G)$. By Lemma 2.1 p is odd, so Lemma 2.2 guarantees that ϕ induces a fixed-point-free automorphism of every quotient $D_i(G)/D_{i+1}$; $i = 1, 2, \ldots$. This yields that the induced automorphism of $L_p(G)$ has no non-zero fixed element. A theorem of Kreknin [**11**] now tells us that $L_p(G)$ is solvable. Hence by Zelmanov's Theorem cited above [**15**] G must be finite. □

3. Proof of Theorem A

Let G be an arbitrary periodic group. We denote by $|x|$ the order of element $x \in G$. Set $T_1(G) = G$, and for $i = 1, 2, \ldots$ define

$$T_{i+1} = \langle [x, y]; x, y \in T_i(G), (|x|, |y|) = 1 \rangle.$$

Obviously we have

$$T_1(G) \geq T_2(G) \geq \cdots \geq T_i(G) \geq T_{i+1}(G) \geq \ldots.$$

Lemma 3.1. *Let G be a periodic group, N a normal subgroup of G. Then for any positive integer k we have*

$$T_k(G/N) = T_k(G)N/N.$$

Proof. Since $T_{k+1}(G) = T_2(T_k(G))$, it suffices to prove the lemma only for $k = 2$.

Certainly, if x and y have relatively prime orders in G, then xN and yN have relatively prime orders in G/N. Therefore the inclusion $T_2(G/N) \geq T_2(G)N/N$ is obvious.

To prove that $T_2(G/N) \leq T_2(G)N/N$ let us take $xN, yN \in G/N$ of relatively prime orders in G/N and show that $[x, y] \in T_2(G)N$.

Let π_1 be the set of prime divisors of $|xN|$ and π_2 the set of prime divisors of $|yN|$. By assumption $\pi_1 \cap \pi_2 = \emptyset$. Let $|x| = m_1, n_1$, $|y| = m_2 n_2$, where n_1 is the maximal π_1-divisor of $|x|$ and n_2 is the maximal π_2-divisor of $|y|$. Then $\langle x^{m_1}N\rangle = \langle xN\rangle$ and $\langle y^{m_2}N\rangle = \langle yN\rangle$. Therefore there exist integers i, j such that $x \in x^{im_1}N$, and $y \in y^{jm_2}N$. The orders of x^{im_1} and y^{jm_2} are relatively prime. Therefore

$$[x, y] \in [x^{im_1}, y^{jm_2}]N \in T_2(G)N,$$

as required. □

Given a periodic group G, let $t(G)$ denote the minimal number i (possibly ∞) such that $T_{i+1}(G) = 1$. Recall that if G is a finite solvable group, then the *Fitting height* $h(G)$ of G is defined as follows. Let $F(G)$ denote the Fitting subgroup of G, i.e., the subgroup generated by all normal nilpotent subgroups of G. Set

$$F_0(G) = 1, F_{i+1}(G)/F_i(G) = F(G/F_i(G)),\ i = 1, 2, \ldots.$$

Then $h(G)$ is the minimal number h such that $F_h(G) = G$.

Lemma 3.2. *Let G be a finite solvable group. Then $t(G) = h(G)$.*

Proof. Let N be the minimal normal subgroup of G such that G/N is nilpotent. Then, obviously, $T_2(G) = N$. Now use induction on $t(G)$ along with the equalities $t(T_2(G)) = t(G) - 1$ and $h(N) = h(G) - 1$. □

Lemma 3.3. *Let G be a periodic residually-finite group admitting a fixed-point-free automorphism ϕ or order 2^n. Then $t(G) \leq n$.*

Proof. Suppose that $t(G) \geq n + 1$, i.e., $T_{n+2} \neq 1$. Since G is residually-finite, there exists a ϕ-invariant normal subgroup N of finite index in G such that $T_{n+2}(G) \not\leq N$. By Lemma 3.1 $T_{n+2}(G/N) \neq 1$. We remark that by 2.1 and 2.2 G/N is a finite group of odd order on which ϕ acts without non-trivial fixed points. By a result of Th. Berger [3] $h(G/N) \leq n$. Now the previous lemma yields $t(G/N) \leq n$, a contradiction. □

Proof of Theorem A. Given an almost regular element of order 2^n in a periodic residually-finite group G, let ϕ denote the inner automorphism of G induced by this element. Of course, without any loss of generality we may assume that G is finitely-generated. Since $C_G(\phi)$ is finite and G is residually-finite, there exists a normal ϕ-invariant subgroup H of G such that $|G:H| < \infty$ and $H \cap C_G(\phi) = 1$. By Lemma 2.1 H is a $2'$-group. Applying Lemma 3.3 with H in place of G, we conclude that $t = t(H) \leq n$. Suppose first that $t = 1$. In this case H is a direct product of maximal p-subgroups. Proposition B yields that each ϕ-invariant p-subgroup of H is locally finite and, therefore, so is H. Since H is of finite index in G, it follows that G is locally finite and we are done.

Thus, without loss of any generality we may assume that $t \geq 2$ and use induction on t. By the induction hypothesis assume that $T_2(H)$ is locally finite. Set $N = T_2(H)$. Then H/N is a direct product of its maximal p-subgroups. By Lemma 2.2 this group admits a fixed-point-free 2-automorphism. To use Proposition B now we need to know that H/N is residually-finite: but this can be false. So, let H_0 be the intersection of all normal subgroups of finite index in H which contain N. Then H/H_0 is a direct product of its maximal p-subgroups (because $N \leq H_0$) and is residually-finite (by the definition of H_0). Lemma 2.2 shows that H/H_0 admits a fixed-point-free automorphism whose order divides 2^n. Now Proposition B implies that H/H_0 is locally finite. Since H is a subgroup of finite index of a finitely-generated group, H/H_0 also is finitely-generated. Thus, we derive that H/H_0 is finite. In particular each subgroup of H_0 having finite index in H_0 has also finite index in H. Now the definition of H_0 implies that for any subgroup K of finite index in H_0 we have $NK = H_0$. It follows that each finite quotient of H_0 is isomorphic to a finite quotient of N. Since $N = T_2(H)$, it follows that $t(N) = t - 1$. We conclude now that for any finite quotient $\bar{H}$ of H_0 we have $t(\bar{H}) \leq t - 1$. Since H_0 is residually-finite, Lemma 3.1 yields now that $t(H_0) \leq t - 1$. By the induction hypothesis we derive now that H_0 is locally finite. Due to the fact that H_0 has finite index in G it follows that G is locally finite. The proof is complete. □

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