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Shadowing and the basins of terminal chain components

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Abstract. We provide an alternative view of some results in [1, 3, 11]. In particular, we prove that (1) if a continuous self-map of a compact metric space has the shadowing, then the union of the basins of terminal chain components is a dense G_δ-subset of the space; and (2) if a continuous self-map of a locally connected compact metric space has the shadowing, and if the chain recurrent set is totally disconnected, then the map is almost chain continuous.

1 Introduction

Shadowing is an important concept in the topological theory of dynamical systems (see [5, 18] for background). It was derived from the study of hyperbolic differentiable dynamics [4, 6] and generally refers to a situation in which coarse orbits, or *pseudo-orbits*, can be approximated by true orbits. Above all else, it is worth mentioning that the shadowing is known to be *generic* in the space of homeomorphisms or continuous self-maps of a closed differentiable manifold (see [19] and Theorem 1 of [16]) and so plays a significant role in the study of topologically generic dynamics.

Chain components are basic objects for global understanding of dynamical systems [9]. In this paper, we focus on attractor-like, or *terminal*, chain components and the basins of them. By a result (Corollary 6.16) of [11], if a continuous flow on a compact metric space has the so-called *weak shadowing*, then the union of the basins of terminal chain components is a dense G_δ-subset of the space. For any continuous self-map of a compact metric space, we strengthen it by assuming the standard shadowing (Theorem 1.1). Our proof is by a method related to but independent of a result (Proposition 22 in Section 7) of [1]. It is shown in [3] that topologically generic homeomorphisms of a closed differentiable manifold are almost chain continuous (see Introduction of [3] where the word "almost equicontinuous" is used). We also give an alternative proof of this fact by using the genericity of shadowing.

First, we define the chain components. Throughout, X denotes a compact metric space endowed with a metric d.

Definition 1.1 Given a continuous map $f: X \to X$ and $\delta > 0$, a finite sequence $(x_i)_{i=0}^k$ of points in X, where $k > 0$ is a positive integer, is called a *δ-chain* of f if

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$d(f(x_i), x_{i+1}) \leq \delta$ for every $0 \leq i \leq k-1$. A δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x_k$ is said to be a *δ-cycle* of *f*.

Let $f: X \to X$ be a continuous map. For any $x, y \in X$ and $\delta > 0$, the notation $x \to_\delta y$ means that there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$ and $x_k = y$. We write $x \to y$ if $x \to_\delta y$ for all $\delta > 0$. We say that $x \in X$ is a *chain recurrent point* for f if $x \to x$, or equivalently, for every $\delta > 0$, there is a δ-cycle $(x_i)_{i=0}^k$ of f with $x_0 = x_k = x$. Let $CR(f)$ denote the set of chain recurrent points for *f*. We define a relation $\leftrightarrow$ in

$$CR(f)^2 = CR(f) \times CR(f)$$

by the following: for any $x, y \in CR(f)$, $x \leftrightarrow y$ if and only if $x \to y$ and $y \to x$. Note that $\leftrightarrow$ is a closed equivalence relation in $CR(f)^2$ and satisfies $x \leftrightarrow f(x)$ for all $x \in CR(f)$. An equivalence class C of $\leftrightarrow$ is called a *chain component* for *f*. We regard the quotient space

$$\mathcal{C}(f) = CR(f)/\leftrightarrow$$

as a space of chain components.

A subset S of X is said to be f-invariant if $f(S) \subset S$. For an f-invariant subset S of X, we say that $f|_S: S \to S$ is *chain transitive* if for any $x, y \in S$ and $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$ of $f|_S$ with $x_0 = x$ and $x_k = y$.

Remark 1.1 The following properties hold:

- $CR(f) = \bigsqcup_{C \in \mathcal{C}(f)} C$,
- every $C \in \mathcal{C}(f)$ is a closed f-invariant subset of $CR(f)$,
- $f|_C: C \to C$ is chain transitive for all $C \in \mathcal{C}(f)$,
- for any f-invariant subset S of X, if $f|_S: S \to S$ is chain transitive, then $S \subset C$ for some $C \in \mathcal{C}(f)$.

Next, we recall the definition of terminal chain components. For $x \in X$ and a subset S of X, we denote by $d(x, S)$ the distance of x from S:

$$d(x, S) = \inf_{y \in S} d(x, y).$$

Definition 1.2 We say that a closed f-invariant subset S of X is *chain stable* if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-chain $(x_i)_{i=0}^k$ of f with $x_0 \in S$ satisfies $d(x_i, S) \leq \varepsilon$ for all $0 \leq i \leq k$. Following [3], we say that $C \in \mathcal{C}(f)$ is *terminal* if C is chain stable. We denote by $\mathcal{C}_{\mathrm{ter}}(f)$ the set of terminal chain components for *f*.

Remark 1.2 For any continuous map $f: X \to X$, a partial order $\leq$ on $\mathcal{C}(f)$ is defined by the following: for all $C, D \in \mathcal{C}(f)$, $C \leq D$ if and only if $x \to y$ for some $x \in C$ and $y \in D$. We can easily show that for any $C \in \mathcal{C}(f)$, $C \in \mathcal{C}_{\mathrm{ter}}(f)$ if and only if C is maximal with respect to $\leq$; that is, $C \leq D$ implies $C = D$ for all $D \in \mathcal{C}(f)$.

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Given a continuous map $f\colon X \to X$ and $x \in X$, the *ω-limit set* $\omega(x, f)$ of x for f is defined as the set of $y \in X$ such that

$$\lim_{j\to\infty} f^{i_j}(x) = y$$

for some sequence $0 \le i_1 < i_2 < \cdots$. Note that $\omega(x, f)$ is a closed f-invariant subset of X and $f|_{\omega(x,f)}\colon \omega(x, f) \to \omega(x, f)$ is chain transitive. We denote by $C(x, f)$ the unique $C(x, f) \in \mathcal{C}(f)$ such that $\omega(x, f) \subset C(x, f)$. For each $C \in \mathcal{C}(f)$, we define the *basin* $W^s(C)$ of C by

$$W^s(C) = \{x \in X\colon \lim_{i\to\infty} d(f^i(x), C) = 0\}.$$

For every $x \in X$, since

$$\lim_{i\to\infty} d(f^i(x), \omega(x, f)) = 0,$$

we have $x \in W^s(C)$ if and only if $C = C(x, f)$. This implies

$$\{x \in X\colon C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)\} = \bigsqcup_{C\in\mathcal{C}_{\mathrm{ter}}(f)} W^s(C).$$

We also define the *chain ω-limit set* $\omega^*(x, f)$ of x for f as the set of $y \in X$ such that for any $\delta > 0$ and $N > 0$, there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$, $x_k = y$, and $k \ge N$. Note that $\omega^*(x, f)$ is a closed f-invariant subset of X and chain stable. We have

$$\omega(x, f) \subset C(x, f) \subset \omega^*(x, f).$$

Remark 1.3 The chain ω-limit set is denoted in [3] as $\omega\mathcal{C}(x, f)$ instead of $\omega^*(x, f)$.

The following lemma is obvious (see Section 1.4 of [3]).

Lemma 1.1 *Let $f\colon X \to X$ be a continuous map.*

(A) *For any $x \in X$, the following properties are equivalent:*
- $C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)$,
- $\omega^*(x, f) \subset C(x, f)$,
- $\omega^*(x, f) = C(x, f)$,
- $f|_{\omega^*(x,f)}\colon \omega^*(x, f) \to \omega^*(x, f)$ *is chain transitive.*

(B) *For any $x \in X$, the following properties are equivalent:*
- $\omega(x, f) = C(x, f) = \omega^*(x, f)$,
- $C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)$ *and* $\omega(x, f) = C(x, f)$.

We give the definition of shadowing.

Definition 1.3 Let $f\colon X \to X$ be a continuous map and let $\xi = (x_i)_{i\ge0}$ be a sequence of points in X. For $\delta > 0$, ξ is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) \le \delta$ for all $i \ge 0$. For $\varepsilon > 0$, ξ is said to be *ε-shadowed* by $x \in X$ if $d(f^i(x), x_i) \le \varepsilon$ for all $i \ge 0$. We say that f has the *shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit of f is ε-shadowed by some point of X.

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For a topological space Z, a subset S of Z is called a G_δ*-subset* of Z if S is a countable intersection of open subsets of Z. If Z is completely metrizable, then by Baire Category Theorem, every countable intersection of open dense subsets of Z is dense in Z. We know that a subspace Y of a completely metrizable space Z is completely metrizable if and only if Y is a G_δ-subset of Z (see Theorem 24.12 of [20]).

For any continuous map $f\colon X \to X$ and $x \in X$, let $\Omega(x, f)$ denote the set of $y \in X$ such that

$$\lim_{j\to\infty} f^{i_j}(x_j) = y$$

for some sequence $0 \le i_1 < i_2 < \cdots$ and $x_j \in X$, $j \ge 1$, with

$$\lim_{j\to\infty} x_j = x.$$

Note that

$$\omega(x, f) \subset \Omega(x, f) \subset \omega^*(x, f)$$

for all $x \in X$. By Proposition 22 in Section 7 of [1], we know that

$$\{x \in X\colon \omega(x, f) = \Omega(x, f)\}$$

is a dense G_δ-subset of X. The proof of this result in [1] is based on a nontrivial fact that the set of continuity points of a lower semicontinuous (lsc) set-valued map is a dense G_δ-subset. If f has the shadowing property, then we have

$$\Omega(x, f) = \omega^*(x, f)$$

for all $x \in X$. This can be proved as follows. Let $(\varepsilon_j)_{j\ge1}$ be a sequence of positive numbers with $\lim_{j\to\infty} \varepsilon_j = 0$. Since f has the shadowing property, for each $j \ge 1$, there is $\delta_j > 0$ such that every δ_j-pseudo orbit of f is ε_j-shadowed by some point of X. Let $x \in X$ and $y \in \omega^*(x, f)$. Since $y \in \omega^*(x, f)$, we have a sequence $(x_i^{(j)})_{i=0}^{k_j}$, $j \ge 1$, of δ_j-chains of f with $x_0^{(j)} = x$, $x_{k_j}^{(j)} = y$, and $k_j < k_{j+1}$ for all $j \ge 1$. By the choice of δ_j, we obtain $x_j \in X$, $j \ge 1$, such that $d(x_j, x) = d(x_j, x_0^{(j)}) \le \varepsilon_j$ and $d(f^{k_j}(x_j), y) = d(f^{k_j}(x_j), x_{k_j}^{(j)}) \le \varepsilon_j$ for all $j \ge 1$. It follows that $0 < k_1 < k_2 < \cdots$,

$$\lim_{j\to\infty} x_j = x,$$

and

$$\lim_{j\to\infty} f^{k_j}(x_j) = y.$$

Thus, $y \in \Omega(x, f)$. Since $x \in X$ and $y \in \omega^*(x, f)$ are arbitrary, we conclude that

$$\omega^*(x, f) \subset \Omega(x, f)$$

for all $x \in X$, completing the proof. It follows that if a continuous map $f\colon X \to X$ has the shadowing property, then

$$\{x \in X\colon \omega(x, f) = \Omega(x, f) = \omega^*(x, f)\}$$

is a dense G_δ-subset of X; therefore,

$$\{x \in X : \omega(x,f) = C(x,f) = \omega^*(x,f)\} = \{x \in X : C(x,f) \in \mathcal{C}_{\mathrm{ter}}(f) \text{ and } \omega(x,f) = C(x,f)\}$$

is a dense G_δ-subset of X (see [11] and [17] for related results). The main aim of this paper is to give an alternative proof of the following statement.

Theorem 1.1 *If a continuous map $f : X \to X$ has the shadowing property, then*

$$V(f) = \{x \in X : C(x,f) \in \mathcal{C}_{\mathrm{ter}}(f)\}$$

and

$$W(f) = \{x \in V(f) : \omega(x,f) = C(x,f)\}$$

are dense G_δ-subsets of X.

Given a continuous map $f : X \to X$ and $x \in X$, we say that f is *chain continuous* at x if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i \geq 0}$ of f with $x_0 = x$ is ε-shadowed by x [2]. We denote by $CC(f)$ the set of chain continuity points for f. The notion of chain continuity is closely related to *odometers*. An *odometer* (or an *adding machine*) is defined as follows. Let $m = (m_j)_{j \geq 1}$ be an increasing sequence of positive integers with $m_j | m_{j+1}$ for all $j \geq 1$. Let X_j, $j \geq 1$, denote the quotient group $\mathbb{Z}/m_j\mathbb{Z}$ with the discrete topology. Let $\pi_j : X_{j+1} \to X_j$, $j \geq 1$, be the natural projections and let

$$X_m = \{x = (x_j)_{j \geq 1} \in \prod_{j \geq 1} X_j : \pi_j(x_{j+1}) = x_j \text{ for all } j \geq 1\}.$$

As a closed subspace of $\prod_{j \geq 1} X_j$ with the product topology, X_m is a compact metrizable space. Consider the map $g_m : X_m \to X_m$ defined by

$$g_m(x)_j = x_j + 1$$

for all $x = (x_j)_{j \geq 1} \in X_m$ and $j \geq 1$. Note that g_m is a homeomorphism. We say that (X_m, g_m) is an odometer with the periodic structure m. We say that a closed f-invariant subset S of X is an *odometer* if $(S, f|_S)$ is topologically conjugate to an odometer. This is equivalent to that S is a Cantor space and

$$f|_S : S \to S$$

is a minimal equicontinuous homeomorphism (see Theorem 4.4 of [15]). By Theorem 7.5 of [3], we know that for any $x \in X$, $x \in CC(f)$ if and only if

$$\omega(x,f) = C(x,f) = \omega^*(x,f)$$

and $C(x,f)$ is a periodic orbit or an odometer. By Lemma 1.1, this is equivalent to that $C(x,f) \in \mathcal{C}_{\mathrm{ter}}(f)$ and $C(x,f)$ is a periodic orbit or an odometer. We say that X is *locally connected* if for any $x \in X$ and any open subset U of X with $x \in U$, we have $x \in V \subset U$ for some open connected subset V of X. A subspace S of X is said to be *totally disconnected* if every connected component of S is a singleton. If X is locally connected and $CR(f)$ is totally disconnected, then due to Theorem 5.1 of [8] or Theorem B of [10], every $C \in \mathcal{C}_{\mathrm{ter}}(f)$ is a periodic orbit or an odometer. By these facts, we obtain the following lemma.

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Lemma 1.2 *Let $f\colon X \to X$ be a continuous map. If X is locally connected and $CR(f)$ is totally disconnected, then for any $x \in X$, the following properties are equivalent:*

- $x \in CC(f)$,
- $\omega(x, f) = C(x, f) = \omega^*(x, f)$,
- $C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)$.

Let $f\colon X \to X$ be a continuous map. For any $j, l \geq 1$, let $C_{j,l}$ denote the set of $x \in X$ such that there is a neighborhood U of x for which every $\frac{1}{j}$-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 \in U$ is $\frac{1}{l}$-shadowed by x_0. We see that $C_{j,l}$ is an open subset of X for all $j, l \geq 1$ and

$$CC(f) = \bigcap_{l\geq 1}\bigcup_{j\geq 1} C_{j,l}.$$

Thus, $CC(f)$ is a G_δ-subset of X. We say that f is *almost chain continuous* if $CC(f)$ is a dense G_δ-subset of X. By Theorem 1.1 and Lemma 1.2, we obtain the following theorem.

Theorem 1.2 *Let $f\colon X \to X$ be a continuous map. If X is locally connected, f has the shadowing property, and if $CR(f)$ is totally disconnected, then f is almost chain continuous.*

We present a corollary of Theorem 1.2. For a closed differentiable manifold M, let $\mathcal{H}(M)$ (resp. $\mathcal{C}(M)$) denote the set of homeomorphisms (resp. continuous self-maps) of M, endowed with the C^0-topology. It is shown in [3] that generic $f \in \mathcal{H}(M)$ (resp. $f \in \mathcal{C}(M)$, if $\dim M > 1$) is almost chain continuous (see Introduction of [3] where the word “almost equicontinuous” is used). Note that the shadowing is generic in $\mathcal{H}(M)$ [19] and also generic in $\mathcal{C}(M)$ [16, Theorem 1]. Moreover, by results of [3, 14], we know that for generic $f \in \mathcal{H}(M)$ (resp. $f \in \mathcal{C}(M)$), $CR(f)$ is totally disconnected (see Introduction of [3] and Theorem 3.3 of [14]). Thus, by Theorem 1.2, we obtain the following corollary.

Corollary 1.1 *Generic $f \in \mathcal{H}(M)$ (resp. $f \in \mathcal{C}(M)$) is almost chain continuous.*

Our results also apply to the case where X is not a manifold. We say that X is a *dendrite* if X is connected, locally connected, and contains no simple closed curves. The shadowing is proved to be generic in the space of continuous self-maps of a dendrite (see [7] and [13, Theorem 19]). However, by Corollary 5.2 of [14], a generic continuous self-map of a dendrite has the totally disconnected chain recurrent set. By Theorem 1.2, we conclude that a generic continuous self-map of a dendrite is almost chain continuous.

This paper consists of two sections. In the next section, we prove Theorem 1.1.

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The proof is based on the following lemma in [12].

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Lemma 2.1 [12, Lemma 2.1] *For any continuous map $f: X \to X$ and $x \in X$, there is $C \in \mathcal{C}_{\mathrm{ter}}(f)$ such that for every $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$ and $x_k \in C$.*

We need one more lemma. In what follows, for $x \in X$ and a subset S of X, we denote by $d(x, S)$ the distance of x from S:

$$d(x, S) = \inf_{y \in S} d(x, y).$$

We also denote by $U_r(S)$, $r > 0$, the open r-neighborhood of S:

$$U_r(S) = \{x \in X: d(x, S) < r\}.$$

Lemma 2.2 *For any continuous map $f: X \to X$ and $x \in X$, if $C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)$, then $C(\cdot, f): X \to \mathcal{C}(f)$ is continuous at x.*

Proof Let $x \in X$ and $C = C(x, f)$. If $C \in \mathcal{C}_{\mathrm{ter}}(f)$ (i.e., C is chain stable), then for any $\varepsilon > 0$, we have $\delta > 0$ such that every δ-chain $(x_i)_{i=0}^k$ of f with $d(x_0, C) \le \delta$ satisfies $d(x_i, C) \le \varepsilon/2$ for all $0 \le i \le k$. It follows that $d(y, C) \le \delta$ implies

$$\omega^*(y, f) \subset U_\varepsilon(C)$$

for all $y \in X$. Since

$$\lim_{i \to \infty} d(f^i(x), C) = 0,$$

we have $d(f^i(x), C) \le \delta/2$ for some $i \ge 0$. By taking $\gamma > 0$ such that $d(x, z) \le \gamma$ implies $d(f^i(x), f^i(z)) \le \delta/2$ for all $z \in X$, we obtain $d(f^i(z), C) \le \delta$ and so

$$C(z, f) \subset \omega^*(z, f) = \omega^*(f^i(z), f) \subset U_\varepsilon(C)$$

for all $z \in X$ with $d(x, z) \le \gamma$. Since $\varepsilon > 0$ is arbitrary, this implies that $C(\cdot, f): X \to \mathcal{C}(f)$ is continuous at x, completing the proof. ■

By using these lemmas, we prove Theorem 1.1.

Proof of Theorem 1.1 First, we show that $V(f)$ is a dense G_δ-subset of X. Fix a sequence $(\varepsilon_j)_{j \ge 1}$ of positive numbers such that $\varepsilon_1 > \varepsilon_2 > \cdots$ and

$$\lim_{j \to \infty} \varepsilon_j = 0.$$

For any $j \ge 1$ and $C \in \mathcal{C}_{\mathrm{ter}}(f)$, we take $\delta_{j,C} > 0$ such that $x \in U_{\delta_{j,C}}(C)$ implies

$$\omega^*(x, f) \subset U_{\varepsilon_j}(C)$$

for all $x \in X$. Let

$$U_{j,C} = U_{\delta_{j,C}}(C)$$

for all $j \ge 1$ and $C \in \mathcal{C}_{\mathrm{ter}}(f)$. We define a subset V of X by

$$V = \bigcap_{j \ge 1} \bigcup_{C \in \mathcal{C}_{\mathrm{ter}}(f)} \bigcup_{m \ge 0} f^{-m}(U_{j,C}).$$

Note that V is a G_δ-subset of X. Since f has the shadowing property, by Lemma 2.1, we see that for every $x \in X$, there is $C \in \mathcal{C}_{\mathrm{ter}}(f)$ such that

$$x \in \overline{\bigcup_{m\geq 0} f^{-m}(U_{j,C})}$$

for all $j \geq 1$. This can be proved as follows. For $x \in X$, fix $C \in \mathcal{C}_{\mathrm{ter}}(f)$ as in Lemma 2.1 and $\gamma_l > 0$, $l \geq 1$, with $\lim_{l\to\infty}\gamma_l = 0$. There are $\beta_l > 0$, $l \geq 1$, and a sequence $(x_i^{(l)})_{i=0}^{k_l}$, $l \geq 1$, of β_l-chains of f such that for each $l \geq 1$,

- every β_l-pseudo orbit of f is γ_l-shadowed by some point of X,
- $x_0^{(l)} = x$ and $x_{k_l}^{(l)} \in C$.

By taking $x_l \in X$, $l \geq 1$, with $d(x_l, x) = d(x_l, x_0^{(l)}) \leq \gamma_l$ and $d(f^{k_l}(x_l), C) \leq d(f^{k_l}(x_l), x_{k_l}^{(l)}) \leq \gamma_l$, we obtain $\lim_{l\to\infty} x_l = x$ and

$$x_l \in f^{-k_l}(U_{j,C}) \subset \bigcup_{m\geq 0} f^{-m}(U_{j,C})$$

for any fixed $j \geq 1$ and all sufficiently large $l \geq 1$, implying

$$x \in \overline{\bigcup_{m\geq 0} f^{-m}(U_{j,C})}$$

for all $j \geq 1$. This proves the claim. It follows that

$$X \subset \bigcup_{C\in\mathcal{C}_{\mathrm{ter}}(f)} \bigcap_{j\geq 1} \overline{\bigcup_{m\geq 0} f^{-m}(U_{j,C})} \subset \bigcup_{C\in\mathcal{C}_{\mathrm{ter}}(f)} \overline{\bigcup_{m\geq 0} f^{-m}(U_{j,C})} \subset \overline{\bigcup_{C\in\mathcal{C}_{\mathrm{ter}}(f)} \bigcup_{m\geq 0} f^{-m}(U_{j,C})}$$

for all $j \geq 1$. With the aid of Baire Category Theorem, this implies that V is a dense G_δ-subset of X. It remains to prove that $V(f) = V$. Given any $x \in V(f)$, by $C(x,f) \in \mathcal{C}_{\mathrm{ter}}(f)$ and

$$x \in \bigcap_{j\geq 1}\bigcup_{m\geq 0} f^{-m}(U_{j,C(x,f)}) \subset V,$$

we have $x \in V$. It follows that $V(f) \subset V$. Conversely, let $x \in V$. For each $j \geq 1$, we take $C_j \in \mathcal{C}_{\mathrm{ter}}(f)$ and $m_j \geq 0$ such that

$$x \in f^{-m_j}(U_{j,C_j}).$$

Then, because $\mathcal{C}(f) = CR(f)/\leftrightarrow$ is a compact metrizable space, there are a sequence $1 \leq j_1 < j_2 < \cdots$ and $C \in \mathcal{C}(f)$ such that

$$\lim_{l\to\infty} C_{j_l} = C$$

in $\mathcal{C}(f)$. Note that for every $\varepsilon > 0$, we have

$$C_{j_l} \subset U_\varepsilon(C)$$

for all sufficiently large $l \geq 1$. For every $l \geq 1$, by

$$f^{m_{j_l}}(x) \in U_{j_l, C_{j_l}},$$

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we have

$$\omega^*(x, f) = \omega^*(f^{m_{j_l}}(x), f) \subset U_{\varepsilon_{j_l}}(C_{j_l}).$$

By

$$\lim_{l\to\infty} \varepsilon_{j_l} = 0,$$

we obtain

$$\omega^*(x, f) \subset U_{2\varepsilon}(C)$$

for all $\varepsilon > 0$; thus, $\omega^*(x, f) \subset C$. From Lemma 1.1, it follows that $C = C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)$, implying $x \in V(f)$. Since $x \in V$ is arbitrary, we conclude that $V \subset V(f)$, proving the claim.

Next, we show that $W(f)$ is a dense G_δ-subset of X. Since $V(f)$ is a dense G_δ-subset of X, it suffices to show that $W(f)$ is a dense G_δ-subset of $V(f)$. Letting

$$W = \bigcap_{j\geq 1}\bigcap_{m\geq 0}\{x \in V(f): C(x, f) \subset U_{\frac{1}{j}}(\{f^i(x): i \geq m\})\},$$

we have $W = W(f)$. Let

$$W_{j,m} = \{x \in V(f): C(x, f) \subset U_{\frac{1}{j}}(\{f^i(x): i \geq m\})\}$$

for all $j \geq 1$ and $m \geq 0$. Given any $x \in W_{j,m}$, $j \geq 1$, $m \geq 0$, by compactness of $C(x, f)$, there are $0 < r < \frac{1}{j}$ and $n \geq m$ such that

$$C(x, f) \subset U_r(\{f^i(x): m \leq i \leq n\}).$$

We take $\varepsilon > 0$ with $r + 2\varepsilon < \frac{1}{j}$. Since $x \in V(f)$ and so $C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)$, by Lemma 2.2, there is $a > 0$ such that $d(x, y) < a$ implies

$$C(y, f) \subset U_\varepsilon(C(x, f))$$

for all $y \in X$. By continuity of f, we have $b > 0$ such that $d(x, y) < b$ implies

$$\{f^i(x): m \leq i \leq n\} \subset U_\varepsilon(\{f^i(y): m \leq i \leq n\})$$

for all $y \in X$. It follows that $d(x, y) < \min\{a, b\}$ implies

$$C(y, f) \subset U_{r+2\varepsilon}(\{f^i(y): m \leq i \leq n\}) \subset U_{\frac{1}{j}}(\{f^i(y): m \leq i \leq n\}) \subset U_{\frac{1}{j}}(\{f^i(y): i \geq m\})$$

for all $y \in X$. Since $x \in W_{j,m}$ is arbitrary, $W_{j,m}$ is an open subset of $V(f)$. Since $j \geq 1$ and $m \geq 0$ are arbitrary, we conclude that W is a G_δ-subset of $V(f)$. It remains to prove that W is a dense subset of $V(f)$. Let $j \geq 1$ and $m \geq 0$. Given any $x \in V(f)$ and $\varepsilon > 0$, since $C(x, f) \in \mathcal{C}_{\mathrm{ter}}(f)$, by Lemma 2.2, there is $0 < a < \varepsilon/2$ such that $d(x, y) < 2a$ implies

$$C(y, f) \subset U_{\frac{1}{3j}}(C(x, f))$$

for all $y \in X$. Since f has the shadowing property, we see that

$$C(x, f) \subset U_{\frac{1}{3j}}(\{f^i(p): i \geq m\})$$

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for some $p \in X$ with $d(x,p) < a$. By compactness of $C(x,f)$, we obtain

$$C(x,f) \subset U_{\frac{1}{3j}}(\{f^i(p): m \le i \le n\})$$

for some $n \ge m$. By continuity of f, we have $b > 0$ such that $d(p,q) < b$ implies

$$\{f^i(p): m \le i \le n\} \subset U_{\frac{1}{3j}}(\{f^i(q): m \le i \le n\})$$

for all $q \in X$. Since $V(f)$ is a dense subset of X, we have $d(p,q) < \min\{a,b\}$ for some $q \in V(f)$. Note that

$$d(x,q) \le d(x,p) + d(p,q) < 2a < \varepsilon.$$

It follows that

$$C(q,f) \subset U_{\frac{1}{3j}}(C(x,f)) \subset U_{\frac{1}{j}}(\{f^i(q): m \le i \le n\}) \subset U_{\frac{1}{j}}(\{f^i(q): i \ge m\}),$$

implying $q \in W_{j,m}$. Since $x \in V(f)$ and $\varepsilon > 0$ are arbitrary, $W_{j,m}$ is an open dense subset of $V(f)$. Since $j \ge 1$ and $m \ge 0$ are arbitrary, we conclude that W is a dense subset of $V(f)$, proving the claim. Thus, the theorem has been proved. ■

We conclude with a remark on the proof.

Remark 2.1

- The proof shows that $V(f)$ and $W(f)$ are G_δ-subsets of X for every continuous map $f: X \to X$.
- For any continuous map $f: X \to X$, we can show that if f has the shadowing property, then

$$V(f) = \{x \in X: C(\cdot, f): X \to \mathcal{C}(f) \text{ is continuous at } x\}.$$

  By this, since $\mathcal{C}(f)$ is a compact metrizable space, we can show that $V(f)$ is a G_δ-subset of X.
- Let $f: X \to X$ be a continuous map and let $\xi = (x_i)_{i \ge 0}$ be a sequence of points in X. For $\delta > 0$, ξ is called a *δ-limit-pseudo orbit* of f if $d(f(x_i), x_{i+1}) \le \delta$ for all $i \ge 0$, and

$$\lim_{i \to \infty} d(f(x_i), x_{i+1}) = 0.$$

  For $\varepsilon > 0$, ξ is said to be *ε-limit shadowed* by $x \in X$ if $d(f^i(x), x_i) \le \varepsilon$ for all $i \ge 0$, and

$$\lim_{i \to \infty} d(f^i(x), x_i) = 0.$$

  We say that f has the *s-limit shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that every δ-limit-pseudo orbit of f is ε-limit shadowed by some point of X. When f has the s-limit shadowing property, by Lemma 2.1, we can easily show that $W(f)$ is a dense subset of X.

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