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LINEARISED MOSER-TRUDINGER INEQUALITY

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As a limiting case of the Sobolev imbedding theorem, the Moser-Trudinger inequality was obtained for functions in $\dot{W}_q^1(\Omega)$ with resulting exponential class integrability. Here we prove this inequality again and at the same time get sharper information for the bound. We also generalise the linearised Moser inequality to higher dimensions, which was first introduced by Beckner for functions on the unit disc. Both of our results are obtained by using the method of Carleson and Chang. The last section introduces an analogue of each inequality for the Laplacian instead of the gradient under some restricted conditions.

1. INTRODUCTION

Let Ω be an open bounded domain in the n-dimensional space $\mathbb{R}^n, n \geqslant 2$. Let $\dot{W}_q^1(\Omega)$ be the completion of the function class $C_0^1(\Omega)$ equipped with the norm

$$\|u\|_{\dot{W}_q^1} = \left(\int_D |\nabla u|^q \, dx\right)^{1/q} \quad \text{for all } u \in C_0^1(\Omega)$$

where ∇u is the gradient of u and $|\nabla u|$ is its Euclidean norm.

As a limiting case $q = n$ of the Sobolev imbedding theorem, Trudinger [**12**] introduced an exponential Sobolev inequality and then, Moser [**8**] improved it as follows: if $u \in \dot{W}_q^1(\Omega)$, $n \geqslant 2$, with $\int_\Omega |\nabla u|^n \, dx \leqslant 1$; then there exists a constant c_n such that

$$\int_\Omega e^{\alpha u^p} \, dx \leqslant c_n m(\Omega), \tag{1}$$

where $p = n/(n-1)$, $\alpha \leqslant \alpha_n = n\omega_{n-1}^{1/(n-1)}$, $m(\Omega) = \int_\Omega dx$ and ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere. The integral on the left actually is finite for any positive α, but if $\alpha > \alpha_n$ it can be made arbitrarily large by an appropriate choice of u.

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In the proof, Moser used a symmetrisation technique and change of variables to change the n-dimensional problem to a one-dimensional problem. Using the corresponding inequality for the function which is defined on a compact manifold of dimension two and having vanishing mean value, Moser [8] also showed that

$$\ln \int_{S^2} e^F d\xi \leqslant \int_{S^2} F d\xi + \frac{1}{4}\int_{S^2} |\nabla F|^2 d\xi + K \tag{2}$$

where K is some positive constant and $d\xi$ is the normalised surface measure on S^2. Later on, Onofri [10] obtained the best constant $K = 0$ and Beckner [2] generalised the Moser-Onofri inequality to higher dimensions. We see applications of those inequalities in several different geometry problems [4, 5, 9, 11]. On the other hand, in [3], using conformal equivalence between the sphere and the plane Beckner proved that the Moser-Onofri inequality on S^2, (2) with $K = 0$, is equivalent to the linearised Sobolev inequality

$$\ln \frac{1}{\pi}\int_{|x|\leqslant 1} e^{2f}\,dx + \left(\frac{1}{\pi}\int_{|x|\leqslant 1} e^{2f}\,dx\right)^{-1} \leqslant 1 + \frac{1}{4\pi}\int_{|x|\leqslant 1} |\nabla f|^2\,dx \tag{3}$$

for non-negative functions with zero boundary-value on the unit disk in $\mathbb{R}^2$. He developed the relation of the sharp inequality (3) with Carleson-Chang's work. In Section 2, we show the analogue of (3) for bounded domains Ω in $\mathbb{R}^n$. The advantage of studying the linearised exponential Sobolev inequality in $\mathbb{R}^n$ is the fact that we can reduce it to a one dimensional problem by using symmetrisation and change of variables, as we see from Moser [8].

As an indication of the richness of the Moser inequality (1), there have been several alternative proofs of it, see [1, 6, 7]. Adams [1] showed an analogue of inequality (1) for higher order derivatives by using Riesz potentials. Specially in [6], Carson-Chang proved the inequality (1) by using the same method as was used to show the existence of an extremal function for the same inequality in the case when Ω is the Euclidean ball in $\mathbb{R}^n$. In Section 3, we give another proof of the Moser inequality (1) by using Carleson-Chang's method and at the same time we obtain a functional form of the Moser-Trudinger inequality. As a consequence, in Section 4, we also obtain an inequality similar to the Adams [1] inequality (when $n = 2$, $m = 4$) which was applied in [5].

2. Linearised Moser-Trudinger Inequality in n-dimensions

The Moser-Onofri inequality

$$\ln \int_{S^2} e^F d\xi \leqslant \int_{S^2} F d\xi + \frac{1}{4}\int_{S^2} |\nabla F|^2 d\xi \tag{4}$$

which played an important role in understanding some geometry problems (see [9, 10]) was generalised to higher dimensions by Beckner [2]. More recently, in [3], the following sharp inequality for non-negative function f with zero boundary value on the unit disk in $\mathbb{R}^2$

$$\ln \frac{1}{\pi}\int_{|x|\leqslant 1} e^{2f}\,dx + \left(\frac{1}{\pi}\int_{|x|\leqslant 1} e^{2f}\,dx\right)^{-1} \leqslant 1 + \frac{1}{4\pi}\int_{|x|\leqslant 1} |\nabla f|^2\,dx \tag{5}$$

which is equivalent to the Moser-Onofri inequality (4) was obtained by using the conformal equivalence between the sphere and the plane.

Prior to the above inequality (5), we find the following inequality from the lemma by Carleson-Chang [**6**]

$$\ln \frac{1}{\pi}\int_{|x|\leqslant 1} e^{2f}\,dx \;\leqslant\; 1 + \frac{1}{4\pi}\int_{|x|\leqslant 1} |\nabla f|^2\,dx.$$

In this section, we generalise the inequality (5) to every 2-dimensional domain and at the same time establish an analogue of it for higher dimensions by using Carleson-Chang's method. And we obtain the general form of the function which satisfies the equality (5). Also we shall give the proof of the equivalence between (4) and (5) which is due to Beckner for better understanding the relation between the two inequalities.

Theorem 2.1.

$$\begin{aligned} \ln \frac{1}{m(\Omega)}\int_\Omega e^{c_\circ f(x)}\,dx &+ \left(\frac{1}{m(\Omega)}\int_\Omega e^{c_\circ f(x)}\,dx\right)^{-1} \\ &\leqslant \sum_{i=1}^{n-1}\frac{1}{i} + \frac{c_\circ^n}{\omega_{n-1}}\frac{(n-1)^{n-1}}{n^{2n-1}}\int_\Omega |\nabla f|^n\,dx \end{aligned} \tag{6}$$

for all non-negative functions f in $\dot{W}_n^1(\Omega)$ and each positive constant $c_\circ$. Here Ω is a bounded domain in $\mathbb{R}^n$ and $m(\Omega) = \int_\Omega dx$.

Since $C_0^1(\Omega)$ functions are dense in $\dot{W}_n^1(\Omega)$, proving the theorem for the function in $C_0^1(\Omega)$ is sufficient. We use symmetrisation and a change of variable as was used in [**8**] to reduce the problem as a one-dimensional problem. By the results of symmetrisation we obtain a rearranged function f^* of f which is defined on the ball Ω^* centred at the origin with radius R, $\int_{|x|\leqslant R} dx = m(\Omega)$. Since f^* is radial, to change the problem to a one dimensional one we set

$$\varphi(t) = n^{(n-1)/n}(\omega_{n-1})^{1/n} f^*\big(|x|\big) \tag{7}$$

$$\frac{|x|^n}{R^n} = e^{-t}. \tag{8}$$

Thus our problem becomes the following: for all C^1 functions φ defined on $0 \leqslant t < \infty$ with $\varphi(0) = 0$ and $\varphi'(t) \geqslant 0$ we have

$$\begin{aligned} \ln \int_0^\infty e^{c\varphi(t)-t}\,dt &+ \left(\int_0^\infty e^{c\varphi(t)-t}dt\right)^{-1} \\ &\leqslant \sum_{i=1}^{n-1}\frac{1}{i} + \frac{c^n}{n}\left(\frac{n-1}{n}\right)^{n-1}\int_0^\infty \varphi'(t)^n\,dt \end{aligned}$$

where $c = c_\circ n^{-(n-1)/n}(\omega_{n-1})^{-(1/n)}$.

For the proof of the theorem we need the lemma which follows from the argument in [**6**, Lemma 1].

Lemma 2.2. *Let*

$$K_\delta = \left\{\varphi : \text{if } C^1 \text{is a function on } 0 \leqslant t < \infty,\ \varphi(0) = 0,\ \varphi'(t) \geqslant 0,\ \int_0^\infty \varphi'(t)^n\,dt = \delta\right\}$$

and let $\varphi_\circ \in K_\delta$ be the extremal function for $\sup_{K_\delta} \int_0^\infty e^{c\varphi(t)-t}\,dt$. Then for each $c > 0$

$$\int_0^\infty e^{c\varphi_\circ(t)-t}\,dt = \frac{1+B}{B} \tag{9}$$

where B is positive and satisfies

$$c^n\delta = (n-1)\left(\frac{n}{n-1}\right)^n\left(\ln\frac{1+B}{B} - \sum_{i=1}^{n-1}\frac{1}{i(1+B)^i}\right).$$

Remark. The general form of the extremal function $\varphi_\circ$ is the following:

$$\varphi_\circ(t) = \frac{n}{c}\left[\ln(1+B) - \ln\left(e^{t/(1-n)} + B\right)\right].$$

Proof of Theorem: Let $\varphi_\circ$ be the extremal function for $\sup_{K_\delta} \int_0^\infty e^{c\varphi(t)-t}\,dt$. Then by Lemma 2.2 we have

$$\begin{aligned}\int_0^\infty e^{c\varphi_\circ(t)-t}\,dt &= e^{((n-1)/n)^{n-1}(c^n/n)\delta} e^{\sum_{i=1}^{n-1}(1/i(1+B)^i)}\\ &\leqslant e^{((n-1)/n)^{n-1}(c^n/n)\delta} e^{(1-B/(1+B))\sum_{i=1}^{n-1}1/i}\end{aligned}$$

where $\delta = \int_0^\infty \varphi'(t)^n\,dt$. Thus by (9) we get

$$\ln\int_0^\infty e^{c\varphi_\circ(t)-t}\,dt + \left(\int_0^\infty e^{c\varphi_\circ(t)-t}\,dt\right)^{-1} \leqslant \left(\frac{n-1}{n}\right)^{n-1}\left(\frac{c^n}{n}\right)\delta + \sum_{i=1}^{n-1}\frac{1}{i}. \tag{10}$$

Note that for all $\varphi \in K_\delta$, $\int_0^\infty e^{c\varphi(t)-t}\,dt \geqslant 1$, and the fact that $1/x + \ln x$ is an increasing function for $x \geqslant 1$. Hence, from (10) we obtain

$$\ln\int_0^\infty e^{c\varphi(t)-t}\,dt + \left(\int_0^\infty e^{c\varphi(t)-t}\,dt\right)^{-1} \leqslant \left(\frac{n-1}{n}\right)^{n-1}\left(\frac{c^n}{n}\right)\delta + \sum_{i=1}^{n-1}\frac{1}{i}$$

for all $\varphi \in K_\delta$. □

Remark. When $n = 2$, and $\Omega = B_2$, by letting $c_\circ = 2$ in (6), we again obtain the sharp inequality (5)

$$\log\frac{1}{\pi}\int_{B_2} e^{2f}\,dx + \left(\frac{1}{\pi}\int_{B_2} e^{2f}\,dx\right)^{-1} \leqslant 1 + \frac{1}{4\pi}\int_{B_2}|\nabla f|^2\,dx, \tag{11}$$

where the equality is attained for radial functions of the form

$$f\left(|x|\right) = \ln(1+B) - \ln\left(|x|^2 + B\right).$$

COROLLARY 2.3. *Let $F_n(t) = \int_1^t (y-1)^{n-1} y^{-n} dy$ for $t \geqslant 1$ and $n \geqslant 2$. Then for $f \geqslant 0$ in $\dot{W}_n^1(\Omega)$*

$$F_n\left[\frac{1}{m(\Omega)} \int_\Omega e^{c_0 f(x)}\, dx\right] \leqslant \frac{c_0^n}{\omega_{n-1}} \frac{(n-1)^{n-1}}{n^{2n-1}} \int_\Omega |\nabla f|^n\, dx.$$

PROOF: This result follows from Lemma 2.2. Observe that

$$F_2(t) = \ln t + \frac{1}{t} - 1$$

which corresponds to equation (6) for $n = 2$. ∎

Before we finish this section, for better understanding about the sharp inequality (11) we introduce Beckner's proof for the equivalence between the Moser-Onofri inequality

$$\ln \int_{S^2} e^F d\xi \leqslant \int_{S^2} F d\xi + \frac{1}{4} \int_{S^2} |\nabla F|^2 d\xi \tag{12}$$

($d\xi$: normalised measure on S^2) and the sharp inequality (11).

By using the result of symmetrisation we assume that $F(\xi)$ depends only on the polar angle θ on S^2 and similarly f is radial on $\mathbb{R}^2$. First, for each $\delta > 0$ set

$$F(\xi) = \begin{cases} 2f\left(\dfrac{x}{\delta}\right) + 2\log\left(1+|x|^2\right), & |x| \leqslant \delta \\ 2\log\left(1+\delta^2\right), & |x| > \delta. \end{cases} \tag{13}$$

Since our function F depends only on θ, notice that we can evaluate the integrals on S^2 by first integrating over the parallel $Z_\theta = \{\xi \in S^2 : e \cdot \xi = \cos\theta\}$ orthogonal to e and where the measure of Z_θ is $2\pi \sin\theta$ (that is, $\int_{S^2} F(\xi) d\xi = (1/2) \int_0^\pi F(\theta) \sin\theta d\theta$). Thus, by using streographic projection from $\mathbb{R}^2$ to $S^2 - \{0, 0, -1\}$ and (13) we obtain the following inequality from (12)

$$\frac{\delta^2}{1+\delta^2} \frac{1}{\pi} \int_{B_2} e^{2f}\, dx + 1 \leqslant e^{1/(4\pi) \int_{B^2} |\nabla f|^2\, dx + \delta^2/(1+\delta^2)}.$$

Let $\gamma = \delta^2/(1+\delta^2)$, then $0 < \gamma \leqslant 1$ and from the above inequality we have

$$1 \leqslant e^{1/(4\pi) \int_{B_2} |\nabla f|^2\, dx} e^\gamma - \gamma\left(\frac{1}{\pi} \int_{B_2} e^{2f}\, dx\right). \tag{14}$$

So we obtain the following sharp inequality by considering the minimum of the right hand side of the inequality (14) as a function of γ on $(0, 1]$,

$$\ln \frac{1}{\pi} \int_{B_2} e^{2f}\, dx + \left(\frac{1}{\pi} \int_{B_2} e^{2f}\, dx\right)^{-1} \leqslant 1 + \frac{1}{4\pi} \int_{B_2} |\nabla f|^2\, dx.$$

To show the other side of the inequality we need the following

LEMMA 2.4. (Beckner) Let K, Λ be positive definite self adjoint operators defined on a σ-finite measure space satisfying the relation $\Lambda K = K\Lambda = 1$. Then the following inequalities are equivalent

$$\int g(Kg)\,d\nu \leqslant c + \int g \ln g\,d\nu, \quad \text{for } g \geqslant 0,\ \int g\,d\nu = 1$$

$$\ln \int e^{2f} d\nu \leqslant c + \int \left|\Lambda^{1/2} f\right|^2 d\nu\,.$$

Now, assume the sharp inequality (11) holds. So we also have

$$\ln \frac{1}{\pi} \int_{B_2} e^{2f}\,dx \leqslant 1 + \frac{1}{4\pi} \int_{B_2} |\nabla f|^2\,dx. \tag{15}$$

Notice that since f has zero boundary-value we have

$$\frac{1}{4\pi} \int_{B_2} |\nabla f|^2\,dx = -\frac{1}{4\pi} \int_{B_2} f(\Delta f)\,dx$$

and the Green's function of $-(1/4\pi)\Delta$ is

$$G(x,y) = \begin{cases} -2\,\ln|x-y| + 2\,\ln\left|\dfrac{x}{|x|} - |x|y\right|, & x \neq 0 \\ -2\,\ln|y|, & x = 0. \end{cases}$$

Thus, from the inequality (15) with the above Lemma we obtain

$$\int_{B_2 \times B_2} g(x)G(x,y)g(y)\,dxdy \leqslant \ln \pi e + \int_{B_2} g \ln g\,dx$$

where $g \geqslant 0$, $\int_{B_2} g\,dx = 1$. To change the inequality for the function defined on $\mathbb{R}^2$ instead of B_2 replace g by the function $g_\varepsilon(x) = (1/\varepsilon^2)g(x/\varepsilon)$ supported in a ball of radius $1/\varepsilon$ such that $\int_{|x|\leqslant(1/\varepsilon)} g_\varepsilon(x)\,dx = 1$. So we get

$$\begin{aligned} \int_{B_{1/\varepsilon} \times B_{1/\varepsilon}} g(x)\Big(-\ln|x-y|^2 + \ln\big(1 - 2\varepsilon^2 x \cdot y + \varepsilon^4 |x|^2 |y|^2\big)\Big) g(y)\,dxdy \\ \leqslant \ln \pi e + \int_{B_{1/\varepsilon}} g \ln g\,dx. \end{aligned} \tag{16}$$

Hence, we have

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} g(x)\big(-\ln|x-y|^2\big) g(y)\,dxdy \leqslant \ln \pi e + \int_{\mathbb{R}^2} g \ln g\,dx \tag{17}$$

since the size of the support of function g was arbitrary, so by taking ε sufficiently small in (16), the inequality holds everywhere. And then by using the conformal equivalence between $\mathbb{R}^2$ and S^2, an inequality which is equivalent to (17) is obtained for the sphere (see [**3**, Theorem 2]):

$$\int_{S^2 \times S^2} F(\xi)\big(-\ln|\xi-\eta|^2\big) F(\eta)\,d\xi d\eta \leqslant \int_{S^2} -\ln|\xi-\eta|^2 d\xi + \int_{S^2} F(\xi) \ln|F(\xi)|\,d\xi \tag{18}$$

where $F \geqslant 0$, $\int_{S^2} F\,d\xi = 1$. Thus, by using duality starting from the Hardy-Littlewood-Sobolev fractional inequality for S^2, (18) is proved to be equivalent to the Moser-Onofri inequality (12) (see [**3**]).

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3. Functional form and numurical estimates for the Moser-Trudinger inequality

Theorem 3.5. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$. Let $u \in \dot{W}_n^1(\Omega)$ with*

$$\int_\Omega |\nabla u|^n \, dx \leqslant 1 .$$

then there exist constants A_0, A_1, A_2 which depend only on n such that

$$\frac{1}{m(\Omega)} \int_\Omega e^{\alpha_n u^p} \, dx \leqslant A_0 + A_1 e^{A_2 \int_\Omega |\nabla u|^n \, dx}$$

where $p = n/(n-1)$, $\alpha_n = n(\omega_{n-1})^{1/(n-1)}$, $m(\Omega) = \int_\Omega dx$ and ω_{n-1} is the $(n-1)$-dimensional surface of the unit sphere.
In particular, when $n = 2$ or 3, we have

$$\frac{1}{m(\Omega)} \int_\Omega e^{\alpha_n u^p} \, dx \leqslant A_1 e^{A_2 \int_\Omega |\nabla u|^n \, dx}.$$

As earlier studies in the direction of the above theorem we found several other proofs of the Moser inequality (1) in [**1, 6, 7**]. Usually they proved the boundedness of the integral by using some constant c_n which depends only on n, with a variety of proofs. In [**6**], they estimated the value $c_2 = 4.3556$ by using some computer experiment.

Remark. By letting $\delta = 1$ in the resulting inequalities of the theorem we can get a constant bound c_n in the Moser inequality (1), for example $c_2 = 4.63, c_3 = 12.28, c_4 = 85.86$. But for large n it is not sharp.

Our proof relies on symmetrisation and a change of variable, as in Theorem 2.1, and uses a sharp lemma by Carleson and Chang [**6**]. Consider the symmetrised function u^* of u which is defined on $\Omega^* = \left\{x : |x| \leqslant R, \int_{|x| \leqslant R} dx = m(\Omega)\right\}$. Without loss of generality, assume that $m(\Omega) = m(\text{unit ball in } \mathbb{R}^n)$ (that is, $R = 1$). Thus, by (7) and (8), it suffices to prove: if $\varphi(t)$ is a C^1-function defined on $0 \leqslant t < \infty$ with

$$\varphi(0) = 0, \ \ \varphi'(t) \geqslant 0, \ \ \int_0^\infty \varphi'(t)^n \, dt = \delta \tag{19}$$

where $\delta \leqslant 1$, $n \geqslant 2$, then there exist constants A_0, A_1, A_2 such that

$$\int_0^\infty e^{\varphi^p(t) - t} \, dt \leqslant A_0 + A_1 e^{A_2 \delta}.$$

In particular, when $n = 2$ or 3,

$$\int_0^\infty e^{\varphi^p(t) - t} \, dt \leqslant A_1 e^{A_2 \delta}. \tag{20}$$

To prove the theorem we shall estimate the integral $\int e^{\varphi^p(t) - t} \, dt$ on each interval $[0, a]$ and $[a, \infty)$ separately by using some specific point $a \in [0, \infty)$ which satisfies the following.

CLAIM. For each φ which satisfies (19) we can choose the point a to be the first point such that

$$\varphi(a) = \left[1 - \left((n-1)/n\right)^{n-1}\right]^{1/n} a^{(n-1)/n}. \tag{21}$$

PROOF OF CLAIM: Suppose not, then there will be two cases;

(i) for all $t \geqslant 0$, $\varphi(t) < \left[1 - \left((n-1)/n\right)^{n-1}\right]^{1/n} t^{(n-1)/n}$;

(ii) for all $t \geqslant 0$, $\varphi(t) > \left[1 - \left((n-1)/n\right)^{n-1}\right]^{1/n} t^{(n-1)/n}$.

In case (i) we have

$$\int_0^\infty e^{\varphi^p(t)-t}\,dt \leqslant \int_0^\infty e^{[(1-((n-1)/n)^{n-1})^{1/(n-1)}-1]t}\,dt$$
$$= \left[1 - \left(1 - \left(\frac{n-1}{n}\right)^{n-1}\right)^{1/(n-1)}\right]^{-1}.$$

If we assume that (ii) is true, and since

$$\varphi(t) \leqslant \left(\int_0^t \varphi'(s)^n\,ds\right)^{1/n} t^{(n-1)/n} \quad \text{for all } t \geqslant 0$$

by Holder's inequality, we have

$$\left(1 - \left(\frac{n-1}{n}\right)^{n-1}\right)^{1/n} \leqslant \left(\int_0^t \varphi'(s)^n\,ds\right)^{1/n}$$

for all $t \geqslant 0$. But this is a contradiction for sufficiently small t. ∎

Now assuming the existence of the point a, let

$$\delta_1 = \int_0^a \varphi'(t)^n\,dt$$
$$\delta_2 = \int_a^\infty \varphi'(t)^n\,dt.$$

By the property (21) of a and the fact that $\varphi^n(a) \leqslant a^{n-1}\int_0^a \varphi'(s)^n\,ds$, δ_1 and δ_2 satisfy

$$\delta_1 \geqslant 1 - \left(\frac{n-1}{n}\right)^{n-1} \tag{22}$$

$$\delta_2 \leqslant 1 - \delta_1 \leqslant \left(\frac{n-1}{n}\right)^{n-1}. \tag{23}$$

We need the following lemma to estimate the integral $\int_a^\infty e^{\varphi^p(t)-t}\,dt$ in the proof of the theorem. At the end of this section we shall prove Lemma 3.7 by using Lemma 3.6

LEMMA 3.6. (Carleson and Chang) Let

$$K = \left\{\psi : \ \psi \text{ is a } C^1 \text{ function defined on } 0 \leqslant t < \infty,\ \psi(0) = 0,\ \int_0^\infty \psi'(t)^n\,dt \leqslant \beta\right\}.$$

Then for each $c > 0$ we have

$$\sup_K \int_0^\infty e^{c\psi(t)-t}\,dt < e^{((n-1)/n)^{n-1}(c^n/n)\beta} e^{1+1/2+\cdots+1/(n-1)}.$$

Also when $c^n\beta \to \infty$, the inequality tends asymptotically to an equality.

Lemma 3.7. *For each C^1 function φ which satisfies (19) and with the fixed point a let $\int_a^\infty \varphi'(t)^n\,dt = \delta_2$; then*

$$\int_a^\infty e^{\varphi^p(t)-t}\,dt \leqslant c_1 e^{c_2\delta_2} e^{[(1-(n-1/n)^{n-1})^{1/(n-1)}-1]a}$$

where

$$c_1 = n^{(2n/(n+1))} e^{(1/(n+1))(1+1/2+\cdots+1/(n-1))}$$
$$c_2 = \frac{(n+1)^{n-1}}{(n-1)^2 e}\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{1/(n-1)}.$$

Proof of the Theorem. Since $\delta_1 + \delta_2 = \delta$ and $\delta_1 \geqslant 1 - \left((n-1)/n\right)^{n-1}$, by Lemma 3.7 we have

$$\int_a^\infty e^{\varphi^p(t)-t}\,dt \leqslant c_1 e^{c_2(\delta_1+\delta_2)} e^{-c_2(1-((n-1)/n)^{n-1})} e^{[(1-((n-1)/n)^{n-1})^{1/(n-1)}-1]a}$$

$$(24)\qquad = c_3 e^{c_2\delta} e^{[(1-((n-1)/n)^{n-1})^{1/(n-1)}-1]a}$$

where $c_3 = c_1 e^{-c_2(1-((n-1)/n)^{n-1})}$. Thus by (24) and property (21) of a, we get

$$\int_0^\infty e^{\varphi^p(t)-t}\,dt = \int_0^a e^{\varphi^p(t)-t}\,dt + \int_a^\infty e^{\varphi^p(t)-t}\,dt \leqslant \int_0^a e^{[(1-((n-1)/n)^{n-1})^{1/(n-1)}-1]t}\,dt$$

$$(25)\qquad + c_3 e^{c_2\delta} e^{[(1-((n-1)/n)^{n-1})^{1/(n-1)}-1]a}.$$

Since $\left(1-\left((n-1)/n\right)^{n-1}\right)^{1/(n-1)} - 1 < 0$, we have

$$\int_0^\infty e^{\varphi^p(t)-t}\,dt \leqslant \left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{1/(n-1)}\right]^{-1} + c_3 e^{c_2\delta}$$

for all C^1 functions φ which satisfy (19). This proves the first part of the theorem.

On the other hand, if we rewrite (25) as

$$\int_0^\infty e^{\varphi^p(t)-t}\,dt \leqslant \left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{1/(n-1)}\right]^{-1}$$
$$+ e^{[(1-((n-1)/n)^{n-1})^{1/(n-1)}-1]a}\left\{c_3 e^{c_2\delta} - \left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{1/(n-1)}\right]^{-1}\right\}$$

and notice that, when $n = 2, 3$,

$$0 \leqslant c_3 e^{c_2\delta} - \left[1-\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{1/(n-1)}\right]^{-1}$$

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for all $0 \leqslant \delta \leqslant 1$ then we have

$$\int_0^\infty e^{\varphi^p(t)-t}\,dt \leqslant c_3 e^{c_2\delta}.$$

This finishes the proof of theorem. ∎

Before we use Lemma 3.6 to prove Lemma 3.7, we rewrite it for functions defined on B_2 as follows; let $v \in C_0^1$ be a function defined on the unit ball B_n, then for each $c > 0$ we have

$$\frac{1}{m(B_n)}\int_{B_n} e^{cv(x)}\,dx \leqslant e^{((n-1)^{n-1}/n^{2n-1})\,((c^n\beta)/(\omega_{n-1}))} e^{1+1/2+\cdots+1/(n-1)} \tag{26}$$

where $\beta = \int_{B_n} |\nabla v|^n\,dx$.

Proof of Lemma 3.7: To estimate the integral $\int_a^\infty e^{\varphi^p(t)-t}\,dt$ which also can be recognised as the integral

$$\frac{n}{\omega_{n-1}}\int_{|x|\leqslant e^{-a/n}} e^{n(\omega_{n-1})^{1/(n-1)}u^{*(n/(n-1))}(x)}\,dx$$

by (7) and (8), we shall use a change of variable and the result of Lemma 3.6. Set

$$y = xe^{a/n}$$
$$g(y) = u^*(x) - n^{-(n-(1/n))}(\omega_{n-1})^{-(1/n)}\varphi(a).$$

Then g is a radial function defined on the unit ball B_n with zero boundary value having the following properties

$$\delta_2 = e^{(n-1)a}\int_{|y|\leqslant 1} |\nabla g|^n dy \tag{27}$$

and

$$\begin{aligned} g(y) &= n^{-(n-1)/n}(\omega_{n-1})^{-(1/n)}\big(\varphi(t)-\varphi(a)\big)\\ &= n^{-(n-1)/n}(\omega_{n-1})^{-(1/n)}\int_a^t \varphi'(s)\,ds\\ &\leqslant n^{-(n-1)/n}(\omega_{n-1})^{-(1/n)}\delta_2^{1/n}(t-a)^{(n-1)/n}\\ &= n^{-(n-1)/n}(\omega_{n-1})^{-(1/n)}\delta_2^{1/n}\big(-n\ \ln|y|\big)^{(n-1)/n} \end{aligned} \tag{28}$$

for all $y \in B_n$. Thus by using (28), we have

$$\begin{aligned} \int_a^\infty e^{\varphi^p(t)-t}\,dt &= \frac{n}{\omega_{n-1}}\int_{|x|\leqslant e^{-a/n}} e^{n(\omega_{n-1})^{1/(n-1)}u^{*(n/(n-1))}(x)}\,dx\\ &= \frac{n}{\omega_{n-1}}\int_{|y|\leqslant 1} e^{n(\omega_{n-1})^{(1/n-1)}\left(g(y)+n^{-(n-1)/n}(\omega_{n-1})^{-(1/n)}\varphi(a)\right)^{n/(n-1)}} e^{-a}dy\\ &\leqslant \frac{n}{\omega_{n-1}}e^{\varphi^p(a)-a}\int_{|y|\leqslant 1} |y|^{-n\delta_2^{1/(n-1)}} e^{cg(y)}dy \end{aligned} \tag{29}$$

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where $c = n^{(2n-1)/n}\big(1/(n-1)\big)(\omega_{n-1})^{1/n}\varphi(a)^{1/(n-1)}$. In the last estimate we used the fact that $(a+b)^{n/(n-1)} \leqslant a^{n/(n-1)} + \big(n/(n-1)\big)ab^{1/(n-1)} + b^{n/(n-1)}$ if $a, b > 0$. Notice that since $\delta_2 \leqslant \big((n-1)/n\big)^{n-1}$, we have $|y|^{(n-1)-n\delta_2^{1/(n-1)}} \leqslant 1$ for all $y \in B_n$. Thus, from (29) by using Holder's inequality we get

$$\int_a^\infty e^{\varphi^p(t)-t}\,dt \leqslant \frac{n}{\omega_{n-1}} e^{\varphi^p(a)-a} \int_{|y|\leqslant 1} |y|^{-(n-1)} e^{cg(y)}\,dy$$
$$\leqslant n^{(2n+1)/(n+1)}(\omega_{n-1})^{-1/(n+1)} e^{\varphi^p(a)-a}\left(\int_{|y|\leqslant 1} e^{c(n+1)g(y)}\,dy\right)^{1/(n+1)}.$$

Note that by (27) we have

$$\int_{|y|\leqslant 1} |\nabla g|^2\,dx = \delta_2 e^{(1-n)a},$$

so if we apply Lemma 3.6 (see (26)) we obtain

$$\int_a^\infty e^{\varphi^p(t)-t}\,dt \leqslant n^{2n/(n+1)} e^{\varphi^p(a)-a} e^{(1/(n+1))(1+1/2+\cdots+(1/n-1))} e^{c'}$$

where

$$c' = \frac{(n+1)^{n-1}}{n-1}\varphi(a)^{n/(n-1)} e^{(1-n)a}\delta_2\,.$$

By property (21) of a and the fact that $xe^{(1-n)x} \leqslant 1/e(n-1)$ for $x \geqslant 0$,

$$c' \leqslant \frac{(n+1)^{n-1}}{e(n-1)^2}\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{1/(n-1)}\delta_2\,.$$

Thus,

$$\int_a^\infty e^{\varphi^p(t)-t}\,dt \leqslant c_1 e^{c_2\delta_2} e^{[(1-((n-1)/n)^{n-1})^{1/(n-1)}-1]a} \tag{30}$$

where

$$c_1 = n^{2n/(n+1)} e^{(1/(n+1))(1+1/2+\cdots+1/(n-1))}$$
$$c_2 = \frac{(n+1)^{n-1}}{e(n-1)^2}\left(1-\left(\frac{n-1}{n}\right)^{n-1}\right)^{1/(n-1)}.$$

∎

4. Applications of 2-dimensional results to 4-dimensional estimates for Δ

As an application of Theorem 3.1 (when $n = 2$, $\Omega = B_2$), we extend the inequality for the gradient on B_2 to the analogue of it for the Laplacian on B_4 under the assumption that our function is radial. We used the property of the Laplacian for the radial function and a change of variable to obtain the following result.

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Corollary 4.8. *Let $u \in C_0^2(B_4)$ be a radial function with $\int_{B_4} |\Delta u|^2\,dx \leqslant 1$. Then there exist $A_1' (= \pi A_1), A_2$ which depend only on n such that*

$$\frac{1}{m(B_4)} \int_{B_4} e^{24\pi^2 u^2}\,dx \leqslant A_1' e^{A_2 \int_{B_4} |\Delta u|^2\,dx}.$$

Remark. In this case (that is, having the assumption about the L_2 norm of Δu), we may not use the symmetrisation technique as we did in the proofs of the previous theorems. In general the relation between $\|\Delta u\|_2$ and $\|\Delta u^*\|_2$ is unknown. Thus, for using the result of Theorem 3.1, we restricted our u to a radial function.

Remark. In [**1**], Adams showed an analogue of the Moser inequality for higher-order derivatives. Specifically, when $n = 4$, it takes the following form. Let Ω be a bounded domain in $\mathbb{R}^4$, $u \in C_0^2(\Omega)$, $\int_\Omega |\Delta u|^2\,dx \leqslant 1$, then there exists an constant c_0 such that

$$\frac{1}{m(\Omega)} \int_\Omega e^{32\pi^2 u^2}\,dx \leqslant c_0.$$

By using the same technique which we used above on the linearised Moser-Trudinger inequality (11) in Section 2, we were able to extend it for the Laplacian on B_4.

Corollary 4.9.

$$\ln \frac{1}{m(B_4)} \int_{B_4} e^{2f}\,dx + \left(\frac{1}{m(B_4)} \int_{B_4} e^{2f}\,dx\right)^{-1} \leqslant 1 + \frac{1}{48 \cdot m(B_4)} \int_{B_4} |\Delta f|^2\,dx$$

for any non-negative radial function f in $C_0^2(B_4)$.

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