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LOCALLY COMPACT HJELMSLEV PLANES AND RINGS

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Introduction. Affine and projective Hjelmslev planes are generalizations of ordinary affine and projective planes where two points (lines) may be joined by (may intersect in) more than one line (point). The elements involved in multiple joinings or intersections are neighbours, and the neighbour relations on points respectively lines are equivalence relations whose quotient spaces define an ordinary affine or projective plane called the canonical image of the Hjelmslev plane. An affine or projective Hjelmslev plane is a topological plane (briefly a TH-plane and specifically a TAH-plane respectively a TPH-plane) if its point and line sets are topological spaces so that the joining of non-neighbouring points, the intersection of non-neighbouring lines and (in the affine case) parallelism are continuous maps, and the neighbour relations are closed.

In this paper we continue our investigation of such planes initiated by the author in [**38**] and [**39**].

One of the main objectives in [**38**] was to present necessary and sufficient conditions for the canonical image of a TH-plane to be an ordinary topological plane. In Section 1, besides collecting some general results, we prove that the canonical image is always a topological plane (1.9).

There are two equivalent ways to define H-planes (see [**33**], [**34**]; [**41**]). We use the previous result to obtain a topological analogue to this fact (1.11). The section ends by correcting some statements from [**39**] (1.12) and then studying connectedness in TH-planes with connected neighbour classes (1.13; 1.14).

In Section 2 we generalize a result of Salzmann and prove that every locally compact hausdorff H-plane is a σ-compact separable metric space (2.7, 2.8).

The main result of Section 3 is: A locally compact hausdorff AH-plane is locally connected (and hence locally arcwise connected, arcwise connected and connected) or the components of each compact neighbourhood are contained in the corresponding neighbour class (3.1.1; 3.2). It then follows that the lines of a locally connected and locally compact hausdorff AH-plane are contractible and locally contractible with trivial homotopy groups (3.4).

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Salzmann ([**48**], page 49) has shown that if an ordinary topological plane is a topological manifold its lines are homeomorphic to real n-space, $\mathbf{R}^n$ ($n = 1, 2, 4, 8$). In TH-planes, however, all dimensionals are possible ([**38**], page 207). In Section 4, using techniques of Löwen [**40**], we prove that TAH-planes which are manifolds have lines homeomorphic to $\mathbf{R}^n$ (4.2); and if the plane is also a translation plane ([**41**]), the lines are n-dimension vector groups (4.3). To conclude Section 4 we present a topological characterization of the (proper) TAH-planes of minimal dimension 4; to be precise the point set is homeomorphic to $\mathbf{R}^4$ if and only if it is a locally compact, locally connected hausdorff space of (topological) dimension four (4.5).

In Section 5 we consider topological Hjelmslev rings (H-ring [**51**], 5.1) whose radicals have a void interior or equivalently whose residue (skew) field is not discrete.

One of the significant results in this section is (5.19): Every locally compact hausdorff H-ring is connected, totally disconnected (equivalently 0-dimensional) or the additive group structure is topologically isomorphic to the cartesian product of an n-dimensional real vector group and a totally disconnected group. The Main Theorem of this section (5.12) characterizes the first two possibilities in 5.19. An H-ring is completely primary and uniserial (for short E-ring) if it has a nilpotent radical. 5.12 then states: A locally compact hausdorff H-ring is an E-ring if and only if it is connected or 0-dimensional. The proof of this result is rather involved and depends on many deep results from the theory of topological rings. (Eg. [**20**] and Kaplansky [**27**] to [**32**]). On the way to proving it we obtain many ancillary results which are of interest in themselves. (See 5.8, 5.10(g), 5.14, 5.16, 5.25.)

In Section 6 the Main Theorem (5.12) is translated into geometric terms as follows. The most significant and widely studied class of H-planes are the H-planes of level n (n-ter Stufe) introduced by Artmann. Moreover, Artmann ([**3**], [**4**]) showed that if $\mathscr{H}$ is an H-ring, then the desarguesian PH-plane, $H(\mathscr{H})$, defined via homogeneous coordinates by Klingenberg ([**33**], Definition 10) is of level n if and only if $\mathscr{H}$ is an E-ring. In [**13**] Drake showed that all finite desarguesian PH-planes are of level n. Combining these results with our Main Theorem 5.12 and replacing finiteness with local compactness we obtain the result (6.3): A locally compact hausdorff desarguesian PH-plane is of level n if and only if the point set is connected or 0-dimensional. Moreover if it is connected, the point set is a topological manifold of dimension $2n(2^m)$ ($m = 0, 1, 2$) and the canonical image is the real plane, the complex plane or the quaternion plane.

We end the paper by constructing in Section 7:

(I) Examples of connected and 0-dimensional desarguesian PH-planes of level n.

(II) Examples of topological desarguesian PH-planes of level 2 (uniform PH-planes) where the neighbour class of a point is its connected component. Hence, these planes are neither connected nor totally disconnected.

Notation. If X is a topological space, $x \in X$, then $\Omega(x)$ is the neighbourhood filter of x; if $A \subseteq X$, then int A and ΓA are the interior and closure of A respectively. $\mathscr{C}(x)$ and $\mathscr{Q}(x)$ are the component and quasicomponent of x; and if $x \in A \subseteq X$, then $\mathscr{C}_A(x)$ and $\mathscr{Q}_A(x)$ are the relative component and quasi-component of x in A. If X is a regular hausdorff (T_2) space, then by the topological dimension of X we mean the small inductive dimension; however, in most cases, we will be dealing with separable metric spaces and so all the usual dimension theories coincide ([**18**]).

Finally, the symbols $\mathbf{Z}^+$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{T}$ denote the set of positive integers, real numbers, complex numbers, quaternions and the torus respectively.

1. Axioms and elementary results for topological Hjelmslev planes. Incidence structures and their homomorphisms are defined as in Dembowski's "Finite Geometries". We only consider infinite incidence structures; the blocks are called lines here. For any incidence structure, $\langle \mathbf{P}, \mathbf{L}, I\rangle$: points are denoted by $P, Q, R, \ldots$ and lines $l, m, n, \ldots$. $\mathbf{L}_p$ denotes the set of lines incident with P and $l \wedge m$ is the set of points incident with both l and m. A *parallelism* of an incidence structure is an equivalence relation $\| \subseteq \mathbf{L} \times \mathbf{L}$. If $\langle \mathbf{P}, \mathbf{L}, I, \|\rangle$ is an incidence structure with parallelism, then Λ_l denotes the equivalence class of $\|$ containing the line l. Equivalence classes of $\|$, Λ, are parallel pencils. A homomorphism $\alpha = (\alpha_1; \alpha_2)$ of incidence structures with parallelism also preserves parallelism, i.e., $l \| m \Rightarrow \alpha_2(l) \| \alpha_2(m)$. Finally, $\langle \mathbf{P}, \mathbf{L}, I\rangle$ is a *topological incidence structure* if $\mathbf{P}$ and $\mathbf{L}$ are topological spaces; and a topological homomorphism of topological incidence structures is a homomorphism (α_1, α_2) so that α_1 and α_2 are continuous.

There are two equivalent definitions for an affine or projective Hjelmslev plane (see [**41**], Definition 2.3 and Satz 2.6 or respectively [**33**], D.0 and D.1 and [**34**], pp. 99-100). The main result of this section is to prove a topological analogue of these results for topological Hjelmslev planes. Consequently, we begin by restating the original definitions of Hjelmslev planes.

1.2 *Definition*. (A) ([**33**]). H $= \langle \mathbf{P}, \mathbf{L}, I\rangle$ is a *projective Hjelmslev plane* (PH-plane for short) if the following axioms hold:

(PH1) If $P, Q \in \mathbf{P}$, then there exists $g \in \mathbf{L}$ so that P, QIg. P and Q are neighbours, $P \sim_{\mathbf{P}} Q$, if and only if there exist $g, h \in \mathbf{L}$, $g \neq h$, so that P, QIg, h. If P is not a neighbour of Q, $P \nsim_{\mathbf{P}} Q$, then $P \vee Q = PQ$ denotes the unique line through P and Q.

(PH2) = the dual of (PH1).

Neighbour lines, $g \sim_{\mathbf{L}} h$, are defined dually and $g \wedge h$ is the unique point of intersection of 2 non-neighbouring lines. In general we write $\sim$ for both $\sim_{\mathbf{P}}$ and $\sim_{\mathbf{L}}$.

(PH3) There exist $P_1, P_2, P_3, P_4 \in \mathbf{P}$ such that $P_i \nsim P_k$ and $P_iP_k \nsim P_tP_j$ for $i \neq j \neq k \neq i$, $i, j, k \in \{1, 2, 3, 4\}$.

(PH4) Let $P \in \mathbf{P}$, $f, g, h \in \mathbf{L}$ and PIf, g, h. If $f \sim g$ and $g \nsim h$ then $f \nsim h$.

(PH5) Let $f, g, h \in \mathbf{L}$. If $f \sim g$ and $g \nsim h$, then $f \wedge h \nsim g \wedge h$.

(PH6) = the dual of (PH5).

It then follows that $l \sim h$ if and only if any point of one line has a neighbour on the other.

$P \sim l$ means there exists XIl so that $P \sim X$.

(B) ([**41**]). An incidence structure with parallelism, $H = \langle \mathbf{P}, \mathbf{L}, I, \| \rangle$ is an affine Hjelmslev plane (AH-plane for short) if the following eight axioms hold:

(AH1) = (PH1). Neighbour points are defined as in (A) and 2 lines l, m are neighbours, $l \sim m$, if and only if any point of one line has a neighbour on the other.

(AH2) There exist three points P_1, P_2, P_3 with $P_i \nsim P_k$ and $P_iP_k \nsim P_tP_j$ for $i \neq j \neq k \neq i$ and $i, j, k \in \{1, 2, 3\}$.

(AH3) If $P \sim Q$ and $Q \sim R$, then $P \sim R$.

(AH4) If PIg, h, then $g \nsim h$ if and only if $|g \wedge h| = 1$.

(AH5) If $P \sim Q$; P, RIg; Q, RIh and $h \nsim g$, then $R \sim P, Q$.

(AH6) If $g \sim h$; PIg, j; QIh, j and $j \nsim g$, then $P \sim Q$.

(AH7) If $g \| h$; PIg, j; $g \nsim j$, then $j \nsim h$ and there exists QIh, j.

(AH8) For each pair $(P, l) \in \mathbf{P} \times \mathbf{L}$, there exists a unique line $h \| l$ so that PIh.

If l lies in the parallel pencil Λ, then $\mathfrak{L}(P, l) = \mathfrak{L}(P, \Lambda)$ is the unique line through P in the pencil $\Lambda = \Lambda_l$.

Two pencils are neighbours, $\Lambda_l \sim \Lambda_m$, if and only if each line of one is neighbour to a line of the other or equivalently $\Lambda_l \nsim \Lambda_m$ if and only if $|l \wedge m| = 1$. If $\Lambda_l \nsim \Lambda_m$, then $l \wedge m$ is the unique point incidence with l and m ([**41**]).

Without loss of generality we may assume that each line of a H-plane is a set of points or $I = \epsilon$.

If l is any line of a PH-plane H then $H_l = \langle \mathbf{P}_l, \mathbf{L}_l, \epsilon \rangle$ is the *derived* AH-*plane associated with* l ([**33**]; [**38**]).

In general we speak of PH, AH or H-planes for short. An H-plane is proper if $\sim \neq (\Delta_{\mathbf{P}}, \Delta_{\mathbf{L}})$. ($\Delta_X$ is the diagonal relation on a set X.)

1.2 *Remark*. For the benefit of the reader we collect the following facts from [**33**], [**34**] and [**41**], and establish some essential notation.

The neighbour relation of a H-plane $H = \langle \mathbf{P}, \mathbf{L}, \epsilon \rangle$ is an equivalence relation on $\mathbf{P}$ and $\mathbf{L}$. $\bar{P}$ and $\bar{l}$ are equivalence classes of $\sim_{\mathbf{P}}$ and $\sim_{\mathbf{L}}$. In $\bar{H} = H/\sim = \langle \mathbf{P}/\sim, \mathbf{L}/\sim, \epsilon \rangle$, the canonical image of H, we consider lines as point sets and identify $\bar{l}$ with $l/\sim = \langle \bar{P} | P \sim l \}$. $\bar{H}$ is an ordinary affine or projective plane and $\pi: H \to \bar{H}$, the (canonical) projection, is an epimorphism with the properties:

(i) $P \sim Q$ if and only if $\pi(P) = \pi(Q)$ for all $P, Q \in \mathbf{P}$,

(ii) $l \sim m$ if and only if $\pi(l) = \pi(m)$ for all $l, m \in \mathbf{L}$, and in the affine case,

(iii) $l \wedge m = \emptyset$ implies $\pi(l) \| \pi(m)$ for all $l, m \in \mathbf{L}$.

1.3 *Definition* ([**38**]). A PH(AH)-*plane* $H = \langle \mathbf{P}, \mathbf{L}, \epsilon \rangle$ $(\langle \mathbf{P}, \mathbf{L}, \epsilon, \| \rangle)$ is a *topological* PH(AH)-*plane* (briefly TH-*plane* and specifically TPH and TAH-*plane*) if the following axioms hold:

(TH1) $\mathbf{P}$ and $\mathbf{L}$ are topological spaces.

(TH2) $\wedge: \mathbf{P} \times \mathbf{P} \backslash \sim_{\mathbf{P}} \to \mathbf{L}$

and

$$\vee: \mathbf{L} \times \mathbf{L} \backslash \sim_{\mathbf{L}} \to \mathbf{P} (\vee: \mathbf{L} \times \mathbf{L} \backslash \{(l, m) | \Lambda_l \sim \Lambda_m\} \to \mathbf{P})$$

are continuous maps. In the AH-case we also assume $\mathfrak{L}: \mathbf{P} \times \mathbf{L} \to \mathbf{L}((P, l) \rightsquigarrow \mathfrak{L}(P, l)$ is a continuous map.

(TH3) $\sim_{\mathbf{P}}$ and $\sim_{\mathbf{L}}$ and closed subsets of $\mathbf{P}$ and $\mathbf{L}$ respectively.

With this definition an ordinary topological plane has a hausdorff point and line set since the neighbour relation is the diagonal relation in this case ([**8**], I, 8.1, Proposition 1). In the Hjelmslev case, the plane is hausdorff if the lines are closed sets (see 1.6(e)).

Every affine plane has a unique projective extension but in the topological setting this is only known to be true if the affine plane is locally compact T_2 and connected ([**47**], § 7). This allows a strong interaction between the affine and projective cases. However in the Hjelmslev situation there are AH-planes with no projective extensions ([**14**]) and so frequently we have to treat the two cases separately.

For TPH-planes, (TH3) means the domain of $\wedge$ is open. We next verify that this is also true for TAH-planes. For ordinary planes this is known as the stability axiom (cf. Löwen, [**40**]).

1.4 Lemma. *IF H is a* TAH-*plane, then the domain of* $\wedge$, $\mathscr{D} = \{(a, b) | |a \wedge b| = 1\}$, *is an open set.*

Proof. We show that the complement of $\mathscr{D}$ is closed. Let $(a_\alpha, b_\alpha)_{\alpha \in I}$ be a net in $\mathbf{L} \times \mathbf{L} \backslash \mathscr{D}$, converging to (a, b). We verify that $|a \wedge b| \neq 1$. If this is false, then $a \wedge b = P$. Also, $|a_\alpha \wedge b_\alpha| \neq 1$ implies $|L(P, a_\alpha) \wedge \mathfrak{L}(P, b_\alpha)| \neq 1$ and so $\mathfrak{L}(P, a_\alpha) \sim \mathfrak{L}(P, b_\alpha)$ by (AH4). But, (TH2) yields $\mathfrak{L}(P, a_\alpha) \to \mathfrak{L}(P, a) = a$ and $\mathfrak{L}(P, b_\alpha) \to \mathfrak{L}(P, b) = b$. Since $\sim_{\mathbf{L}}$

is closed by (TH3) and $(\mathfrak{L}(P, a_\alpha), (\mathfrak{L}(P, b_\alpha)) \in \sim_L$ we have that $(a, b) \in \sim_L$ or (a, b) lies in $\mathbf{L} \times \mathbf{L} \backslash \mathscr{D}$.

1.5 LEMMA. *Let H be a* TAH-*plane. If $l \in \mathbf{L}_P$, the $\mathbf{L}_P \backslash \bar{l}$ is homeomorphic to a line of H.*

Proof. Choose a line a so that $P \nsim a$ and $a \| l$. The map $\vee^P: a \rightarrow \mathbf{L}_{\mathbf{P}} \backslash \bar{l}$ $(X \rightsquigarrow X \vee P)$ has inverse $\vee^a: \mathbf{L}_P \backslash \bar{l} \rightarrow a$ $(x \rightsquigarrow a \wedge x)$. By (TH2) $\vee^P$ and $\wedge^a$ are continuous and so $\mathbf{L}_\mathbf{P} \backslash \bar{l}$ is homeomorphic to a.

The next two results from [**38**] will be needed later, and are listed here for the convenience of the reader.

1.6 THEOREM. *Let H be a* TH-*plane. Then*

(a) *All $\bar{P}$ and $\bar{l}$ are closed sets of* **P** *and* **L** *respectively.*

(b) *All lines are homeomorphic.*

(c) **P** *is a* T_1-*space if and only if* **P** *is a* T_2-*space if and only if all lines are closed subsets of* **P**.

(d) *The point set of H is regular.*

(e) *If H is a* TPH-*plane, then each derived* AH-*plane, H_l, has an open point set.*

If $\{l_1, l_2, l_3\}$ are chosen so that $|l_1 \wedge l_j| = 1$, $i \neq j$, $i, j \in \{1, 2, 3\}$ then $\mathbf{P} = \bigcup_{i=1}^{3} \mathbf{P}_{l_i}$.

(f) $\wedge$ *and* $\vee$ *are open maps.*

If k is any line, then π_k: $l \rightarrow l/\sim$ is the canonical projection restricted to k.

1.7 THEOREM. *If k is a line of a* TH-*plane H, and $k/\sim$ is endowed with the quotient topology, then π_k is an open-continuous map. Hence, $k/\sim$ is* T_2.

Moreover, $k/\sim$ is discrete if and only if there exists $P \in k$ so that $\operatorname{int}(\bar{P} \cap k) \neq \emptyset$ *or equivalently $\bar{P} \cap k$ is an open set.*

1.8 *Coordinates and coordinate maps in* AH-*planes* (see [**37**], [**38**] for details).

Let H be an AH-plane and $\{O, Y, Y\}$ three points as in (AH2). Put $E = L(X, OY) \wedge L(Y, OX)$ and $k = OE$. The elements of k are written as $a, b, c, \ldots,$ and $E = 1$, $O = 0$. The point P can be assigned coordinates $P(x, y)$. Also $\mathbf{L} = \mathbf{L}_1 \cup \mathbf{L}_2$ where $\mathbf{L}_1 = \{l \in \mathbf{L} \mid |l \wedge OY| \neq 1\}$ are the lines of the first kind and $\mathbf{L}_2 = \{l \in \mathbf{L} \mid |l \wedge OY| = 1\}$ are the lines of the second kind. Each $l \in \mathbf{L}_i$ is assigned coordinates $l[m, n]_i$ $(i = 1, 2)$ where $m \sim 0$ if $i = 1$.

From [**38**], 2.4, the coordinate maps of $\{O, X, Y\}$ are denoted by κ: $k \times k \rightarrow \mathbf{P}$, ψ_1: $k \cap \bar{0} \times k \rightarrow \mathbf{L}_1$, and ψ_2: $k \times k \rightarrow \mathbf{L}_2$.

Two ternary operations, T and T', are defined on k so that $(x, y) \in [m, n]_1$ if and only if $x = T'(y, m, n)$; $(x, y) \in [m, n]_2$ if and only if $y = T(x, m, n)$. $\mathscr{B} = \langle k, T, T', 0, 1 \rangle$ is the biternary ring of H with respect to $\{O, X, Y\}$. $\mathscr{R} = \langle k, T, 0, 1 \rangle$ and $\mathscr{R}' = \langle k, T', 0, 1 \rangle$ are the ternary rings of $\{O, X, Y\}$.

$a + b = T(a, 1, b)$ and $a \cdot b = T(a, b, 0)$ are the associated addition and multiplication of $\mathscr{R}$. $\mathscr{A}(R) = \langle k, +, \cdot, 0, 1 \rangle$ is the associated algebra of $\{O, X, Y\}$.

1.8.1 THEOREM ([**38**]). *If H is a* TAH-*plane, and $\{O, X, Y\}$ is a fixed coordinate frame then,*

(a) *The coordinate maps of $\{O, X, Y\}$ are homeomorphisms.*

(b) *The ternary operators of $\{O, X, Y\}$, and hence their associated addition and multiplication are all continuous.*

In ([**38**], § 5, 7) necessary and sufficient conditions were given for the canonical image of a TH-plane to be an ordinary topological affine or projective plane under the identification (quotient) topologies of $\mathbf{P}/\sim$ and $\mathbf{L}/\sim$. We now prove that, in fact, the canonical image is always a topological plane.

1.9 THEOREM. *If H is a* TAH (TPH) *plane, then $H/\sim$, endowed with the quotient topologies of $\mathbf{P}/\sim$ and $\mathbf{L}/\sim$, is a topological affine* (*projective*) *plane. Moreover, the projection $\pi: H \to H/\sim$ is continuous and open.*

Proof. From [**38**], 4.2 and 7.10 it suffices to show that $\pi: \mathbf{P} \to \mathbf{P}/\sim$ is continuous and open. Moreover the proof of 7.10 shows that it suffices to prove the affine case only. Consequently, assume H is a TAH-plane. By definition of the quotient topology π is continuous on points and lines. Let l be a line of H. $\kappa: l \times l \to \mathbf{P}$ and $\bar{\kappa}: l/\sim \times l/\sim \to \mathbf{P}/\sim$ are (1.8) coordinate maps of H and $\bar{H}$ respectively. Then

$$\pi \circ \kappa = \bar{\kappa} \circ \pi_l^2 \quad \text{or} \quad \pi = \bar{\kappa} \circ \pi_l^2 \circ \kappa^{-1}$$

by [**38**], 4.2. Now, endow $l/\sim$ with the identification topology of $\pi_l: l \to l/\sim$. By 1.7, π_l is continuous and open. Hence, by [**42**], Proposition 3.1, p. 249, the product topology of $l/\sim \times l/\sim$ is the identification topology of $\pi_l^2: l \times l \to l/\sim \times l/\sim$. Let $\mathscr{O}$ be the identification topology on $\mathbf{P}/\sim$ with respect to the coordinate map $\bar{\kappa}: l/\sim \times l/\sim \to \mathbf{P}/\sim$. Then, $\bar{\kappa}$ is a homeomorphism with respect to $\mathscr{O}$. Hence $\pi = \bar{\kappa} \circ \pi_l^2 \circ \kappa^{-1}: \mathbf{P} \to \mathbf{P}/\sim$ is open and continuous with respect to $\mathscr{O}$ and so $\mathscr{O}$ is the identification topology on $\mathbf{P}/\sim$ with respect to the canonical map π.

Let $\mathscr{X}$ be a topological space and R an equivalence relation on $\mathscr{X}$. Endow $\mathscr{X}/R$ with the usual quotient topology from the canonical map $\nu: \mathscr{X} \to \mathscr{X}/R$. If $\mathscr{A}$ is a subset of $\mathscr{X}/R$, then $\mathscr{A}$ has two natural topologies:

(i) the subspace topology from $\mathscr{X}/R$.

(ii) the quotient topology from $\nu|\nu^{-1}(A): \nu^{-1}(A) \to A$.

These two topologies are not necessarily equivalent. However, we can prove the following.

1.10 LEMMA. *Let H be a* TH-*plane and k any line of H. Endow $H/\sim$*

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with its quotient topologies. Then, the quotient topology on $k/\sim$ *from* $\pi_k: k \to k/\sim$ *is equal to the subspace topology from* $\mathbf{P}/\sim$.

Proof. As in the proof of 1.9, it suffices to prove the affine case. We can choose a coordinate frame $\{O, X, Y\}$ so that $k = OE$ and $\pi_k: k \to k/\sim$ is continuous and open by 1.7. From the proof of 1.9, the coordinate map $\bar{\kappa}: k/\sim \times k/\sim \to \mathbf{P}/\sim$ is a homeomorphism, with respect to the quotient topologies. Since $H/\sim$ is a topological plane, the coordinate map

$$\bar{\kappa}: k/\sim \times k/\sim \to \mathbf{P}/\sim$$

is also a homeomorphism where $k/\sim$ has the subspace topology from $\mathbf{P}/\sim$. It follows that the quotient topology coincides with the subspace topology on $l/\sim$.

From [**44**], page 262 an ordinary topological projective plane without (TH3) is either indiscrete or hausdorff. Hence it is customary to assume the plane is always hausdorff. If H is a TH-plane, then by our theorem $H/\sim$ is an ordinary hausdorff plane. Moreover it is customary to assume that ordinary planes are not discrete either ([**48**], p. 47). It is thus useful to know when our canonical image is discrete.

1.10.1 COROLLARY. *Let H be a* TH-*plane. The following statements are equivalent.*

(1) *$H/\sim$ is discrete.*

(2) $\text{int}(\bar{P}) \neq \emptyset$ *for each point P.*

(3) $\text{int}(\bar{P} \cap l) \neq \emptyset$ *for some flag (P, l).*

(4) $\text{int}(\bar{P}) \neq \emptyset$ *for some point P.*

Proof. This follows from the fact that the canonical projection $\pi: H \to \bar{H}$ is open-continuous, 1.7 and [**39**] 3.5, 6.4.

Our previous results now allow us to present an equivalent formulation for TH-planes.

1.11 THEOREM. (I) $H = \langle \mathbf{P}, \mathbf{L}, \epsilon \rangle$ *is a* TPH-*plane if and only if*

(a) *H satisfies* (PH1) *and* (PH2).

(b) *H satisfies* (TH1), (TH2) *and* (TH3).

(c) *There exists a topological projective plane H^* and a continuous-open epimorphism $\phi: H \to H^*$ so that $\phi(P) = \phi(Q)$ if and only if $P \sim Q$ for all $P, Q \in \mathbf{P}$; $\phi(l) = \phi(m)$ if and only if $l \sim m$ for all $l, m \in \mathbf{L}$. In this case H^* is topologically isomorphic to $H/\sim$.*

(II) $H = \langle \mathbf{P}. \mathbf{L}, \epsilon, \| \rangle$ *is a* TAH-*plane if and only if*

(a) *H satisfies* (AH1), (AH4), (AH8).

(b) *H satisfies* (TH1), (TH2) *and* (TH3).

(c) *There exists a topological affine plane H^* and a continuous-open epimorphism $\phi: H \to H^*$ so that $\phi(P) = \phi(Q)$ if and only if $P \sim Q$ for all*

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$P, Q \in \mathbf{P}$, $\phi(l) = \phi(m)$ if and only if $l \sim m$ for all $l, m \in \mathbf{L}$ and $l \wedge m = \emptyset \Rightarrow \phi(l) \| \phi(m)$ for all $l, m \in \mathbf{L}$. In this case H^* is topologically isomorphic to $H/\sim$.

Proof. The geometric part of I (II) is found in [**33**] and [**34**] ([**41**] Theorem (2.3)). The topological part follows from 1.9 and [**15**], VI 1.4 and 7.2.

1.12 *Definition.* A topological space X with an equivalence relation $\sim$ is $\sim$*-disconnected* if $X = \mathscr{U} \cap \mathscr{V}$ where $\mathscr{U}$ and $\mathscr{V}$ are open and $(\mathscr{U} \times \mathscr{V}) \cap (\sim) = \emptyset$; otherwise X is $\sim$-connected. $\tilde{\mathscr{C}}(p)$, the $\sim$-component of $p \in X$, is the largest $\sim$-connected set containing p, $\tilde{\mathscr{Q}}(p)$, the quasi $\sim$-component of p, is the intersection of all clopen saturated sets containing P. X is totally $\sim$-disconnected if $\tilde{\mathscr{Q}}(p) = \{x | x \sim p\}$ for all p. $\tilde{\mathscr{C}}_Y(p)$ and $\tilde{\mathscr{Q}}_Y(p)$ are the relative $\sim$-components of $Y \subseteq X$.

Using the numbering (in italics) from [**39**] we now list some corrections to [**39**]:

(a) *5.11(4)*: $\{X | X \sim P\} \subseteq \tilde{\mathscr{Q}}(P)$ and $\mathscr{C}(P) \subseteq \tilde{\mathscr{C}}(P) \subseteq \tilde{\mathscr{Q}}(P)$.

When $\mathscr{H}$ is a TAH-plane with $\mathscr{H}/\sim$ non-discrete,

(b) *5.12 (1)*: For each line l: l is $\sim$-connected or totally $\sim$-disconnected where

$$\tilde{\mathscr{C}}_l(P) \subseteq \tilde{\mathscr{Q}}_l(P) = \bar{P}.$$

5.12 (2): $\mathbf{P}$ is $\sim$-connected or $\tilde{\mathscr{C}}(P) \subseteq \tilde{\mathscr{Q}}(P) = \bar{P}$.

Finally we remark on 7.5 of [**39**].

(c) *7.5 (2)*: The assumption that each $\bar{P}$ is connected must be added.

7.5 (3): The statement is correct but the proof requires a simple addition in line seven. It should read:

"(2) and 5.12 imply that $\bar{P} = \tilde{\mathscr{Q}}(P) \supseteq \tilde{\mathscr{C}}(P) \supseteq \mathscr{C}(P)$ and so $\bar{P} = \mathscr{C}(P) = \mathscr{Q}(P) = \tilde{\mathscr{C}}(P) = \tilde{\mathscr{Q}}(P)$ since $\bar{P}$ is connected."

In [**39**] the author had not proved 1.9. In view of this and our preceding corrections to statements in [**39**] we now collect some results on $\sim$-connectedness in TH-planes.

1.13 THEOREM. *Let H be a* TH-*plane.*

(1) *If $H/\sim$ is non-discrete, then the point set $\mathbf{P}$ is connected or each neighbour class $\bar{P}$ contains the quasicomponent of the point P. Moreover, if H is an* AH-*plane, then $\mathbf{P}$ is connected if and only if one line (and hence all lines) of H is connected.*

(2) *If $H/\sim$ is non-discrete, then the point set $\mathbf{P}$ is $\sim$-connected, or for each point P, $\bar{P} = \tilde{\mathscr{Q}}(P) \supseteq \tilde{\mathscr{C}}(P)$ and so $\mathbf{P}$ is totally $\sim$-disconnected. Moreover, $\mathbf{P}$ is $\sim$-connected if and only if one line (and hence all lines) of H is $\sim$-connected.*

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(3) *The point set* $\mathbf{P}$ *is* $\sim$ *-connected if and only if* $\mathbf{P}/\sim$ *is connected.*

(4) *If* $H/\sim$ *is discrete then* H *is totally* $\sim$ *-disconnected; indeed* $\mathscr{C}(P) \subseteq \tilde{\mathscr{Q}}(P) = \bar{P}$ *and* $\mathscr{C}(P) \subseteq \mathscr{Q}(P) \subseteq \tilde{\mathscr{Q}}(P) = \bar{P}$ *for each point* P.

Proof. (1) and (2) are 3.10 and 6.12 of [**39**]. Since $\pi: H \to H/\sim$ is open and continuous (1.9), (3) follows from [**39**] 5.6; and (4) is [**39**] 8.1.

In Section 4 we will be especially interested in TH-planes whose neighbour classes are connected. We consider some results for such planes now. Now recall that for any point P of a H-plane H, $\mathbf{L}_p = \{l \in \mathbf{L} | P \in l\}$ is the pencil of lines through P; also for any line l of a PH-plane, $\mathbf{P}_l = \{X \in \mathbf{P} | X \nsim l\}$ is the point set of the derived AH-plane H_l.

1.14 THEOREM. *Let* H *be a* TH-*plane and suppose one neighbour class (and hence all* [**39**] 4.1) $\bar{P}$ *is connected.*

(1) *If* H *is a* TAH-*plane, then* $\sim$ *-connectedness is equivalent to connectedness in the sets* $\mathbf{P}$, $\mathbf{L}$ *or* $l \in \mathbf{L}$. *Moreover,* $\mathbf{P}$ *is connected if and only if* l *is connected if and only if* $l/\sim$ *is connected if and only if* $\mathbf{P}/\sim$ *is connected.*

(2) *If* H *is a* TPH-*plane, then* $\sim$ *-connectedness is equivalent to connectedness in the sets* $\mathbf{P}$, $\mathbf{L}$, $\mathbf{L}_p$ $(P \in \mathbf{P})$, $l \in \mathbf{L}$, $\mathbf{P}_l$ $(l \in \mathbf{L})$ *and* $l \backslash \bar{S}$ $(S \in l \in \mathbf{L})$. *Moreover, if one of these sets is connected then so are all the others.*

(3) *For each line* l, $\bar{l}$ *is connected in* $\mathbf{L}$; *and if* $P \in l$ *then* $\bar{P} \cap l$ *and* $\mathbf{L}_P \cap \bar{l}$ *are connected in* l *and* $\mathbf{L}_P$ *respectively.*

Proof. (1) If $X \in \{\mathbf{P}, \mathbf{L}, l(l \in \mathbf{L})\}$, then the quotient maps $\pi: X \to X/\sim$ are continuous-open by 1.7 and 1.9. Our result then follows from [**39**], 5.6 and [**15**] VI 3.4.

(2) By the same argument as in (1) we have our result for $\mathbf{P}$, $\mathbf{L}$ and $l \in \mathbf{L}$. By duality we have it for $\mathbf{L}_P$. Since H_l is a TAH-plane (1.6(e)) we have it for $\mathbf{P}_l$ and $l \backslash \bar{S}$ $(S \in l)$. By [**39**] 6.12, if one of the six sets is $\sim$ -connected so are all the others. The last statement then follows immediately.

(3) Let l be any line and $P \in l$. Choose $X \in l$, $X \nsim P$. Now

$$\vee_P: \bar{X} \to \mathbf{L}_P \cap \bar{l}(Q \rightsquigarrow Q \vee P)$$

is a continuous epimorphism, and so each $\mathbf{L}_P \cap \bar{l}$ is connected. Next we prove $\bar{l}$ is connected. If H is a PH-plane, then by (PH2) $\mathbf{L} = \bigcup_{P \in l} \mathbf{L}_P$ and so

$$\bar{l} = \bigcup_{P \in l}(\mathbf{L}_P \cap \bar{l}).$$

Since $l \in \bigcap_{P \in l} (\mathbf{L}_P \cap \bar{l})$ it follows that $\bar{l}$ is connected. By duality $\bar{P} \cap l$ is connected. Finally, assume H is a TAH-plane. By [**39**] 4.1, $\bar{P} \cap l$ is connected. To show $\bar{l}$ is connected we introduce coordinates in H so that $l = [0, 0]_2$ and $k = [1, 0]_2$. Then $\bar{l} = \{[u, v]_2 | u, v \sim 0]$ is homeomorphic to $(\bar{0} \cap k) \times (\bar{0} \cap k)$ by the coordinate map, ψ_2, of 1.8. Hence, $\bar{l}$ is connected.

2. Locally compact T_2 Hjelmslev planes. A TH-plane is said to possess a topological property (*) if the point set has the property (*).

Our main objective in this section is to prove that every locally compact T_2 TH-plane is a separable metric space. For ordinary planes, this was first proved by Salzmann [**46**] (see also [**48**], § 7) for topological projective planes by first showing that it holds for ternary rings of locally compact projective planes ([**48**] 7.9, 7.10) and then proving it for affine planes by observing that every locally compact affine plane (of positive dimension) has a (unique) compact projective extension ([**48**] 7.17).

However, as mentioned previously, there exist AH-planes with no projective extensions. We thus prove our result first for AH-planes, using some techniques of Löwen [**40**] and then deduce the result for PH-planes using 1.6 (e).

2.1 THEOREM. *If H is a locally compact* T_2 TH-*plane, then*

(a) *all lines of H are locally compact* T_2 *spaces.*

(b) **L** *is locally compact.*

(c) *$\bar{H}$ is a locally compact (affine or projective) plane.*

Proof. (a) follows from 1.6 (c) and [**15**] XI, 6.5 (3).

(b) The domain of $\vee$ is open by (TH3) or 1.4; and $\vee$ is an open continuous map by 1.6 (f). Hence **L** is locally compact by [**15**] XI 6.5(1).

(c) follows like (b) using the fact that the projection $\pi: H \to \bar{H}$ is open-continuous (1.9).

2.2 LEMMA. *Let $\mathscr{A}(\mathscr{R}) = \langle k, +, \cdot, 0, 1\rangle$ be an associated algebra of a* TAH-*plane H. Then,*

(a) *Addition $(+)$ and multiplication $(\cdot)$ are continuous maps.*

(b) *If $-a$ is the unique solution of the equation $a + x = 0$, then $-: k \to k$ is continuous.*

(c) *If a^* is the unique solution of the equation $ax = 1$ $(a \nsim 0)$, then $^*: k\backslash\bar{0} \to k\backslash\bar{0}$ is continuous.*

(d) *If $a \nsim 0$, then the map $f_a: k \to k$ $(x \rightsquigarrow ax)$ is a homeomorphism and* $f_a(0) = 0$.

Proof. (a), (b) and (c) follow from the proof of [**38**] 6.4 ((1) $\Rightarrow$ (2)).

(d). From 1.8.1, $f_a(x) = T(a, x, 0)$ is continuous. Also $f_a^{-1}(x) = x'$ if and only if $ax' = x$ by [**37**] 2.14 (3), and so

$$f_a^{-1}(x) = k \wedge \mathfrak{L}(0[\mathfrak{L}(a, 0Y) \wedge \mathfrak{L}(y, 0x)] \wedge \mathfrak{L}(1, h), g)$$

is continuous.

2.3. LEMMA. *Let H be a locally compact* T_2 TAH-*plane and $\mathscr{A}(\mathscr{R}) = \langle k, +, \cdot, 0, 1\rangle$, the associated algebra of a coordinate frame $\{0, X, Y\}$. If $H/\sim$ is non-discrete, then the following statements hold.*

(a) *There exists a sequence $\{a_n\}$ in k, $a_n \nsim 0, 1$ so that $a_n \to 0$.*

(b) *If* $\mathscr{W} \in \Omega(0)$, *and* $\mathscr{K} \subseteq k$ *is compact, then there exists* $\mathscr{V} \in \Omega(0)$ *such that* $\mathscr{K} \cdot \mathscr{V} \subseteq \mathscr{W}$.

(c) *If* $a_n \to 0$ $(a_n \not\sim 0)$ *and* $\mathscr{V} \in \Omega(0)$ *so that* $\Gamma\mathscr{V}$ *is compact, then* $\{a_n \Gamma \mathscr{V}\}_{n \in \mathbf{Z}^+}$ *is a neighbourhood basis for* 0.

Proof. These results are shown for ternary fields in [**46**], p. 440. The proofs, except for (a), are essentially the same. We now verify (a). Since $k/\sim$ is non-discrete (1.10.1), k has no isolated points by 1.7. Hence, each neighbourhood of k has infinitely many points. We now claim that if $P \not\sim 0, 1$, then there exists a compact neighbourhood $\mathscr{C}$ of P so that $(\bar{0} \cup \bar{1}) \cap \mathscr{C} = \emptyset$. Since $k/\sim$ is T_2 and k is regular (0.6 (d)) there exist closed neighbourhoods $\mathscr{U}, \mathscr{V} \in \Omega(P)$, $\mathscr{W} \in \Omega(0)$ and $\mathscr{Z} \in \Omega(1)$ so that $\mathscr{U} \not\sim \mathscr{W}$ and $\mathscr{V} \not\sim \mathscr{Z}$. Hence $\bar{0} \cap \mathscr{V} = \bar{1} \cap \mathscr{V} = \emptyset$. By 2.1 (a) there is a compact $\mathscr{K} \in \Omega(P)$. Since $\mathscr{U}, \mathscr{V}$ and $\mathscr{K}$ are closed, $\mathscr{C} = \mathscr{U} \cap \mathscr{V} \cap \mathscr{K}$ is a compact neighbourhood of P so that $(\bar{0} \cup \bar{1}) \cap \mathscr{C} = \emptyset$.

Now, take such a $\mathscr{C} \in \Omega(P)$. Then, select a sequence $\{b_n\}$ in $\mathscr{C}$. Hence $b_n \not\sim 0, 1$. Because $\mathscr{C}$ is compact, $\{b_n\}$ has a cluster point $b \in \mathscr{C}$, and a subsequence $\{c_n\}$ converging to b. By 2.2 (b), $a_n = c_n - b \to 0$.

(b) follows as in [**46**] and so does (c), by 2.2 (c), since each $a_n \not\sim 0$.

2.3.1 COROLLARY. *Let H be a locally compact* T_2 TAH-*plane with* $H/\sim$ *non-discrete.* k *is any line. If* $\{X_0, \ldots, X_m\}$ *is a finite subset of* k, *and* $A \in k$ *so that* $A \not\sim X_i$ $(i = 0, \ldots, m)$, *then there exists a compact neighbourhood of* A, $\mathscr{C}$, *so that* $\mathscr{C} \cap \bar{X}_i = \emptyset$ $(i = 0, \ldots, m)$ *and a sequence* $\{A_n\}_{n \in \mathbf{Z}^+}$ *converging to* X_0 *so that for each* n, $A_n \not\sim X_i$ $(i = 0, \ldots, m)$.

Proof. Let $A \not\sim X_i$ $(i = 0, \ldots, m)$. Without loss of generality we can assume l is the point set of an associated algebra $\mathscr{A}(\mathscr{R})$, where $0 = X_0$. Then, there exist closed $\mathscr{U}_i \in \Omega(A)$ so that $\mathscr{U}_i \cap \bar{X}_i = \emptyset$; and compact $\mathscr{K} \in \Omega(A)$. $\mathscr{C} = \mathscr{K} \cap \bigcap_{i=1}^{m} U_i$ is also a compact neighbourhood of A and $\mathscr{C} \cap \bar{X}_i = \emptyset$ $(i = 0, \ldots, m)$. The argument then follows as in (a) of the theorem.

2.4 LEMMA. *Every line of a locally compact* T_2 TAH-*plane* H *with* $H/\sim$ *non-discrete is metrizable.*

Proof. We may assume our line is the point set of an associated algebra $\mathscr{A}(\mathscr{R}) = \langle k, +, \cdot, 0, 1\rangle$. By 2.1 (a) and 2.3 (c), k is a locally compact T_2 space with the first axiom of countability. Because of the continuity of addition, multiplication, and their inverses (where defined) we may argue as in [**48**] 7.8, p. 48, and so we just sketch the remainder of the proof. From 2.2 we can choose a filter base for $\Omega(0)$, $\{a_n \mathscr{V}\}_{n \in \mathbf{Z}^+}$ $(a_n \not\sim 0)$ so that $\Gamma\mathscr{V}$ is compact. Put $\mathscr{V}_n = a_n \cdot \mathscr{V}$. Then, for each n, the sets $\{x + \mathscr{V}_n\}$ form an open covering $\mathscr{B}_n$, and, for each $y \in k$, the stars ([**15**], p. 167) $\{\mathrm{St}(y + \mathscr{V}_n, \mathscr{B}_n)\}$ form a neighbourhood basis for y. Hence, by [**15**]: *IX*, 9.5(3), $\mathscr{A}(\mathscr{R})$ is metrizable.

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As an immediate consequence of 2.4 we have

2.5 THEOREM. *Every locally compact* T_2 TAH-*plane*, $\mathscr{H}$, *with* $H/\sim$ *non-discrete, is metrizable.*

Proof. If H is an AH-plane, then for any line k the point set $\mathbf{P}$ is homeomorphic to $k \times k$ via the coordinate map κ (1.8). Hence, $\mathbf{P}$ is metrizable by [**15**] IX, 7.2.

Next, we prove that local compactness also implies that the plane is separable. First, we need a simple result.

2.6 LEMMA. *Let H be a* TAH-*plane. If $l \| m$ and $Y \not\sim l, m$, then the map* $\wedge(l, Y, m): l \to m (X \rightsquigarrow (X \vee Y) \wedge l)$ *is a homeomorphism.*

Proof. Use (AH7) and (TH2).

2.7 THEOREM. *Let H be a locally compact* T_2 TAH-*plane with* $H/\sim$ *non-discrete. Then,* $\mathbf{P}$ *and* $\mathbf{L}$ *satisfy the second axiom of countability. Hence, H is a separable metric space.*

Proof. It suffices to show any line is second countable since, $\mathbf{P} \cong l \times l$ (1.8) and by (TH3) and 1.b(f) $\vee: \mathbf{P} \times \mathbf{P} \backslash \sim_{\mathbf{P}} \to \mathbf{L}$ is open-continuous with an open domain ([**15**]: VIII 6.2).

Now each line is metrizable by 2.4. Hence, to show l has the second axiom of countability it suffices to show l has a countable dense subset. Let $\mathscr{U}$ be a relatively compact, open subset of any line l where $\Gamma\mathscr{U} \neq l$. In fact, we may choose $\Gamma\mathscr{U}$ so that $\Gamma\mathscr{U} \cap \bar{X} = \emptyset$ for some $X \in l$ by 2 3.1. Now, choose a line y so that $y \wedge l = L$. Let $Y, A \in y$ so that $Y \not\sim A \not\sim L \not\sim Y$. Then there exists a sequence $\{Y_n\}$ so that $Y_n \to Y$ and $Y_n \not\sim Y, L, A$ by 2.3.1.

Let $k_n = \mathfrak{L}(Y_n, l)$, $k = \mathfrak{L}(Y, l)$ and $a = \mathfrak{L}(A, l)$. Then, $k_n \not\sim k, l, a$ and $k \not\sim l \not\sim a \not\sim k$. Next, if $\mathscr{S}(\Gamma\mathscr{U})$ is the saturation of $\Gamma\mathscr{U}$ then, for each n,

$$k \cap \mathscr{S}(\Gamma\mathscr{U}) = k_n \cap \mathscr{S}(\Gamma\mathscr{U}) = \emptyset$$

since $k, k_n \| l$. Now, $\wedge(l, Y, a): l \to a$ is a homeomorphism by 2.6. For each $X \in \Gamma\mathscr{U}$, let

$$X' = \wedge(l, Y, a)(X).$$

Then, $\wedge(k_n, X', l)$ is also defined since $a \not\sim k_n, l$. Hence,

$$\mathscr{U}_n(X) = \Gamma\mathscr{U} \cap [\wedge(k_n, X', l) \circ \wedge(l, Y, k_n)(\mathscr{U})]$$

is an open neighbourhood of X in $\Gamma\mathscr{U}$.

For each fixed n: $\Gamma\mathscr{U} \subseteq \bigcup_{i=1}^{r_n} \mathscr{U}_n{}^i$ where $\mathscr{U}_n{}^i = \mathscr{U}_n(X_n{}^i)$ since $\Gamma\mathscr{U}$ is compact.

Claim. $\{X_n{}^i: n \in \mathbf{Z}^+, 1 \leqq i \leqq r_n\}$ is a dense subset of $\Gamma\mathscr{U}$. Take

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$X \in \Gamma\mathscr{U}$. Then, for each n, there exists $\mathscr{U}_n(X_n^{j(X)}) \ni X$ and so

$$X = \wedge\,(k_n, X_n^{j(X)\prime}, l) \circ \wedge\,(l, Y, k_n)(U_n), \text{ where } U_n \in \mathscr{U}.$$

We show $X_n^{j(X)} \to X$.

Now, without loss of generality, we may assume, because of the compactness of $\Gamma\mathscr{U}$, that $X_n^{j(X)} \to A$ and $U_n \to U$. Then,

$$t_n = U_n \vee Y \to U \vee Y = t$$

implies

$$T_n = t_n \wedge k_n \to t \cap k = Y.$$

Also $X_n^{j(X)\prime} \to A'$. Hence,

$$T_n \vee X_n^{j(X)\prime} \to A' \vee Y$$

implies

$$(T_n \vee X_n^{j(X)\prime}) \cap l = X \to A$$

or $X = A$. Thus, $(X_n^{j(X)}) \to X$.

Since $\mathbf{P} \cong l_1 \times l_2$, for $l_1, l_2 \in \mathbf{L}_P$ $(l_1 \not\sim l_2)$ by 1.8, we can take $\mathscr{W}$, an open relatively compact neighbourhood of a point P so that $\mathscr{W}$ has a countable basis. Then, by 1.6 (f), $\vee_P: \mathscr{W}\backslash\bar{P} \to \mathbf{L}_P$ $(X \rightsquigarrow X \vee P)$ is open and continuous. Also, $\vee_P$ is surjective; otherwise there exists $l \in \mathbf{L}_P$ such that $l \cap \mathscr{W} \cap (\mathbf{P}\backslash\bar{P}) = \emptyset$ or $l \cap \mathscr{W} \subseteq \bar{P} \cap l$, which implies that $\operatorname{int}(\bar{P} \cap l) \neq \emptyset$, a contradiction to 1.10.1. Hence, $\mathbf{L}_P$ has a countable basis. But, $\mathbf{L}_P\backslash\bar{l}$ $(P \in l)$ is isomorphic to a line, by 1.5, and so all lines have a countable basis.

2.7.1 COROLLARY. *A locally compact* T_2 PH-*plane* H *with* $H/\sim$ *non-discrete is a separable metric space.*

Proof. The derived AH-planes of H are open (1.6(e)) and hence locally compact T_2 with non-discrete canonical images. By the theorem, they have the second axiom of countability. Hence, H has the second axiom of countability and is regular (1.6 (d)). By [**15**]: IX, 9.2 our result follows.

We end this section with the following observation.

2.8 THEOREM. *The lines and point set of a locally compact* T_2 TH-*plane with non-discrete canonical image are* σ-*compact.*

Proof. This follows immediately from 2.6 and [**15**] (VIII 6.3; XI 7.2).

3. Connectedness in locally compact T_2 AH-planes. First we remind the reader that for any point P, $\widetilde{\mathscr{C}}(P)$ is the $\sim$-connected component of P (§ 1.12). We can then prove

3.1 THEOREM. *Let k be any line of a locally compact* T_2 TAH-*plane H with $H/\sim$ non-discrete. Then either*

(i) *For each point $P \in k$: if $\mathscr{W} \in \Omega(P)$ is compact, then $\tilde{\mathscr{C}}_{\mathscr{W}}(P) \subseteq \bar{P}$.*

or

(ii) *For each point P: $\Omega(P)$ has a basis of $\sim$-connected sets.*

Proof. If (i) is false, then there is a point P_0 with a compact neighbourhood $\mathscr{W}$ so that $\tilde{\mathscr{C}}_{\mathscr{W}}(P_0) = \mathscr{C} \not\subseteq \bar{P}_0$. By [**39**] 5.7 $\mathscr{C}$ is closed and hence compact. Now, we may assume k is the point set of an algebra $\mathscr{A}(\mathscr{R}) = \langle k, +, \cdot, 0, 1\rangle$ where $0 = P_0$. By 2.1, k is locally compact T_2. Due to 2.3 (b) there exists $\mathscr{V} \in \Omega(0)$ so that $\mathscr{C}\cdot\mathscr{V} \subseteq \mathscr{W}$. But then, $\mathscr{C} \not\subseteq \bar{0}$ implies there is a $c_0 \in \mathscr{C}$, $c_0 \nsim 0$ and so by 2.2 (d), $c_0\mathscr{V} \in \Omega(0)$. Hence, $\mathscr{C}\mathscr{V} \in \Omega(0)$. Since multiplication is continuous and preserves the neighbour relation ([**37**] 2.12) $\mathscr{C}\cdot v$ is $\sim$-connected for each $v \in \mathscr{V}$. Because $0 \in \bigcap_{v\in\mathscr{V}} \mathscr{C}\cdot v$, [**37**] 5.5.2 implies that $\bigcup_{v\in\mathscr{V}} \mathscr{C}\cdot v = \mathscr{C}\cdot\mathscr{V}$ is $\sim$-connected. By [**39**] 5.7, $\Gamma(\mathscr{C}\cdot\mathscr{V})$ is $\sim$-connected and, being closed in $\mathscr{W}$, is compact. Consequently 2.2 allows us to construct the desired neighbourhood basis, $\{a_n\Gamma(\mathscr{C}\cdot\mathscr{V})\}_{n\in\mathbf{Z}^+}$ ($a_n \to 0$ and $a_n \nsim 0$).

3.1.1 COROLLARY. *Let H be a locally compact* T_2 TAH-*plane with $H/\sim$ non-discrete. Then, for each line k either*

(i) *For each point $P \in k$: if $\mathscr{W} \in \Omega(P)$ is compact, then $\mathscr{C}_{\mathscr{W}}(P) \subseteq \bar{P}$.*

or

(ii) *k is locally connected.*

Proof. This is identical to the theorem with connectedness replacing $\sim$-connectedness.

In ordinary locally compact affine planes condition (i) above means there are no totally disconnected compact neighbourhoods. This is true ([**47**], p. 49) if the plane has positive dimension or equivalently is connected.

We were not able to prove this for H-planes, but we do have the following result.

3.2 LEMMA. *Let H be a locally compact* T_2 TAH-*plane with $H/\sim$ non-discrete. The following statements are equivalent.*

(1) *H is locally connected.*

(2) *H is locally arcwise connected.*

(3) *H is arcwise connected.*

If any of the above conditions hold, then H is connected.

Proof. For any line k, $\mathbf{P} \cong k \times k$. Hence, by [**15**], V: 4.3 and Problem 2, Section 5, p. 119, it suffices to show the lemma for a line k.

First we recall again that H is a separable metric space (2.7). Now, [**56**] (p. 38: 5.2 and 5.3) implies the following important result for our proof:

(*) A locally compact, locally connected, connected separable metric space is arcwise and locally arcwise connected.

We now proceed with the proof of the lemma.

(1) $\Rightarrow$ (2): In view of (*) we need only show k is connected. If this is false then 1.12 (1) implies that $\mathscr{C}(P) \subseteq \bar{P}$. Since k is locally connected, there exists an open-connected $\mathscr{K} \in \Omega(P)$, and so $\mathscr{K} \subseteq \mathscr{C}(P) \subseteq \bar{P}$. This yields $\mathrm{int}(\bar{P}) \neq \emptyset$, a contradiction to 1.10.1.

(2) $\Rightarrow$ (3). By [**17**], p. 260, problem L, k is locally connected, and hence connected from the above argument. (*) again yields our result.

(3) $\Rightarrow$ (1). By [**15**], V (5.5), H is connected and each point $P \in k$ has an arcwise connected and hence connected neighbourhood $\mathscr{K}$ so that $\Gamma(\mathscr{K})$ is a compact connected neighbourhood. Considering k as the point set of an algebra $\mathscr{A}(\mathscr{R})$, 2.3 (c) yields our claim.

The last assertion follows from (3) and [**15**], Theorem 5.3, p. 115.

Summing up the previous two results we have

3.3 THEOREM. *Let H be a locally compact* T_2 TAH-*plane with $H/\sim$ non-discrete. Then H is locally connected (and hence locally arcwise connected, arcwise connected and connected) or for each point P: if $\mathscr{W} \in \Omega(P)$ is compact, then $\mathscr{Q}_{\mathscr{W}}(P) = \mathscr{C}_{\mathscr{W}}(P) \subseteq \bar{P}$.*

For the definition of a contractible and locally contractible space, see [**22**].

3.4 THEOREM. *The lines of a locally connected, locally compact* T_2 TAH-*plane H with $H/\sim$ non-discrete are contractible and locally contractible with trivial homotopy groups.*

Proof. We show each line is locally contractible first. By 3.3 H is connected. Moreover, any line k has the same topological properties as H. Let $A \in k$ and take an arbitrary $\mathscr{U} \in \Omega(A)$. Then $\mathscr{U} \cap (\mathbf{P}\backslash\bar{A}) = \emptyset$ or else $\mathrm{int}(\bar{A}) \neq \emptyset$. Hence, there exists $B \in \mathscr{U}$ such that $B \nsim A$. We may then assume k is the point set of an algebra $\mathscr{A}(\mathscr{R}) = \langle k, +, \cdot, 0, 1\rangle$ where $0 = A$ and $1 = B$. In addition, since k is locally compact and locally connected, we may assume, without loss of generality, that $\mathscr{U}$ is open, connected and relatively compact. By 2.3(b) there exists $\mathscr{W} \in \Omega(0)$ so that $\Gamma\mathscr{U}\cdot\mathscr{W} \subseteq \mathscr{U}$. Also by [**56**] (14.3, p. 20 and 5.2, p. 38) $\mathscr{U}$ is arcwise connected. Hence $0, 1 \in \mathscr{U}$ implies there exists a continuous curve e_t: $[0, 1] \rightarrow \mathscr{U}$ so that $e_0 = 1$ and $e_1 = 0$. Now, define $\mathscr{F}: \mathscr{W} \times I \rightarrow k$ by $\mathscr{F}(x, t) = e_t\cdot x$. Then, $\mathscr{F}$ is continuous and

$$\mathscr{F}(\mathscr{W} \times I) \subseteq \mathscr{U}\cdot\mathscr{W} \subseteq \Gamma\mathscr{U}\cdot\mathscr{W} \subseteq \mathscr{U}.$$

Moreover, for all $x \in \mathscr{W}$, $\mathscr{F}(X, 0) = e_0\cdot x = 1\cdot x = x$ and $\mathscr{F}(x, 1) = e_1\cdot x = 0\cdot x = 0$. This means k is locally contractible.

To show that k is contractible we first observe by 3.2 that k is arcwise connected. Then, again taking $\mathscr{A}(\mathscr{R}) = \langle k, +, \cdot, 0, 1\rangle$ we, as above, find a continuous curve e_t: $[0, 1] \to k$ where $e_0 = 1$ and $e_1 = 0$. The homotopy $\mathscr{F}: k \times I \to k$ $((x, t) \mapsto e_t \cdot x)$ then shows k is contractible to $\{0\}$.

The last statement follows from [**22**], Theorem 4-40.

4. TH-planes which are topological manifolds. We use techniques of [**40**] to show that the lines of a TAH-plane, which is a topological manifold, are homeomorphic to $\mathbf{R}^n$ ($n = 1, 2, \ldots$). As mentioned in [**38**] there are TAH-planes whose lines are homeomorphic to $\mathbf{R}^n$ for any $n \in \mathbf{Z}^+$.

4.1 Lemma ([**40**] 1.18). *Let $\mathscr{X}$ be a σ-compact* T_2 *space so that each compact subset of $\mathscr{X}$ is contained in an open, relatively compact n-cell $\mathscr{C} \cong \mathbf{R}^n \subseteq \mathscr{X}$. Then, $\mathscr{X}$ is homeomorphic to $\mathbf{R}^n$.*

4.2 Theorem. *If H is a* TAH-*plane with $H/\sim$ non-discrete, whose point set is a topological manifold, then each line of H is homeomorphic to $\mathbf{R}^n$ for some $n \in \mathbf{Z}^+$. Moreover,* $\mathbf{P}$ *and* $\mathbf{L}$ *have dimension $2n$.*

Proof. Let k be any line of H. Then $k \times k \cong \mathbf{P}$, and so k is also a topological manifold. Hence, k is locally compact T_2 and thus (2.6) a separable metric space. By 2.8 k is σ-compact. 1.5 implies that k is homeomorphic to a set of the form $\mathbf{L}_P\backslash\bar{l}$ $(P \in l)$. We now show a set $\mathbf{L}_P\backslash\bar{l}$ satisfies the assumptions of 4.1, and hence is homeomorphic to some $\mathbf{R}^n$.

Let $\mathscr{K} \subseteq \mathscr{L}_P\backslash\bar{l}$ be compact. Then, by [**15**] XI 7.2, Ex. 2, there exists an open, relatively compact subset of $\mathbf{L}_P\backslash\bar{l}$, $\mathscr{U}$, so that $\mathscr{K} \subseteq \mathscr{U}$. For each $a \in \Gamma\mathscr{U}$, $a \not\sim l$, and so $(a, l) \in \mathscr{D}$, the domain of $\vee$. By 1.4 $\mathscr{D}$ is open, and so there exist $\mathscr{A}_a \in \Omega(a)$ and $\mathscr{B}_a \in \Omega(l)$ so that $\mathscr{A}_a \times \mathscr{B}_a \subseteq \mathscr{D}$. Since $\Gamma\mathscr{U}$ is compact,

$$\Gamma\mathscr{U} \subseteq \bigcup_{i=1}^{m} \mathscr{A}_{a_i}.$$

Put $\mathscr{B} = \bigcap_{i=1}^m \mathscr{B}_{a_i}$. Then

$$\Gamma\mathscr{U} \times \mathscr{B} \subseteq \bigcup_{i=1}^{m} \mathscr{A}_{a_i} \times \mathscr{B} \subseteq \mathscr{D}.$$

$\mathscr{B}$ is thus a neighbourhood of l with the property: $a \in \Gamma\mathscr{V}$ and $x \in \mathscr{B}$ implies $|a \wedge x| = 1$. Also, $\mathscr{B}$ contains lines $x \not\sim P$; otherwise the openness of $\pi: H \to \bar{H}$ implies $\operatorname{int}(\mathbf{L}_P) = \emptyset$ or that $\bar{H}$ is discrete, a contradiction.

If $\Gamma\mathscr{U} \in \Omega(h_0)$, then for each $x \in \mathscr{B}$, $x \not\sim P$, the map $\wedge^x$: $\Gamma\mathscr{U} \to x$ $(U \rightsquigarrow u \wedge x)$ is a homeomorphism with inverse $\wedge^P$: $\wedge_x[\Gamma\mathscr{U}] \to \Gamma\mathscr{U}$ $(X \rightsquigarrow X \vee P)$. Hence $\mathscr{K}_x = \wedge_x[\Gamma\mathscr{V}]$ is a compact neighbourhood of

$X_0 = h_0 \wedge x$, in x. Next, choose a so that $a \| l$, $a \nsim P$ and $a \notin \mathscr{B}$. Now the parallel projection

$$\mathfrak{L}^{h_0}: x \rightarrow a(X \rightsquigarrow \mathfrak{L}(X, h_0) \wedge a)$$

is a homeomorphism for each x near l, by [**39**] 3.5. Thus, $T_x = \mathfrak{L}^{h_0}(K_x)$ is a compact neighbourhood of $A_0 = \mathfrak{L}^{h_0}(X_0)$ in a. We can make T_x arbitrarily near A_0 by taking x arbitrarily close to l, $x \nsim P$. Since a is a manifold, then for a suitable x near l, T_x is contained in an open, relatively compact n-cell, $\mathscr{C}$. Thus

$$\Gamma \mathscr{U} \subseteq \wedge^P \circ \mathfrak{L}^{h_0}(\mathscr{C}).$$

The last claims of the theorem follow from the facts that $\mathbf{P} \cong l \times l$ and in any coordinate system $\mathbf{L}_2$ is an open set homeomorphic to $l \times l$ (1.8).

We refer the reader to [**41**] for the definition of an AH-translation plane. We can then get a result analogous to 4.2.

4.3 THEOREM. *Let H be a locally connected, locally compact* T_2 TAH-*translation plane with $H/\sim$ non-discrete. Suppose $A(R) = \langle k, +, \cdot, 0, 1\rangle$ is an associated algebra. Then $\langle k, +\rangle$ is topologically isomorphic to the real vector group $\mathbf{R}^n$.*

Proof. $k^+ = \langle k, +\rangle$ is a locally connected, connected, locally compact T_2 separable metric abelian group by 3.2, 2.7 and [**37**] 3.8. By a result of [**45**] (Theorem 43, p. 170) $k^+ = \mathbf{R}^n \times \mathbf{T}^q$ where $0 \leqq q \leqq \aleph_0$ and $0 \leqq n < \aleph_0$. It is well known that the Torus, $\mathbf{T}$, has a non-trivial fundamental group ([**22**], p. 129). We use this fact to show that $q = 0$, and hence obtain our result. Suppose $q > 0$. Now the continuity of projections implies that the Torus $\mathbf{T}$ is a retract of k. By 3.4 and [**22**], Theorem 4.11 $\mathbf{T}$ is contractible and hence has a trivial fundamental group, a contradiction.

For ordinary topological affine planes which are manifolds, the only possible dimensions for a line is 1, 2, 4 or 8. This is true, since any ternary field K with multiplication, $\cdot: K\backslash\{0\} \times K\backslash\{0\} \rightarrow K\backslash\{0\}$, can be projected onto the sphere $(x \rightsquigarrow x/\|x\| (x \neq 0))$ to form a Hopf space. By Adam's result the only spheres which are Hopf spaces have dimension 1, 2, 4 or 8 (cf. [**48**], p. 49). This procedure fails for H-planes because of the existence of zero divisors in our biternary ring (cf. [**37**] 2.7(e) and 2.11). However, we can obtain results if the neighbour classes are connected. In Section 6 we study special planes where this is the case and obtain some dimension results (cf. 6.4).

We consider the general case next.

4.4 THEOREM. *Let H be a proper locally compact* T_2, *locally connected*

TAH-*plane with connected neighbour classes and* $H/\sim$ *non-discrete. Then,* $\dim(\mathbf{P}) \geqq 4$ *and for any line* k $\dim(k) \geqq 2$.

Proof. By 2.7 H is a separable metric space. Since each TPH-plane contains an open TAH-plane, it suffices to prove the AH-case. Suppose $n = \dim(\mathbf{P}) < 4$. Since $\mathbf{P} \cong l \times l$, l is also a locally compact T_2, locally connected separable metric space. By [**19**] $2 \dim(l) - 1 < 4$ or $\dim(l) \leqq 2$. Also 3.3 and 3.4 imply that k is connected and locally contractible. Moreover from [**39**] 3.4, 3.5 k is homogeneous. Hence [**7**], Corollary 8.2 implies that k is a topological manifold, and by 4.2 homeomorphic to $\mathbf{R}^n$ ($n = 1, 2$). Now for $P \in k$, $\bar{P} \cap k$ is connected, by 1.11 (3). Hence $\dim(\bar{P} \cap k) \geqq 1$ or else by [**23**], p. 15, $\bar{P} \cap k = \{P\}$ which implies H is an ordinary plane, a contradiction. Also, $\operatorname{int}(\bar{P} \cap k) = \emptyset$ means $\bar{P} \cap k$ is not open and so by [**23**], IV 3 $\dim(\bar{P} \cap k) < n$. Hence $\dim(\bar{P} \cap k) = 1$. Consequently, $\dim(k) = 2$ and thus $\dim(\mathbf{P}) = 4$, a contradiction. Finally, notice that the above argument also proves that $\dim(k) \geqq 2$ for any line k.

We next give a structure theorem for the smallest dimension, 4.

4.5 Theorem. *Let H be a proper* TAH-*plane whose neighbour classes are connected and suppose* $H/\sim$ *is non-discrete. Then, the following statements are equivalent.*

(1) *H is a locally connected, locally compact* T_2 TAH-*plane of topological dimension* 4.

(2) $\mathbf{P}$ *is homeomorphic to* $\mathbf{R}^4$.

(3) *Each line of H is homeomorphic to* $\mathbf{R}^2$.

(4) *Each line of H is a locally connected, locally compact* T_2 *space of topological dimension* 2.

Proof. A simple modification of the argument in the proof of 4.4 yields [(1) ⇒ (2)] and [(2) ⇒ (3)]. [(3) ⇒ (4)] is immediate from the well known topological properties of $\mathbf{R}^n$. (4) ⇒ (1): Since $\mathbf{P} \cong l \times l$, $\mathbf{P}$ has the same topological properties as l except for dimension. By 4.4 and [**23**], p. 33, $\dim(\mathbf{P}) = 4$.

4.5.1 *Comment.* For the projective analogue of 4.5 we must replace $\mathbf{R}^4$ and $\mathbf{R}^2$ in conditions (2) and (3) of 4.5 by 4- and 2-dimensional manifolds.

5. Locally compact T_2 H-rings.

All rings, $\mathscr{H}$, are associative with unit 1 and $0 \neq 1$. $\mathscr{J}(\mathscr{H}) = \mathscr{J}$ is the jacobson radical and $U(\mathscr{H}) = U$ is the group of units. $\mathscr{H}$ is local (= completely primary) if H/J is a division ring (= skew field) ([**35**], p. 75).

5.1 *Definition* ([**51**], 5.1). A (not necessarily commutative) ring, $\mathscr{H}$ is a *Hjelmslev* ring (H-ring) if the following conditions hold:

(1) The lattice of left ideals and the lattice of right ideals both form a chain.

(2) The zero-divisors, $\mathscr{N}$, form a two-sided ideal and each zero-divisor is a two sided zero-divisor.

(3) Each nonzero-divisor is a unit.

5.1 PROPOSITION ([**51**] 5.1; [**35**] 3.7, Proposition 1). *Let $\mathscr{H}$ be an* H-*ring. Then*

(i) *$\mathscr{J}$ is the unique maximal left (right) ideal of $\mathscr{H}$ and equals the two sided ideal of zero-divisors $\mathscr{N}$. Hence $\mathscr{J}$ contains every proper left (right) ideal.*

(ii) *$U(\mathscr{H}) = \mathscr{H} \setminus \mathscr{J}$ or equivalently $\mathscr{J}$ is the set of non-units.*

(iii) *$\mathscr{H}/\mathscr{J}$ is a skew field; that is $\mathscr{H}$ is local.*

(iv) *For each element $h \in \mathscr{H}$: h is a unit or $1 - h$ is a unit. Moreover h is a left unit if and only if h is a right unit if and only if h is a unit.*

$\bar{\mathscr{H}} = \mathscr{H}/\mathscr{J}$ is the residue (skew) field of H and $\nu: \mathscr{H} \to \mathscr{H}/\mathscr{J}$ is the canonical homomorphism ($h \mapsto h + \mathscr{J}$).

5.2 *Definition* ([**8**] *I*). A ring $\mathscr{R}$ is a *topological* ring if it possesses a topology so that the mappings $(a, b) \mapsto a + b$, $a \mapsto -a$ and $(a, b) \mapsto ab$ are continuous. $\mathscr{R}^+ = (\mathscr{R}, +)$ is then an abelian topological group.

If, in addition, the (multiplicative) group of units of $\mathscr{R}$, $U(\mathscr{R}) = U$, is an open topological group then $\mathscr{R}$ is a *Gelfand* ring ([**8**], I, p. 317, Ex. 11). $\mathscr{R}$ is a *topological division* ring if $\mathscr{R}$ is a topological ring and $a \mapsto a^{-1}$ ($a \neq 0$) is also continuous.

Clearly a hausdorff topological division ring is a Gelfand ring.

5.3 *Remark*. If $\mathscr{H}$ is a topological H-ring, then $\nu: \mathscr{H} \to \bar{\mathscr{H}}$ is an open-continuous map and $\bar{\mathscr{H}}$ is a topological ring (inversion may not be continuous). Also $\bar{\mathscr{H}}$ is discrete if and only if the interior of the radical $\mathscr{J}$ is non-empty ([**39**], 3.5, [**38**] 6.4 and [**24**] Proposition 16, p. 58).

Hence we will usually assume $\operatorname{int} \mathscr{J} = \emptyset$ or equivalently that $\bar{\mathscr{H}}$ is non-discrete (see 1.10.1).

Next we obtain some useful results on the ideals of $\mathscr{H}$.

5.4 PROPOSITION. *Let $\mathscr{H}$ be a topological* H-*ring so that* $\operatorname{int} J = \emptyset$. *Then*

(i) *$\mathscr{H}$ has no proper open left (right) ideals.*

(ii) *$\mathscr{H}$ has no nonzero discrete left (right) ideals.*

(iii) *$\mathscr{H}$ has no nonzero compact right (left) ideals. In particular $\mathscr{H}$ and $\mathscr{J}$ are neither discrete nor compact.*

Proof. By 5.1 (i) $\mathscr{J}$ contains every left (right) ideal of $\mathscr{H}$. Because $\operatorname{int} \mathscr{J} = \emptyset$, (i) follows immediately. Now a left (right) ideal of $\mathscr{H}$ is a left (right) module over $\mathscr{H}$ and so (ii) and (iii) follow from [**20**], Proposition (1.1).

5.5 PROPOSITION. *Let* $\mathscr{H}$ *be a topological* H-*ring.*

(1) $\mathscr{H}_0$, *the component of* 0, *is an ideal of* $\mathscr{H}$.

(2) $\mathscr{H}$ *is connected or* $\mathscr{H}_0 \subseteq \mathscr{J}$.

Proof. (1) is [**8**], Vol. I, exercise 3, p. 316 and (2) is then immediate from 5.1 (i).

5.6 *Definition. Let* $\mathscr{R}$ *be a ring and* $\mathscr{I} \subseteq \mathscr{R}$.

(a) $\mathscr{I}^r = \{x \in \mathscr{R} | \mathscr{I}x = \{0\}\}$ is the right annihilator of $\mathscr{I}$. Similarly we define the left annihilator $\mathscr{I}^l$.

(b) If $\mathscr{I}_1$ and $\mathscr{I}_2$ are 2 left (right) ideals of $\mathscr{R}$, then $\mathscr{I}_1 \sqsubset \mathscr{I}_2$ means if $I_1 \subseteq \mathscr{K} \subseteq \mathscr{I}_2$ for a left (right) ideal $\mathscr{K}$, then $\mathscr{I}_1 = \mathscr{K}$ or $\mathscr{I}_2 = \mathscr{K}$. If $\mathscr{I}_1 \sqsubset I_2$, then I_1 is a lower neighbour of $\mathscr{I}_2$ ([**10**]).

5.7 PROPOSITION. *Let* $\mathscr{R}$ *be a* T_2 *topological ring.*

(a) *For each left (right) ideal* $\mathscr{I}$, $\mathscr{I}^{rl}(\mathscr{I}^{lr})$ *is a closed left (right) ideal.* ([**32**] *page* 689).

(b) *If* $\mathscr{R}$ *is an* H-*ring, then all principal left (right) ideals are closed, i.e.,* $(\mathscr{R}_a)^{rl} = \mathscr{R}_a$.

(c) *If* $\mathscr{R}$ *is an* H-*ring, then for any left (right) ideal* $\mathscr{I}$, *either* $\mathscr{I}$ *is closed or* $\mathscr{I}^{rl}(\mathscr{I}^{lr})$ *is the closure of* $\mathscr{I}$. *In fact one of the following holds;* $\mathscr{I} = \Gamma\mathscr{I} = \mathscr{I}^{rl}$, $\mathscr{I} = \Gamma\mathscr{I} \sqsubset \mathscr{I}^{rl}$ *or* $\mathscr{I} \sqsubset \Gamma\mathscr{I} = \mathscr{I}^{rl}$.

Proof. Now $\Gamma\mathscr{I}$ is a left (right) ideal and so $\mathscr{I} \subseteq \Gamma\mathscr{I} \subseteq \mathscr{I}^{rl}(\mathscr{I}^{lr})$. [**51**] 5.8, 5.9 then yield (b) and (c) respectively.

We now obtain some essential properties of $\mathscr{H}$ for our investigations, all of which arise from rather deep results.

5.8 THEOREM. *Let* $\mathscr{H}$ *be a locally compact* T_2 H-*ring. Then*

(1) $\mathscr{J}$ *is a closed ideal.*

(2) $U = U(\mathscr{H})$ *is an open topological group under multiplication; that is* H *is a Gelfand ring.*

In addition if int $J = \emptyset$, *then*

(3) $\mathscr{H}$ *is a separable metric space.*

(4) $\overline{\mathscr{H}} = \mathscr{H}/\mathscr{J}$ *is a non-discrete locally compact* T_2 *division ring (skew field). Hence* $\overline{\mathscr{H}}$ *is connected or zero-dimensional. In the first case,* $\overline{\mathscr{H}}$ *is one of the classical (skew) fields* **R**, **C** *or* **H**.

Proof. (1) is Theorem 1 of [**29**] and (2) follows from 5.1 (ii) and [**16**]. Because of (1) and (2), $\mathscr{H}$ is a line of a locally compact T_2 TAH-plane ([**38**], 6.4) and so (3) follows from 2.7. Next (4) results from 5.1, 5.3, [**24**] Proposition 15, p. 58 and Theorem 11, p. 60; and [**9**], pp. 433-434.

In [**13**] Lemma 5.2 Drake showed that every finite H-ring has a nilpotent radical. Our main objective in this section is to characterize those locally compact T_2 H-rings with nilpotent radicals. We begin by collecting some pertinent facts about such rings from [**3**], [**4**] and [**51**], some of which are not explicitly stated in these papers.

5.9 *Definition.* (1) A ring $\mathscr{A}$ with unit is *completely primary and uniserial* (E-ring) if there is a (two-sided) ideal $\mathscr{I}$ of $\mathscr{A}$ such that every left or right ideal is of the form $\mathscr{I}^i$ (where $\mathscr{I}^0 = \mathscr{A}$). The rank of such a ring is the smallest integer n such that $\mathscr{I}^n = \{0\}$ ([**4**]).

Clearly all ideals of such a ring are two-sided.

(2) A two-sided ideal $\mathscr{P}$ of a ring $\mathscr{A}$ is *prime* if $\mathscr{P} \neq \mathscr{A}$ and for all $a, b \in \mathscr{A}$: $a\mathscr{A}b \subseteq \mathscr{P} \Rightarrow a \in P$ or $b \in \mathscr{P}$. $\mathscr{A}$ is a *prime ring* if $\{0\}$ is a prime ideal. Thus $\mathscr{P}$ is a prime ideal if and only if $\mathscr{A}/\mathscr{P}$ is a prime ring ([**35**], p. 54).

(3) A two-sided ideal $\mathscr{P}$ of a ring $\mathscr{A}$ is *completely prime* if $P \neq \mathscr{A}$ and for all $a, b \in \mathscr{A}$: $ab \in \mathscr{P} \Rightarrow a \in \mathscr{P}$ or $b \in \mathscr{P}$ ([**51**] 5.11).

(4) ([**51**]). Let $\mathscr{I}$ be a two-sided ideal of a ring $\mathscr{A}$.

$$S_l(I) = \{s \in \mathscr{A} | st \in \mathscr{I} \text{ implies } t \in \mathscr{I} \text{ for any } t \in \mathscr{A}\}$$

$$S_r(I) = \{s \in \mathscr{A} | ts \in \mathscr{I} \text{ implies } t \in \mathscr{I} \text{ for any } t \in \mathscr{A}\}.$$

Next, we collect some equivalent definitions of an E-ring, due to Törner and Artmann, and add one new one, which we will require later to characterize locally compact E-rings.

5.10 THEOREM. *Let $\mathscr{H}$ be an* H-*ring with radical $\mathscr{J}$. The following statements are equivalent.*

(a) *$\mathscr{H}$ is an* E-*ring.*

(b) *$\mathscr{J}$ is nilpotent.*

(c) *$\mathscr{H}$ is a left (right) Noetherean ring.*

(d) *$\mathscr{H}$ is a left (right) Artinean ring.*

(e) *All zero divisors are nilpotent and $\mathscr{J} = \mathscr{H}a$, for some $a \in \mathscr{H}$.*

(f) *$\mathscr{H}$ possesses exactly one prime ideal and $\mathscr{J} = \mathscr{H}a$.*

(g) *$\mathscr{H}$ possesses exactly one completely prime ideal and $\mathscr{J} = \mathscr{H}a$.*

Proof. The equivalence of the first four conditions is just [**3**], Satz 2.6 and [**51**] 5.27; and [(e) $\Leftrightarrow$ (f)] follows from [**51**] 5.23. Also [(e) $\Leftrightarrow$ (a)] is 4.23 Satz of [**53**]. To complete the proof we show the equivalence of (f) and (g).

(f) $\Rightarrow$ (g): This is immediate from the fact that every completely prime ideal is prime.

(g) $\Rightarrow$ (f): By 5.1 $\mathscr{J}$ is the unique completely prime ideal.

First we claim that for any two-sided ideal, $\mathscr{I} \neq \mathscr{J}$, $\mathscr{H}/\mathscr{I}$ is an H-ring. By [**51**] 5.17 $\mathscr{H}\backslash S_l(\mathscr{I})$ and $\mathscr{H}\backslash S_r(\mathscr{I})$ are completely prime ideals and hence both are equal to $\mathscr{J}$. Hence, $S_l(\mathscr{I}) = S_r(\mathscr{I}) = U(\mathscr{H})$ and so by [**51**] 5.19 $\mathscr{H}/\mathscr{I}$ is a PH-ring.

Now, let $\mathscr{P}$ be a prime ideal of $\mathscr{H}$. We must show $\mathscr{P} = \mathscr{J}$. Suppose $\mathscr{P} \neq \mathscr{J}$ and so $P \subsetneq \mathscr{J}$. Since $\mathscr{J} = \mathscr{H}a$, [**51**] 5.14 and [**53**] 4.22 yield

$\mathscr{P} \neq \{0\}$. Thus, $\mathscr{H}/\mathscr{P}$ is a prime H-ring with a principal radical $J/\mathscr{P} = (a + \mathscr{P})\mathscr{H}/\mathscr{P}$. Hence, $\{\mathscr{P}\}$ is a prime ideal of $\mathscr{H}/\mathscr{P}$ and again by [**53**] 4.22, $\{\mathscr{P}\}$ is a lower neighbour in the lattice of left ideals. From [**51**] 5.14 $\{\mathscr{P}\} = \mathscr{J}(\mathscr{H}/\mathscr{P}) = J/\mathscr{P}$ or $\mathscr{J} = \mathscr{P}$, a contradiction.

Hence, $\mathscr{J}$ is the unique prime ideal of $\mathscr{H}$.

From [**51**] 5.5, 5.26, 5.8, 7.1 and [**35**], p. 20, Proposition 2 we have

5.11 PROPOSITION. *Let $\mathscr{H}$ be an* E-*ring of rank $n + 1$. Then*

(i) $\mathscr{J} = \mathscr{H}a = a\mathscr{H}$, *for some $a \in \mathscr{H}$.*

(ii) *For every $a \in \mathscr{H}$: $a\mathscr{H} = \mathscr{H}a$ and $a\mathscr{J} = \mathscr{J}a$.*

(iii) *If $J = \mathscr{H}a$, then every ideal of $\mathscr{H}$ has the form $\mathscr{J}^i$ where $\mathscr{J}^i = \mathscr{H}a^i = a^i\mathscr{H} = \mathscr{J}a^{i-1} = a^{i-1}\mathscr{J}$ $(i = 1, \ldots, n)$.*

(iv) *For each ideal $\mathscr{I}$: $\mathscr{I}^{rl} = \mathscr{I}$.*

(v) *If $\mathscr{I} = \mathscr{J}^i$, then $\mathscr{H}/\mathscr{I}$ is an* E-*ring of rank i $(1 \leqq i \leqq n)$, $\mathscr{J}(\mathscr{H}/\mathscr{I}) = \mathscr{J}/\mathscr{I}$ and $(\mathscr{H}/\mathscr{I})/\mathscr{J}(\mathscr{H}/\mathscr{I}) \cong \mathscr{H}/\mathscr{J}$.*

We next state the main result of this section. The proof will consist of a lengthy string of results, many of interest in themselves.

5.12 MAIN THEOREM. *Let $\mathscr{H}$ be a locally compact* T_2 H-*ring with* int $\mathscr{J} = \emptyset$. *Then $\mathscr{H}$ is an* E-*ring if and only if $\mathscr{H}$ is connected or totally disconnected (equivalently* 0-*dimensional).*

We begin our assault on this result by examining topological E-rings of rank $n + 1$ $(n > 0)$.

5.13 PROPOSITION. *Let $\mathscr{H}$ be a topological* E-*ring of rank $n + 1$ so that* int $\mathscr{J} = \emptyset$. *If $\mathscr{I} = \mathscr{J}^i$ $(1 \leqq i \leqq n)$, then $\mathscr{H}/\mathscr{J}^i$ is a topological* E-*ring of rank i, so that its radical has an empty interior. Moreover the residue field of $\mathscr{H}/\mathscr{J}^i$, $(\mathscr{H}/\mathscr{J}^i)/(\mathscr{J}/\mathscr{J}^i)$, is topologically isomorphic to $\mathscr{H}/\mathscr{J}$.*

Proof. Since the canonical homomorphism $\nu_i: \mathscr{H} \rightarrow \mathscr{H}/\mathscr{J}^i$ is open-continuous, the proof follows immediately from 5.11 (iv) and [**24**], Proposition 20, p. 61.

5.14 PROPOSITION. *Let $\mathscr{H}$ be a* T_2 *topological* E-*ring. Then all the ideals of $\mathscr{H}$ are closed. In particular, $\mathscr{H}$ is a dual ring in the sense of* [**32**].

Proof. This is immediate from 5.11 (iv) and 5.7.

5.15 PROPOSITION. *Let $\mathscr{H}$ be a locally compact* T_2 *topological* E-*ring of rank $n + 1$ $(n > 0)$ with* int$(\mathscr{J}) = \emptyset$. *Then, $\mathscr{J}^n$ is a one dimensional topological linear space over $\mathscr{H}/\mathscr{J}$ and is topologically isomorphic to $\mathscr{H}/\mathscr{J}$ as a topological linear space over itself.*

Proof. We define a scalar multiplication on $\mathscr{J}^n$ over $\overline{\mathscr{H}}$ by $\phi: \mathscr{J}^n \times \overline{\mathscr{H}} \rightarrow \mathscr{J}^n$ $((x, h + J) \mapsto xh)$. First we show that ϕ is well defined. Since $\mathscr{J}^n$ is an ideal $xh \in \mathscr{J}^n$; also if $h + \mathscr{J} = \tilde{h} + \mathscr{J}$, then $x(h - \tilde{h}) \in$

$\mathscr{J}^{n+1} = \{0\}$ and so $xh = x\tilde{h}$. Since $\nu: \mathscr{H} \to \overline{\mathscr{H}}$ and multiplication in $\mathscr{H}$ are continuous, so is ϕ. Hence, $\mathscr{J}^n$ is a topological linear space over $\overline{\mathscr{H}}$. By 5.11 $\mathscr{J}^n = \mathscr{H}b = b\mathscr{H}$ and so $\mathscr{J}^n$ is one dimensional over $\overline{\mathscr{H}}$. By 5.8 $\overline{\mathscr{H}}$ is a non-discrete locally compact T_2 skew field. From [**31**] Theorem 8 $\overline{\mathscr{H}}$ possesses a valuation (i.e., absolute value) which preserves the topology of $\overline{\mathscr{H}}$. Then $\overline{\mathscr{H}}$ is of type $\vee$ ([**31**], pp. 529-530) and hence by [**31**] Theorem 15 we conclude that $\mathscr{J}^n$ is topologically isomorphic to $\overline{\mathscr{H}}$ as a topological linear space over itself.

The preceding proposition is crucial in our study of topological E-rings.

5.16 THEOREM. *Let* $\mathscr{H}$ *be a locally compact* T_2 E-*ring of rank* $n + 1$ $(n > 0)$ *with* $\operatorname{int} \mathscr{J} = \emptyset$.

The following statements are equivalent.

(1) $\mathscr{H}$ *is connected.*

(2) $\mathscr{J}$ *is connected.*

(3) $\overline{\mathscr{H}}$ *is connected, and hence is one of the classical division rings* **R**, **C** *or* **H**.

If any of the above conditions hold, then all proper ideals of $\mathscr{H}$ *are connected and locally compact with empty interior.*

Proof. [(1) $\Rightarrow$ (2)] follows from 5.1 (iii).

[(2) $\Rightarrow$ (3)]: If $\mathscr{J} = \mathscr{H}a$ is connected, then by 5.11 (iii) $\mathscr{J}^n = \mathscr{J}a^{n-1}$ is also connected. Then 5.15 and 5.8 (4) give us our claim.

[(3) $\Rightarrow$ (1)]: We proceed by an induction on the rank $n + 1$. If $n = 1$, then the ideals of $\mathscr{H}$ are $\{0\}$, $\mathscr{J}$ and $\mathscr{H}$. By 5.15 $\mathscr{J}$ is connected and hence ([**23**], 7.14) so is $\mathscr{H}$. Now assume the implication is true for all E-rings of rank $i < n + 1$. By 5.13 $\mathscr{H}/\mathscr{J}^n$ is a topological E-ring of rank n whose radical has an empty interior; and its residue field is homeomorphic to $\overline{\mathscr{H}}$ and hence connected. By the induction hypothesis $\mathscr{H}/\mathscr{J}^n$ is connected. Then 5.15 implies that $\mathscr{J}^n$ is also connected and hence ([**23**] 7.14) $\mathscr{H}$ is too.

The last statement follows immediately from 5.11 (ii).

5.17 THEOREM. *Let* $\mathscr{H}$ *be a locally compact* T_2 E-*ring of rank* $n + 1$ $(n \geqq 0)$ *with* $\operatorname{int}(\mathscr{J}) = \emptyset$. *Then* $\mathscr{H}$ *is connected or totally disconnected (equivalently* 0-*dimensional).*

Proof. Suppose $\mathscr{H}$ is not connected. By [**18**], p. 33, 0-dimensionality and totally disconnectedness are equivalent. Since $\mathscr{H}_0$ is an ideal (5.5) it suffices to show that $\mathscr{H}$ possesses no nonzero connected ideals. We proceed by induction on the rank $n + 1$. If $n = 0$, then $\mathscr{J} = \{0\}$ and so $\mathscr{H}$ is a skew field. Consequently the only ideals of $\mathscr{H}$ are $\{0\}$ and $\mathscr{H}$ and our result follows. Now suppose all E-rings of rank $i < n + 1$ $(n > 0)$ which are disconnected have no nonzero connected ideals. Now $\mathscr{H}/\mathscr{J}^n$

is of rank n and is disconnected; otherwise its residue field $\overline{\mathscr{H}}$ (5.13) is connected and hence so is $\mathscr{H}$ by 5.16. By the induction hypothesis $\mathscr{H}/\mathscr{J}^n$ has no nonzero connected ideals, and by 5.16 and 5.15 $\mathscr{J}^n$ is totally disconnected. Let $\mathscr{J}^i \neq \{0\}$ be a connected ideal of $\mathscr{H}$ (5 (iii)). Then, $0 < i < n$ and so $\mathscr{J}^i/\mathscr{J}^n$ is a connected ideal of $\mathscr{H}/\mathscr{J}^n$. Consequently $\mathscr{J}^i/\mathscr{J}^n = \{\mathscr{J}^n\}$ or $i = n$, a contradiction. Hence, $\mathscr{H}$ possesses no nonzero connected ideals.

Since a locally compact T_2 H-ring is a separable metric space (5.8 (3)), all the standard dimension theories coincide ([**18**]). We then obtain

5.18 THEOREM. *Let $\mathscr{H}$ be a locally compact T_2 E-ring of rank $n + 1$ ($n \geqq 0$) with* $\operatorname{int} \mathscr{J} = \emptyset$. *Then $\mathscr{H}$ has topological dimension 0 or $(n + 1)\, 2^m$ ($m = 0, 1, 2$).*

Proof. Suppose $\mathscr{H}$ has positive dimension. Then, by 5.17 $\mathscr{H}$ is connected and $\overline{\mathscr{H}}$ is one of $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. Thus, $\overline{\mathscr{H}}$ has dimension 2^m ($m = 0, 1, 2$). We now proceed by induction on the rank, $n + 1$, of $\mathscr{H}$. If $n = 0$, then $\mathscr{H} = \overline{\mathscr{H}} = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ and $\dim(\mathscr{H}) = 2^m$ ($m = 0, 1, 2$).

Now suppose the conclusion holds for E-rings of rank $i < n + 1$. Then, $\mathscr{H}/\mathscr{J}^n$ satisfies the induction hypothesis and has dimension $n \cdot 2^m$. By [**43**], p. 239,

$$\dim(\mathscr{H}/\mathscr{J}^n) = \dim(\mathscr{H}) - \dim(\mathscr{J}^n) = n \cdot 2^m.$$

Consequently 5.15 implies that

$$\dim(\mathscr{J}^n) = \dim(\overline{\mathscr{H}}) = 2^m$$

and so $\dim(\mathscr{H}) = (n + 1)\, 2^m$.

We have now completed the necessity of 5.12. To begin the assault on the sufficiency we first consider the possible states of connectedness in a topological H-ring.

5.19 THEOREM. *Let $\mathscr{H}$ be a locally compact T_2 H-ring with* $\operatorname{int}(\mathscr{J}) = \emptyset$. *Then one of the following statements holds:*

(1) *$\mathscr{H}$ is totally disconnected or equivalently zero-dimensional.*

(2) *$\mathscr{H}$ is a connected finite dimensional algebra over the reals with a unit.*

(3) *$\mathscr{H}^+$ is topologically isomorphic to the unique cartesian product of a real vector group $\mathbf{R}^n$ and a totally disconnected abelian group $\mathscr{G}$.*

In this case $\mathbf{R}^n$ is the component of $\mathscr{H}$ and also a finite dimensional algebra over the reals without a unit. Moreover $\mathscr{J}^+ = \mathscr{H}_0 \oplus (\mathscr{J} \cap \mathscr{G})$ (as a direct sum of topological groups).

Proof. We begin by following the idea of the proof of a similar result for locally compact fields in [**25**]. Because of 5.8 (3) $\mathscr{H}^+$ is a locally compact T_2 separable topological group, and so by the van-Kampen-Pontrjagin theorem ([**51**]; [**21**] 24.30), [**55**], p. 110 and [**2**], $\mathscr{H}^+ = \mathbf{R}^n \oplus \mathscr{G} \cong$

$\mathbf{R}^n \times \mathscr{G}$ (topological group isomorphism) where $\mathbf{R}^n$ is a (real) vector group and $\mathscr{G}$ is an abelian group whose component of 0, $\mathscr{G}_0$, is the maximal compact-connected subgroup of $\mathscr{H}^+$; and this decomposition is unique.

Because of the continuity of multiplication $\mathscr{G}_0$ is a compact ideal of $\mathscr{H}$. But 5.4 (iii) then implies $\mathscr{G}_0 = \{0\}$. Thus, $\mathscr{G}$ is totally disconnected or equivalently 0-dimensional ([**18**], p. 33); and so by 5.5 (1) $\mathbf{R}^n = \mathscr{H}_0$, the component of 0, and is an ideal. Moreover, the vector group $\mathbf{R}^n$ is a finite dimensional algebra over the reals (possibly without a unit) by [**43**] Theorem 1, p. 104 and [**25**] Lemma (2). Because of 5.1 it is clear that the only idempotents of $\mathscr{H}$ are 0 and 1. Hence the algebra $\mathscr{H}_0$ has a unit if and only if $\mathscr{H}_0 = \mathscr{H}$. Finally we observe that if $\mathscr{H}$ is neither connected nor 0-dimensional, then by 5.5 $\mathscr{H}_0 \subseteq \mathscr{J}$ and so $\mathscr{J} = \mathscr{H}_0 \oplus (\mathscr{J} \cap \mathscr{G}_0)$.

We were not able to decompose $\mathscr{H}$ into the product of a connected and a totally disconnected ring, because $\mathscr{H}_0$ is not necessarily a direct summand in $\mathscr{H}$; for example if $\mathscr{H}_0 = \mathscr{J} \neq \{0\}$ then by [**29**] Theorem (1), and [**20**], 7.9, 7.10, $\mathscr{H}_0$ is not a direct summand of $\mathscr{H}$.

5.19.1 COROLLARY. *Let $\mathscr{H}$ be a locally compact* T_2 H-*ring with* int $\mathscr{J} = \emptyset$.

The following are equivalent.

(1) $\mathscr{H}$ *is connected.*

(2) $\mathscr{H} = \mathbf{R}^n$ *is a finite dimensional algebra over the reals with unit.*

(3) $\mathscr{H}$ *is locally connected.*

Proof. The equivalence of the three statements follows from the proof of the theorem and 3.2.

Clearly every ideal of an algebra $\mathscr{A}$ with unit is a subspace of $\mathscr{A}$. Thus if $\mathscr{A}$ is finite dimensional it satisfies the descending chain condition on left ideals or is left Artinean ([**1**], p. 114). Hence 5.19 and 5.10 immediately give us

5.20 THEOREM. *Every locally compact, connected* T_2 H-*ring with* int $\mathscr{J} = \emptyset$ *is an* E-*ring.*

Next we begin our attack on the totally disconnected case.

5.21 LEMMA. *Let $\mathscr{H}$ be a locally compact* T_2 *totally disconnected* H-*ring. Then, $\mathscr{H}$ possesses a neighbourhood filter of* 0 *consisting of compact open subrings.*

Proof. Use [**28**] Lemma (4).

5.22 *Definition* ([**20**]). A compact open subring $\mathscr{R}$ of a locally compact T_2 ring, $\mathscr{H}$, is called an *order* of *H*.

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5.23 LEMMA. *Let $\mathscr{H}$ be a totally disconnected locally compact* T_2 H-*ring with* $\text{int}(\mathscr{J}) = \emptyset$. *Then, each order of $\mathscr{H}$ has an open radical.*

Proof. By 5.8 $\mathscr{J}$ is closed. 5.1 (i) then implies that $\mathscr{H}$ has no proper dense left ideals. The result then follows from 5.21 and [**20**] Lemma 7.17.

5.24 LEMMA. *Let $\mathscr{H}$ be a totally disconnected locally compact* T_2 H-*ring with* $\text{int}(\mathscr{J}) = \emptyset$. *Then, every order, $\mathscr{R}$, of $\mathscr{H}$ is a local ring. Moreover the centre of $\mathscr{R}$ is a compact commutative local ring contained in the centre of $\mathscr{H}$.*

Proof. To show $\mathscr{R}$ is a local ring, it suffices to show that its radical R is the only maximal left ideal ([**35**], p. 75). By [**20**], Theorem (2.2), the radical is the intersection of all the closed maximal left ideals. Hence, it suffices to show that R is the only closed maximal left ideal of $\mathscr{R}$. Let $\mathscr{I}$ be a closed maximal left ideal of $\mathscr{R}$. Thus, $R \subseteq \mathscr{I}$. But [**20**], p. 385 implies that $\mathscr{I} = \mathscr{R}e + \mathscr{I} \cap \mathscr{R}$ with e an appropriately chosen idempotent. But as mentioned previously the only idempotents in $\mathscr{H}$ (and thus $\mathscr{R}$) are 0 and 1. Since $\mathscr{I} \neq \mathscr{R}$, $e = 0$ and $\mathscr{I} = \mathscr{I} \cap R \subseteq R$. Hence $\mathscr{I} = R$.

Finally let $\mathscr{K}$ be the centre of $\mathscr{R}$. Then, by 5.4 and [**20**] Proposition 4.8 $\mathscr{K}$ is compact and lies in the centre of $\mathscr{H}$. Using the argument following the proof of Proposition 4.18 in [**20**] we have that $\mathscr{K}$ is a cartesian product, $\prod \mathscr{K}_\lambda$, of compact local rings and the kernel of the projection from $\mathscr{K}$ onto $\mathscr{K}_\lambda$ is generated by an idempotent e_λ. Since the only idempotents are 0 and 1, and $1 \notin$ kernel, we have $e_\lambda = 0$. Thus, the kernel is $\{0\}$ and so the projection is a topological isomorphism.

5.25 PROPOSITION. *Let $\mathscr{H}$ be a totally disconnected locally compact* T_2 H-*ring with* $\text{int}(\mathscr{J}) = \emptyset$. *The following statements are equivalent.*

(a) *$\mathscr{I}$ is a closed left ideal.*

(b) *$\mathscr{I}$ is a finitely generated left ideal.*

(c) *$\mathscr{I}$ is a principal left ideal.*

Proof. First we show (a) $\Rightarrow$ (b).

Let $\mathscr{I}$ be a closed left ideal of $\mathscr{H}$. Then, $\mathscr{I}$ is locally compact T_2. By 5.4 and 5.23 $\mathscr{H}$ is a locally compact T_2 ring having no proper open left nor proper open right ideals and the radical of any order of $\mathscr{H}$ is open. Then [**20**] 6.4 says that all locally compact left $\mathscr{H}$-modules are finitely generated and hence so is $\mathscr{I}$.

From [**51**] 5.2 we obtain [(b) $\Rightarrow$ (c)]. The last implication follows from 5.7 (b).

We may now give the

Proof of 5.12. The necessity is 5.17. Next we verify the sufficiency. Let $\mathscr{H}$ be a connected or totally disconnected H-ring. If $\mathscr{H}$ is connected, then by 5.20 $\mathscr{H}$ is an E-ring. Next suppose $\mathscr{H}$ is totally disconnected. By 5.8 (i) $\mathscr{J}$ is a closed ideal and so 5.25 implies $\mathscr{J} = \mathscr{H}a$. By 5.1 the radical $\mathscr{J}$ is a completely prime ideal and $\{0\}$ is not. To complete the proof we need only prove, because of 5.10 (g), that $\mathscr{J}$ is the only completely prime ideal of $\mathscr{H}$. Let $\mathscr{P}$ be any completely prime ideal, $\mathscr{P} \neq \mathscr{J}$. Hence, $\{0\} \subsetneq \mathscr{P} \subsetneq \mathscr{J}$. By [**10**] 3.3, 3.4, $\mathscr{P}$ is not a finitely generated left ideal and $\mathscr{P}^{rl} = \mathscr{P}$. But $\mathscr{P}^{rl}$ is the closure of $\mathscr{P}$ (5.7(a)) and so $\mathscr{P}$ is closed. Then 5.25 implies that $\mathscr{P}$ is finitely generated, a contradiction. Hence, $\mathscr{P} = \mathscr{J}$.

6. Locally compact T_2 desarguesian PH-planes. In this section we translate our results from Section 5 into geometric terminology.

If $\mathscr{H}$ is an H-ring, then $H(\mathscr{H})$ is the PH-plane defined in [**33**] Definition 10 via homogeneous coordinates. We shall, however, use the notation of [**38**] § 8: Points and lines of $H(\mathscr{H})$ are denoted $\langle p_1p_2p_3\rangle$ and $[u_1u_2u_3]$ respectively. The affine H-plane with [001] as its line at infinity ([**33**]) is topologically isomorphic to

$$A(\mathscr{H}) = \langle \mathscr{H} \times \mathscr{H}, \mathbf{L}_1 \cup \mathbf{L}_2, \|, \epsilon\rangle$$

(as defined in [**36**]) by [**38**] 8.3. Note that the points of $A(\mathscr{H})$ are identified with the points $\langle yx1\rangle$ in $H(\mathscr{H})$.

From [**38**] 8.1 and its proof and 1.11 we obtain

6.1 THEOREM. *The desarguesian* PH-*plane* $H(\mathscr{H})$ *is a topological* PH-*plane if and only if* $\mathscr{H}$ *is a topological* H-*ring with the addition properties:*

(i) *The radical* $\mathscr{J}$ *is closed, or equivalently the units,* $U(\mathscr{H})$, *form an open set.*

(ii) $U(\mathscr{H})$ *is a topological (multiplicative) group.*

(i.e., $\mathscr{H}$ *is a Gelfand ring (cf. 5.2)). Moreover,* $H(\mathscr{H})/\sim$ *is topologically isomorphic to* $H(\mathscr{H}/\mathscr{J})$.

Combining 6.1 with 5.8 and 1.6 (e) we immediately obtain

6.2 THEOREM. *Let* $H(\mathscr{H})$ *be a desarguesian* PH-*plane. Then* $H(\mathscr{H})$ *is a locally compact* T_2 TPH-*plane if and only if* $\mathscr{H}$ *is a locally compact* T_2 H-*ring.*

Moreover, $H(\mathscr{H})/\sim$ *is the real, complex or quaternion plane, if* $\mathscr{H}(H)/\sim$ *is non-discrete.*

We now refer the reader to [**4**] or [**52**] for the definitions of a PH-plane of height n and level n. For finite H-planes these two notions are equivalent ([**52**] 2.19). In the desarguesian PH-plane $H(\mathscr{H})$ the two notions of height n and level n are both equivalent to the statement that the radical $\mathscr{J}$ is nilpotent or equivalently (5.10) that $\mathscr{H}$ is an E-ring ([**3**]

3.6, 3.7). Since every finite H-ring is an E-ring ([**13**] 5.2), we have that every finite desarguesian plane is of level n.

Replacing finiteness by local compactness we obtain via 5.12 the following characterization of desarguesian PH-planes of level n.

6.3 THEOREM. *Let* $H(\mathscr{H})$ *be a locally compact* T_2 *desarguesian* PH-*plane with* $H(\mathscr{H})/\sim$ *non-discrete or equivalently* $\operatorname{int}(\mathscr{J}) = \emptyset$. *The following statements are equivalent:*

(1) $H(\mathscr{H})$ *is connected or totally disconnected (equivalently* 0-*dimensional).*

(2) *The affine H-plane* $A(\mathscr{H})$ *is connected or totally disconnected* (0-*dimensional*).

(3) $\mathscr{H}$ *is connected or totally disconnected* (0-*dimensional*).

(4) $\mathscr{H}$ *is an* E-*ring.*

(5) $H(\mathscr{H})$ *is of level* n.

(6) $H(\mathscr{H})$ *is of height* n.

Proof. First we remember ([**18**]) that for locally compact T_2 spaces total disconnectedness is equivalent to 0-dimensionality.

Next we recall that the lines of $A(\mathscr{H})$ are homeomorphic to $\mathscr{H}$ ([**38**] 6.4). Because of the remarks preceding the theorem and 5.12 it suffices to prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). If $H(\mathscr{H})$ is totally disconnected, it is also 0-dimensional and hence so is the subspace $A(\mathscr{H})$. Next suppose $H(\mathscr{H})$ is connected. Thus $H(\mathscr{H})/\sim$ and consequently $H(\mathscr{H}/\mathscr{J})$ (6.1) is connected. If (2) is false then $A(\mathscr{H})$ and hence $\mathscr{H}$ is not connected. By 5.5 $\mathscr{H}_0 \subseteq \mathscr{J}$. Because the quotient map $\nu: \mathscr{H} \rightarrow \mathscr{H}/\mathscr{J}$ is open-continuous,

$$\nu^{-1}(\{\mathscr{J}\}) = \mathscr{J} \subseteq \mathscr{H}_0$$

and $\mathscr{H}/\mathscr{J}$ is T_2 (5.8 (4)). Then [**21**] 7.13 implies that $\nu(\mathscr{H}_0) = \{\mathscr{J}\}$ is the component of $\mathscr{J}$ in $\mathscr{H}/\mathscr{J}$. Therefore $\mathscr{H}/\mathscr{J}$ and consequently $H(\mathscr{H}/\mathscr{J})$ ([**47**], p. 494 and [**45**], p. 448) is 0-dimensional, a contradiction to its connectedness.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). If $\mathscr{H}$ is connected, then so is $A(\mathscr{H})$. Since $A(\mathscr{H})$ is dense in $H(\mathscr{H})$ ([**39**] Theorem 6.7), $H(\mathscr{H})$ is also connected ([**15**] 1.6, p. 109). Finally assume $\mathscr{H}$ is 0-dimensional. Thus $A(\mathscr{H})$ is 0-dimensional ([**18**] 1.36). Moreover, since $\mathscr{J}$ is closed (5.8), $\mathscr{H}/\mathscr{J}$ is 0-dimensional ([**21**] 7.11) and hence so is $H(\mathscr{H}/\mathscr{J})$. Thus, $H(\mathscr{H})$ is not connected and so $\mathscr{C}(\langle 001\rangle) \subseteq \langle\overline{001}\rangle$ (1.12 (i)). But $\langle\overline{001}\rangle = \{\langle xy1\rangle | x, y \in \mathscr{J}\}$ lies in $A(\mathscr{H})$ and hence is 0-dimensional also. Finally by [**39**] 7.4,

$$\mathscr{C}(\langle 001\rangle) = \mathscr{C}_{<\overline{001}>}(\langle 001\rangle) = \{\langle 001\rangle\}.$$

A simple change of coordinates yields $\mathscr{C}(P) = \{P\}$ for all points P and so $H(\mathscr{H})$ is totally disconnected.

Next we observe that $\sim$ -connectedness (see Section 1) is the same as connectedness in our situation. To be precise,

6.3.1 COROLLARY. *Let $H(\mathscr{H})$ be a locally compact* T_2 *desarguesian* PH-*plane of level n with $H(\mathscr{H})/\sim$ non-discrete.*

The following statements are equivalent:

(1) *$H(\mathscr{H})$ is $\sim$ -connected.*

(2) *$H(\mathscr{H})$ is connected.*

(3) *$A(\mathscr{H})$ is connected.*

(4) *$\mathscr{H}$ is connected.*

(5) *$\mathscr{J}$ is connected.*

(6) *$\mathscr{H}/\mathscr{J}$ is connected.*

(7) *$H(\mathscr{H}/\mathscr{J}) \cong H(\mathscr{H})/\sim$ is connected.*

(8) *One neighbour class $\bar{P}$ (and hence all) are connected.*

Proof. From the proof of the theorem and 5.16 we have that conditions (2) to (7) are equivalent. By 1.13 (3), (1) is equivalent to (7). We need only show (5) is equivalent to (8). But, as seen in the proof of the theorem, $\langle\overline{001}\rangle$ is homeomorphic to $\mathscr{J} \times \mathscr{J}$ and consequently the proof is complete.

We end this section by observing the possible dimensions for $H(\mathscr{H})$ under the usual assumptions.

6.4 THEOREM. *Let $H(\mathscr{H})$ be a locally compact* T_2 *desarguesian* PH-*plaen of level n with $H(\mathscr{H})/\sim$ non-discrete. Then, $H(\mathscr{H})$ (and hence $A(\mathscr{H})$) is zero dimensional or a topological manifold of dimension* $(2n)2^m$ $(m = 0, 1, 2)$.

Proof. This follows immediately from 6.2, 4.3 and 1.6 (f).

7. Examples. In this section, via 6.1, we construct Gelfand H-rings concomitantly with topological desarguesian PH-planes of the types mentioned in 5.18, 5.19 and 6.3 respectively.

7.1 *Locally compact* T_2 PH-*planes of level n.* Let $\mathscr{F}$ be a non-discrete topological skew field. Hence $\mathscr{F}$ is a Gelfand ring (cf. 5.8) and so by [**8**] (I, Ex. 11(b), p. 317) $\mathscr{M}_n(F)$, the ring of all $n \times n$ matrices (with the usual cartesian product topology), is also a Gelfand ring.

The subring $\mathscr{D}_n(\mathscr{F}) = \{(d_{ij})|d_{ij} = 0$ for $i > j$ and $d_{i,i+k-1} = d_{1k}$ for $i, k, i + k - 1 \in \{1, 2, \ldots, n\}\}$ is an E-ring of rank n ([**5**]), and also a closed Gelfand subring of $\mathscr{M}_n(\mathscr{F})$. As in [**5**] we denote elements of $\mathscr{D}_n(\mathscr{F})$ by $[d_1, \ldots, d_n]$. The radical of $\mathscr{D}_n(\mathscr{F})$, $\mathscr{J}$, consists of the elements $[0, d_2, \ldots, d_n]$. Clearly $\mathscr{D}_n(\mathscr{F})$ is homeomorphic to $\mathscr{F}^n$. Moreover $\mathscr{D}_n(\mathscr{F})/\mathscr{J}$ is topologically isomorphic to $\mathscr{F}$ and $\text{int}(\mathscr{J}) = \emptyset$. Thus $\mathscr{H}(\mathscr{D}_n(\mathscr{F}))$ is a topological desarguesian PH-plane. Clearly

$\mathcal{D}_n(\mathcal{F})$ is locally compact if $\mathcal{F}$ is. In addition, the above comments, 6.4 and its corollary imply that: $\mathcal{F}$ is connected (0-dimensional) if and only if $\mathcal{D}_n(\mathcal{F})$ is connected (0-dimensional) if and only if $\mathcal{H}(\mathcal{D}_n(\mathcal{F}))$ is connected (0-dimensional).

For the appropriate choices of F see [**8**] I Theorem 1, p. 433.

It is possible to analyze most closely the structure of commutative locally compact T_2 H-rings (see 5.19(2)) using ideas from [**11**] and [**26**] but we leave this for another time and place.

7.2 *Ordered* H-*rings*. We construct a Gelfand H-ring, $\mathcal{H}$, with $\text{int}(\mathcal{J}) \neq \emptyset$. Then $H(\mathcal{H})/\sim$ is discrete in the quotient topology. Thus, $H(\mathcal{H})$ is totally $\sim$-disconnected ([**39**] 8.1) and so disconnected. For one particular example, $H(\mathcal{H})$ is of level 2 but neither connected nor 0-dimensional since $\bar{P}$ is the component of any point P.

An H-ring $\mathcal{H}$ is *ordered* ([**33**] 6.4) if there exists a subset $\mathcal{P}$ of $\mathcal{H}$ with the properties:

(i) $\mathcal{P} \cap -\mathcal{P} = \{0\}$ and $\mathcal{P} \cup -\mathcal{P} = \mathcal{H}$ with $1 \in \mathcal{P}$.

(ii) $\mathcal{P} + \mathcal{P} \subseteq \mathcal{P}$.

(iii) $\mathcal{P} \cdot \mathcal{P} \subseteq \mathcal{P}$.

We define a relation $\leqq$ on $\mathcal{H}$ by $a \leqq b$ if and only if $b - a \in \mathcal{P}$.

Then $\mathcal{P} = \{h \in \mathcal{H} | h \geqq 0\}$. It follows that

(iv) $\leqq$ is a total ordering on $\mathcal{H}$.

Also $\leqq$ is compatible with the ring structure; that is

(v) $a \leqq b$ implies $a + c \leqq b + c$ for all $a, b, c \in \mathcal{H}$.

(vi) $a, b \geqq 0$ implies $ab \geqq 0$ for all $a, b \in \mathcal{H}$.

Thus $(\mathcal{H}, \leqq)$ is a totally ordered ring in the sense of [**49**], Definition A.2.1. Clearly conditions (iv) to (vi) give an equivalent formulation of an ordered H-ring.

If $\mathcal{J}$ is the radical of $\mathcal{H}$, then

(vii) $a \in \mathcal{J}$: $-1 < a < 1$ and $0 \leqq b \leqq a$ implies $b \in \mathcal{J}$ ([**6**] 2.1); that is, $\mathcal{J}$ is an order ideal ([**49**], Definition A.2.3.)

Hence by [**49**] Theorem A.2.4

(viii) $\mathcal{H}/\mathcal{J}$ is an ordered field under the total ordering $a + \mathcal{J} \leqq b + \mathcal{J}$ if and only if there exist $x, y \in \mathcal{H}$ so that $x + \mathcal{J} = a + \mathcal{J}$, $y + \mathcal{J} = b + \mathcal{J}$ and $x \leqq y$.

Also the canonical map $\nu: \mathcal{H} \to \mathcal{H}/\mathcal{J}$ is an order homomorphism, i.e., $a \leqq b$ implies $\nu(a) \leqq \nu(b)$.

In the notation of [**8**] let $\mathcal{C}_0(\mathcal{H})$ be the topology (order topology) generated by the open segments $(a, b) = \{h \in \mathcal{H} | a < x < b\}$. $\mathcal{C}_0(\mathcal{H})$ is regular T_2 ([**8**] I, p. 137, Ex. 17). Since $\mathcal{H}$ is totally ordered, the open segments are actually a base for $\mathcal{C}_0(\mathcal{H})$ ([**8**] I, p. 120, Ex. 5). Moreover using the ideas in [**8**] I, p. 378, Ex. 1, II, p. 26, Ex. 2 and I, p. 387, Ex. 2(b) we have that $\langle \mathcal{H}, \mathcal{C}_0(\mathcal{H}) \rangle$ is a Gelfand H-ring. By (vii) and (viii) $\text{int}\,\mathcal{J} \neq \emptyset$ and so the quotient topology on $\mathcal{H}/\mathcal{J}$ is discrete and distinct from $\mathscr{C}_0(\mathscr{H}/\mathscr{J})$. Also $\nu:\mathscr{H} \rightarrow \mathscr{H}/\mathscr{J}$ is continuous but not open with respect to the order topologies on $\mathscr{H}$ and $\mathscr{H}/\mathscr{J}$.

By the introductory remarks concerning ordered H-rings $H(\mathscr{H})$ is a disconnected TPH-plane with $H(\mathscr{H})/\sim\ = H(\mathscr{H}/\mathscr{J})$ discrete in the quotient topology. If we endow $H(\mathscr{H}/\mathscr{J})$ with the induced order topology from $\mathscr{H}/\mathscr{J}$ (see (viii)) then the canonical projection π: $H(\mathscr{H}) \rightarrow H(\mathscr{H}/\mathscr{J})$ is a continuous but not open map with respect to the two order topologies.

If we let $\mathscr{F} = \mathbf{R}$, then $\mathscr{D}_n(\mathbf{R})$ with elements $[d_1, \ldots, d_n]$ ordered lexicographically is an ordered H-ring. (This is essentially Hjelmslev's original example.) For $n = 2$, $\mathscr{J} = \{[0, x] | x \in \mathbf{R}\}$ is homeomorphic to $\mathbf{R}$ ($[0, x] \mapsto x$). Then, $\mathscr{D}_2(\mathbf{R})$ is disconnected and $\mathscr{J}$ is connected. By 5.5 (2) $\mathscr{J}$ is the connected component of $[0, 0]$. Since in any PH-plane, $H(\mathscr{H})$, $\langle \overline{001} \rangle = \{\langle xy1 \rangle | x, y \in \mathscr{J}\}$ ([**52**], p. 57), then in our example $\langle \overline{001} \rangle$ is homeomorphic to $\mathscr{J} \times \mathscr{J}$ and hence connected. Thus, for each point P of $H(\mathscr{D}_2(R))$ the connected component of P is the neighbour class $\bar{P}$. Hence, $H(\mathscr{D}_2(R))$ is neither connected nor totally disconnected.

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