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# Corrigendum to "Asymptotic prime divisors over complete intersection rings" [Math. Proc. Camb. Phil. Soc. 160 (3) (2016) 423-436]

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*Abstract*

There was a gap in the proof of Theorem 4·1 of [**1**]. In this corrigendum, we correct the error.

## 1. *Introduction*

*Set-up* 1·1. Let $Q$ be a Noetherian ring of finite Krull dimension. Let $\mathbf{f} = f_1, \ldots, f_c$ be a $Q$-regular sequence. Set $A := Q/(\mathbf{f})$. Suppose $M$ and $N$ are finitely generated $A$-modules, where $\operatorname{projdim}_Q(M)$ is finite. Let $I$ be an ideal of $A$.

In [**1**, theorem 3·1], we proved that $\bigcup_{n,i\geqslant 0} \operatorname{Ass}_A\left(\operatorname{Ext}^i_A(M, N/I^nN)\right)$ is a finite set. Complementing this finiteness result, in [**1**, theorem 4·1], we showed the following asymptotic stability: There exist $n_0, i_0 \geqslant 0$ such that for all $n \geqslant n_0$ and $i \geqslant i_0$,

$$\operatorname{Ass}_A\left(\operatorname{Ext}^{2i}_A(M, N/I^nN)\right) = \operatorname{Ass}_A\left(\operatorname{Ext}^{2i_0}_A(M, N/I^{n_0}N)\right),$$
$$\operatorname{Ass}_A\left(\operatorname{Ext}^{2i+1}_A(M, N/I^nN)\right) = \operatorname{Ass}_A\left(\operatorname{Ext}^{2i_0+1}_A(M, N/I^{n_0}N)\right).$$

1·2. Our strategy to prove [**1**, theorem 4·1] is as follows:

(i) choose $\mathfrak{p} \in \bigcup_{n,i\geqslant 0} \operatorname{Ass}_A\left(\operatorname{Ext}^i_A(M, N/I^nN)\right)$;

(ii) for every fixed $l = 0, 1$, show that there exist $n_l, i_l \geqslant 0$ such that

$$\text{either} \quad \mathfrak{p} \in \operatorname{Ass}_A\left(\operatorname{Ext}^{2i+l}_A(M, N/I^nN)\right) \quad \text{for all } n \geqslant n_l \text{ and } i \geqslant i_l;$$
$$\text{or} \quad \mathfrak{p} \notin \operatorname{Ass}_A\left(\operatorname{Ext}^{2i+l}_A(M, N/I^nN)\right) \quad \text{for all } n \geqslant n_l \text{ and } i \geqslant i_l.$$

Localising at $\mathfrak{p}$, and replacing $A_\mathfrak{p}$ by $A$ and $\mathfrak{p}A_\mathfrak{p}$ by $\mathfrak{m}$, we may assume that $A$ is a local ring with maximal ideal $\mathfrak{m}$ and residue field $k$. In [**1**, lemma 4·2], we proved that the lengths $\lambda\left(\operatorname{Hom}_A\left(k, \operatorname{Ext}^{2i}_A(M, N/I^nN)\right)\right)$ and $\lambda\left(\operatorname{Hom}_A\left(k, \operatorname{Ext}^{2i+1}_A(M, N/I^nN)\right)\right)$ are given by polynomials in $n, i$ with rational coefficients for all sufficiently large $n, i$. Using this, we erroneously concluded the fact 1·2(ii). Our assertion would have been correct if $\bigoplus_{n,i\geqslant 0} \operatorname{Hom}_A\left(k, \operatorname{Ext}^i_A(M, N/I^nN)\right)$ is a *finitely generated* module over some appropriate Noetherian bigraded ring. However, we believe that this module is practically never finitely generated over the ring $\mathscr{S} = \mathscr{R}(I)[t_1, \ldots, t_c]$ we worked with (see [**1**, section 2]). In this corrigendum, we correct our oversight. We prove the following:

LEMMA 1·3. *Along with Set-up* 1·1, *further assume that $Q$ is a local ring with the residue field $k$. Then, for every fixed $l = 0, 1$, we have that:*

$$\begin{aligned} &\textit{either} \quad \operatorname{Hom}_A\left(k, \operatorname{Ext}_A^{2i+l}(M, N/I^nN)\right) \neq 0 \quad \textit{for all } n, i \gg 0;\\ &\textit{or} \quad \operatorname{Hom}_A\left(k, \operatorname{Ext}_A^{2i+l}(M, N/I^nN)\right) = 0 \quad \textit{for all } n, i \gg 0. \end{aligned}$$

Using Lemma 1·3, one can easily prove the fact 1·2(ii), and hence [**1**, theorem 4·1].

## 2. *Proof of Lemma* 1·3

With *Set-up* 1·1, further assume that $\mathcal{N} = \bigoplus_{n\geqslant 0} N_n$ is a graded module over the Rees ring $\mathscr{R}(I) := \bigoplus_{n\geqslant 0} I^nX^n$. Then $\mathscr{E}(\mathcal{N}) := \bigoplus_{n,i\geqslant 0} \operatorname{Ext}_A^i(M, N_n)$ turns into a bigraded module over $\mathscr{S} := \mathscr{R}(I)[t_1, \ldots, t_c]$, where $t_j : \operatorname{Ext}_A^i(M, N_n) \to \operatorname{Ext}_A^{i+2}(M, N_n)$, $i \geqslant 0$, are the Eisenbud operators, and we set $\deg(t_j) = (0, 2)$ for all $1 \leqslant j \leqslant c$ and $\deg(uX^s) = (s, 0)$ for all $u \in I^s$, $s \geqslant 0$; see [**1**, section 2·3]. Since $\mathcal{L} := \bigoplus_{n\geqslant 0}(I^nN/I^{n+1}N)$ and $\mathcal{L}' := \bigoplus_{n\geqslant 0}(N/I^nN)$ are graded $\mathscr{R}(I)$-modules, we obtain that

$$U = \bigoplus_{n,i\geqslant 0} U_{(n,i)} := \mathscr{E}(\mathcal{L}) = \bigoplus_{n,i\geqslant 0} \operatorname{Ext}_A^i\left(M, I^nN/I^{n+1}N\right), \tag{2·1 a}$$

$$V = \bigoplus_{n,i\geqslant 0} V_{(n,i)} := \mathscr{E}(\mathcal{L}') = \bigoplus_{n,i\geqslant 0} \operatorname{Ext}_A^i(M, N/I^nN) \tag{2·1 b}$$

are bigraded modules over $\mathscr{S} = \mathscr{R}(I)[t_1, \ldots, t_c]$. To prove Lemma 1·3, we use:

LEMMA 2·1. *Let $A$ be a Noetherian ring and $I$ an ideal of $A$. Let $\mathscr{R}(I)$ be the Rees ring of $I$. Set $\mathscr{S} := \mathscr{R}(I)[t_1, \ldots, t_c]$, where $\deg(t_j) = (0, 2)$ for all $1 \leqslant j \leqslant c$ and $\deg(I^s) = (s, 0)$ for all $s \geqslant 0$. Suppose $L = \bigoplus_{(n,i)\in\mathbb{N}^2} L_{(n,i)}$ is a finitely generated bigraded $\mathscr{S}$-module. Then, for every fixed $l = 0, 1$, we have that either $L_{(n,2i+l)} \neq 0$ for all $n, i \gg 0$; or $L_{(n,2i+l)} = 0$ for all $n, i \gg 0$.*

*Proof.* By virtue of [**3**, proposition 5·1], there is $(n_0, i_0) \in \mathbb{N}^2$ such that

$$\begin{aligned} \operatorname{Ass}_A\left(L_{(n,2i)}\right) &= \operatorname{Ass}_A\left(L_{(n_0,2i_0)}\right) \quad \text{for all } (n, i) \geqslant (n_0, i_0);\\ \operatorname{Ass}_A\left(L_{(n,2i+1)}\right) &= \operatorname{Ass}_A\left(L_{(n_0,2i_0+1)}\right) \quad \text{for all } (n, i) \geqslant (n_0, i_0). \end{aligned}$$

The result now follows from the well-known fact: for an $A$-module $M$, $\operatorname{Ass}_A(M)$ is non-empty if and only if $M \neq 0$.

We now give:

*Proof of Lemma* 1·3. We prove the lemma for $l = 0$ only. For $l = 1$, the proof is similar. Set $f(n, i) := \lambda\left(\operatorname{Hom}_A\left(k, \operatorname{Ext}_A^{2i}(M, N/I^nN)\right)\right)$ for all $n, i \geqslant 0$. By virtue of [**1**, lemma 4·2], $f(n, i)$ is given by a polynomial in $n, i$ with rational coefficients for all $n, i \gg 0$. If $f(n, i) = 0$ for all $n, i \gg 0$, then there is nothing to prove. Suppose this is not the case. Then we claim that $\operatorname{Hom}_A\left(k, \operatorname{Ext}_A^{2i}(M, N/I^nN)\right) \neq 0$ for all $n, i \gg 0$.

For every $n \geqslant 0$, the exact sequence $0 \to I^nN/I^{n+1}N \to N/I^{n+1}N \to N/I^nN \to 0$ yields an exact sequence of $A$-modules (for each $i$):

$$\begin{aligned} &\operatorname{Ext}_A^i\left(M, I^nN/I^{n+1}N\right) \longrightarrow \operatorname{Ext}_A^i\left(M, N/I^{n+1}N\right) \longrightarrow \operatorname{Ext}_A^i\left(M, N/I^nN\right)\\ &\longrightarrow \operatorname{Ext}_A^{i+1}\left(M, I^nN/I^{n+1}N\right). \end{aligned}$$

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Taking direct sum over $n, i$, and using the naturality of the Eisenbud operators $t_j$, we have an exact sequence $U \xrightarrow{\Phi} V(1,0) \xrightarrow{\Xi} V \xrightarrow{\Psi} U(0,1)$ of bigraded modules over $\mathscr{S} = \mathscr{R}(I)[t_1,\ldots,t_c]$, where $U$ and $V$ are as in (2·1$a$) and (2·1$b$) respectively. Hence, setting $X := \text{Image}\,(\Phi)$, $Y := \text{Image}\,(\Xi)$ and $Z := \text{Image}\,(\Psi)$, we obtain the short exact sequences: $0 \to X \to V(1,0) \to Y \to 0$ and $0 \to Y \to V \to Z \to 0$. Applying $\text{Hom}_A(k,-)$ to these short exact sequences, we get the following exact sequences:

$$0 \longrightarrow \text{Hom}_A(k,X) \longrightarrow \text{Hom}_A\big(k,V(1,0)\big) \longrightarrow \text{Hom}_A(k,Y) \longrightarrow C \longrightarrow 0, \qquad (2\cdot2a)$$

$$0 \longrightarrow \text{Hom}_A(k,Y) \longrightarrow \text{Hom}_A(k,V) \longrightarrow D \longrightarrow 0, \qquad (2\cdot2b)$$

where $C := \text{Image}\,\big(\text{Hom}_A(k,Y) \to \text{Ext}^1_A(k,X)\big)$ and $D := \text{Image}\,\big(\text{Hom}_A(k,V) \to \text{Hom}_A(k,Z)\big)$. By virtue of [**2**, theorem 1·1], $U$ is a finitely generated bigraded $\mathscr{S}$-module, and hence $X$ and $Z$ are so. This implies that $\text{Hom}_A(k,X)$, $\text{Ext}^1_A(k,X)$ and $\text{Hom}_A(k,Z)$ are finitely generated bigraded $\mathscr{S}$-modules. Therefore $C$ and $D$ are finitely generated bigraded $\mathscr{S} = \mathscr{R}(I)[t_1,\ldots,t_c]$-modules. Hence, by Lemma 2·1, we get:

$$\left\{\begin{array}{ll}\text{either} & \text{Hom}_A\big(k,X_{(n,2i)}\big) \neq 0 \ \text{ for all } \ n,i \gg 0,\\ \quad\text{or} & \text{Hom}_A\big(k,X_{(n,2i)}\big) = 0 \ \text{ for all } \ n,i \gg 0;\end{array}\right\} \qquad (2\cdot3)$$

$$\left\{\begin{array}{ll}\text{either} & C_{(n,2i)} \neq 0 \ \text{ for all } \ n,i \gg 0,\\ \quad\text{or} & C_{(n,2i)} = 0 \ \text{ for all } \ n,i \gg 0;\end{array}\right\} \quad \left\{\begin{array}{ll}\text{either} & D_{(n,2i)} \neq 0 \ \text{ for all } \ n,i \gg 0,\\ \quad\text{or} & D_{(n,2i)} = 0 \ \text{ for all } \ n,i \gg 0.\end{array}\right\}$$

For $n, i \geqslant 0$, the $(n,2i)$th components of (2·2$a$) and (2·2$b$) yield the exact sequences:

$$0 \longrightarrow \text{Hom}_A\big(k,X_{(n,2i)}\big) \longrightarrow \text{Hom}_A\big(k,V_{(n+1,2i)}\big) \longrightarrow \text{Hom}_A\big(k,Y_{(n,2i)}\big) \longrightarrow C_{(n,2i)} \longrightarrow 0, \qquad (2\cdot4a)$$

$$0 \longrightarrow \text{Hom}_A\big(k,Y_{(n,2i)}\big) \longrightarrow \text{Hom}_A\big(k,V_{(n,2i)}\big) \longrightarrow D_{(n,2i)} \longrightarrow 0. \qquad (2\cdot4b)$$

Now we are in a position to prove our claim that $\text{Hom}_A\big(k,V_{(n,2i)}\big) \neq 0$ for all $n, i \gg 0$. We consider the following four cases:

*Case 1*. Assume that $\text{Hom}_A\big(k,X_{(n,2i)}\big) \neq 0$ for all $n, i \gg 0$. Then, in view of (2·4$a$), we get that $\text{Hom}_A\big(k,V_{(n,2i)}\big) \neq 0$ for all $n, i \gg 0$. So, in this case, we are done.

*Case 2*. Assume that $C_{(n,2i)} \neq 0$ for all $n, i \gg 0$. So again, in view of (2·4$a$), we have that $\text{Hom}_A\big(k,Y_{(n,2i)}\big) \neq 0$ for all $n, i \gg 0$. Hence (2·4$b$) yields that $\text{Hom}_A\big(k,V_{(n,2i)}\big) \neq 0$ for all $n, i \gg 0$. Thus, in this case also, we are done.

*Case 3*. Assume that $D_{(n,2i)} \neq 0$ for all $n, i \gg 0$. In this case, (2·4$b$) gives that $\text{Hom}_A\big(k,V_{(n,2i)}\big) \neq 0$ for all $n, i \gg 0$, and hence we are done.

In view of (2·3), if none of the above three cases holds, then we have the following:

*Case 4*. Assume that $\text{Hom}_A\big(k,X_{(n,2i)}\big) = 0$ for all $n, i \gg 0$, $C_{(n,2i)} = 0$ for all $n, i \gg 0$, and $D_{(n,2i)} = 0$ for all $n, i \gg 0$. Hence the exact sequences (2·4$a$) and (2·4$b$) yield the isomorphisms: $\text{Hom}_A\big(k,V_{(n+1,2i)}\big) \cong \text{Hom}_A\big(k,Y_{(n,2i)}\big) \cong \text{Hom}_A\big(k,V_{(n,2i)}\big)$ for all $n, i \gg 0$. These isomorphisms provide the following equalities:

$$f(n+1,i) = f(n,i) \quad \text{for all } \ n,i \gg 0. \qquad (2\cdot5)$$

We may write the polynomial expression of $f(n,i)$ in the following way:

$$f(n,i) = h_0(i)n^a + h_1(i)n^{a-1} + \cdots + h_{a-1}(i)n + h_a(i) \quad \text{for all } \ n,i \gg 0, \qquad (2\cdot6)$$

where $h_j(i)$, $0 \leqslant j \leqslant a$, are polynomials in $i$ over $\mathbb{Q}$. We may assume without loss of generality that $h_0$ is a non-zero polynomial. Therefore $h_0$ may have only finitely many roots.

Let $i' \geqslant 0$ be such that $h_0(i) \neq 0$ for all $i \geqslant i'$. In view of (2·5) and (2·6), there exist some $n_0$ ($\geqslant 0$) and $i_0$ ($\geqslant i'$, say) such that for all $n \geqslant n_0$ and $i \geqslant i_0$, we have

$$f(n+1, i) = f(n, i) \quad \text{and} \quad f(n, i) = h_0(i)n^a + h_1(i)n^{a-1} + \cdots + h_{a-1}(i)n + h_a(i).$$

These equalities imply that $a$ must be equal to 0, and hence $f(n, i) = h_0(i)$ for all $n \geqslant n_0$ and $i \geqslant i_0$. Thus $f(n, i) \neq 0$ for all $n \geqslant n_0$ and $i \geqslant i_0$, and hence $\operatorname{Hom}_A\left(k, V_{(n,2i)}\right) \neq 0$ for all $n \geqslant n_0$ and $i \geqslant i_0$, which completes the proof of Lemma 1·3.

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