

# CONSISTENT NON-GAUSSIAN PSEUDO MAXIMUM LIKELIHOOD ESTIMATORS OF SPATIAL AUTOREGRESSIVE MODELS

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This paper studies the non-Gaussian pseudo maximum likelihood (PML) estimation of a spatial autoregressive (SAR) model with SAR disturbances. If the spatial weights matrix  $M_n$  for the SAR disturbances is normalized to have row sums equal to 1 or the model reduces to a SAR model with no SAR process of disturbances, the non-Gaussian PML estimator (NGPMLE) for model parameters except the intercept term and the variance  $\sigma_0^2$  of independent and identically distributed (i.i.d.) innovations in the model is consistent. Without row normalization of  $M_n$ , the symmetry of i.i.d. innovations leads to consistent NGPMLE for model parameters except  $\sigma_0^2$ . With neither row normalization of  $M_n$  nor the symmetry of innovations, a location parameter can be added to the non-Gaussian pseudo likelihood function to achieve consistent estimation of model parameters except  $\sigma_0^2$ . The NGPMLE with no added parameter can have a significant efficiency improvement upon the Gaussian PML estimator and the generalized method of moments estimator based on linear and quadratic moments. We also propose a non-Gaussian score test for spatial dependence, which can be locally more powerful than the Gaussian score test. Monte Carlo results show that our NGPMLE with no added parameter and the score test based on it perform well in finite samples.

## 1. INTRODUCTION

The spatial autoregressive (SAR) model, originated in Cliff and Ord (1973, 1981), is a popular spatial econometric model. It has been applied in a range of fields

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in economics to capture spatial dependence.<sup>1</sup> In this paper, we consider the non-Gaussian pseudo maximum likelihood (PML) estimation of the SAR model with SAR disturbances (SARAR model), with no need to correctly specify the distribution of independent and identically distributed (i.i.d.) innovations in the model. We provide conditions for the consistency of the non-Gaussian PML estimator (NGPMLE) and prove its asymptotic distribution. Our applications to several popular datasets in the spatial econometric literature show some evidence of nonnormal and leptokurtic innovations for these datasets.<sup>2</sup> In such situations, our NGPMLE on the basis of leptokurtic distributions can have significant efficiency improvements over existing estimators including the Gaussian PML estimator (GPMLE) (Lee, 2004), and lead to different but more reliable empirical results.

We consider the following SARAR model:

$$Y_n = \lambda_0 W_n Y_n + X_n \beta_0 + U_n, \quad U_n = \rho_0 M_n U_n + \sigma_0 V_n, \quad (1)$$

where  $n$  is the sample size,  $Y_n = [y_{n1}, \dots, y_{nn}]'$  is an  $n \times 1$  vector of observations on the dependent variable,  $X_n$  is an  $n \times k_x$  matrix of exogenous variables,  $W_n = [w_{n,ij}]$  and  $M_n = [m_{n,ij}]$  are spatial weights matrices with zero diagonals,<sup>3</sup> the innovations  $v_i$ 's in  $V_n = [v_1, \dots, v_n]'$  are i.i.d. with mean zero and unit variance,  $\lambda_0$  and  $\rho_0$  are scalar spatial dependence parameters,  $\beta_0$  is a  $k_x \times 1$  parameter vector, and  $\sigma_0$  is a standard deviation (SD) parameter. We formulate an NGPMLE using a chosen density function for  $v_i$  that can differ from its true density function. Our results on the consistency of the NGPMLE for the SARAR model extend those in Newey and Steigerwald (1997) for conditional heteroskedasticity models, by properly taking into account spatial dependence.<sup>4</sup> We show that, when the spatial weights matrix  $M_n$  in the SAR process of disturbances is normalized to have row sums equal to 1,<sup>5</sup> the NGPMLE for model parameters except the intercept term and the variance  $\sigma_0^2$  of i.i.d. innovations is consistent under regularity conditions; without row normalization of  $M_n$ , if the innovations are symmetric, the NGPMLE for model parameters except  $\sigma_0^2$  is consistent; and with neither row normalization of  $M_n$  nor the symmetry of innovations, a location parameter can be added to the pseudo

<sup>1</sup>Reviews on studies about the class of SAR models can be found in, e.g., Anselin and Bera (1998), Anselin (2010), and Arbia (2016).

<sup>2</sup>See Section 5 and the Supplementary Material.

<sup>3</sup>The zero diagonals of the spatial weights matrices exclude self-influence. It is a normalization condition usually maintained in the literature (see, e.g., Kelejian and Prucha, 1998; Lee, 2004). Indeed, it is not used in our theoretical analysis.

<sup>4</sup>Other studies on the NGPMLE include, among others, Gouriéroux, Monfort, and Trognon (1984), Francq, Lepage, and Zakoïan (2011), Fan, Qi, and Xiu (2014), and Fiorentini and Sentana (2019). The results in Gouriéroux et al. (1984) are on the basis of a density function  $f(x, m)$  or  $f(x, m, \Sigma)$ , where  $m$  is the mean and  $\Sigma$  is the variance of the distribution. They focus on the exponential family, for which all moments exist. Our analysis does not restrict the density function to be of the form  $f(x, m)$  or  $f(x, m, \Sigma)$ , and we can use a density function which does not have a finite moment with an order higher than 3. Francq et al. (2011) and Fan et al. (2014) propose modifications of NGPMLEs for GARCH models with zero conditional mean. Fiorentini and Sentana (2019) propose consistent NGPMLEs for GARCH models with nonzero conditional mean and for some other location-scale models such as multivariate regressions.

<sup>5</sup>We refer to a matrix with all row sums equal to 1 as a row-normalized matrix hereafter.

likelihood function to obtain consistent estimators of model parameters except  $\sigma_0^2$ . An important special case of the SARAR model is the SAR model with exogenous variables but with no SAR process of disturbances. Consistent non-Gaussian PML estimation of model parameters except  $\sigma_0^2$  only requires an intercept term in the model. Furthermore, although we only consider SAR models in this paper, consistent NGPMLEs can also be extended to other spatial econometric models.<sup>6</sup> We expect that NGPMLEs for those models can be more efficient than existing estimation methods.

We prove the  $\sqrt{n}$ -consistency and asymptotic normality of our NGPMLE under the condition that the innovations have a finite third moment, which can allow for innovations with relatively heavy tails. By contrast, the  $\sqrt{n}$ -consistency of the GPMLE is established under the existence of a moment of innovations with an order higher than 4 (Lee, 2004). Furthermore, using numerical integration and Student's  $t$  distribution to formulate a likelihood function, we show that the NGPMLE with no added parameter can have a uniform efficiency improvement upon the GPMLE, and can also have a significantly larger efficiency improvement than the best generalized method of moments (GMM) estimator on the basis of linear-quadratic moments (Liu, Lee, and Bollinger, 2010), but the NGPMLE with an added parameter can be less efficient than the GPMLE. An intuitive explanation from the non-Gaussian score is that, unlike GPMLE and the best GMM estimator (BGMME), the NGPMLE with no added parameter does not restrict the moments to be linear and quadratic in innovations. The NGPMLE with an added parameter loses some efficiency since one more parameter has to be estimated. Our Monte Carlo experiments further corroborate the efficiency improvement of the NGPMLE with no added parameter upon the GPMLE and BGMME.

We also propose a non-Gaussian score test for spatial dependence in SAR models, which only requires the restricted NGPMLE. The test statistic generalizes the Moran  $I$  test statistic that is quadratic in estimated innovations (Moran, 1950). If the NGPMLE is asymptotically more efficient than the GPMLE, then the non-Gaussian score test is locally more powerful than the Gaussian score test.

Estimation methods for SAR models include maximum likelihood (ML) (Ord, 1975), generalized spatial two-stage least squares (GS2SLS) (Kelejian and Prucha, 1998), Gaussian PML,<sup>7</sup> GMM (Lee, 2007),<sup>8</sup> best GMM, and adaptive estimation (Robinson, 2010; Lee and Robinson, 2020), among others. GS2SLS is computationally simpler than ML, Gaussian PML, GMM, and best GMM, but is less efficient. Like our NGPMLE, the GPMLE does not need the distribution of

<sup>6</sup>For example, the matrix exponential spatial specification (LeSage and Pace, 2007), spatial moving average models (e.g., Haining, 1978; Cliff and Ord, 1981; Fingleton, 2008; Doğan and Taşpınar, 2013), and high-order versions of those models (e.g., Blommestein, 1983, 1985). See the Supplementary Material for some consistency analysis.

<sup>7</sup>Exact and high-order properties of the GPMLE are studied in Bao (2013) and Hillier and Martellosio (2018). Gupta and Robinson (2018) study the GPMLE of SAR models with increasingly many parameters.

<sup>8</sup>A related estimation method is the generalized empirical likelihood (Jin and Lee, 2019), which is asymptotically as efficient as the GMM with the same moments, but can have smaller higher-order bias.

innovations to be correctly specified, and it is relatively efficient.<sup>9</sup> In addition, whether the NGPMLE or the ML estimator based on nonnormal distributions is consistent or not is not clear according to the existing literature. Thus, the GPMLE is popular in practice (see, e.g., Robinson, 2010). However, it can have a significant efficiency loss compared with the ML estimator, when the innovations are far from normally distributed (Fan et al., 2014). The adaptive estimation in Robinson (2010) requires that each unit is influenced aggregately by a significant portion of units in the population, which is a very stringent condition that may not be reasonable in some practical circumstances.<sup>10</sup>

This paper is organized as follows: In Section 2, we prove the convergence and asymptotic distribution of the NGPMLE for the SARAR model, and compare its efficiency with those of the GPMLE and BGMME. In Section 3, the non-Gaussian score test is investigated. Monte Carlo and application results are reported in Sections 4 and 5, respectively. Section 6 concludes. Proofs and other materials are collected in the Appendix and in the Supplementary Material.

## 2. NGPMLE

Let  $\theta_0 = [\lambda_0, \rho_0, \beta'_0, \sigma_0^2]'$  be the true parameter vector in model (1), and let  $\theta = [\lambda, \rho, \beta', \sigma^2]'$  be a general parameter vector. We consider a density function  $f(x, \eta)$  of a random variable with mean zero and unit variance, where  $\eta$  is a  $k_\eta \times 1$  parameter vector. For example,  $f(x, \eta)$  can be the density function of a standardized Student's  $t$  distribution with  $\eta$  degrees of freedom. The pseudo log-likelihood function of the SARAR model (1), as if  $v_i$  had the density function  $f(v_i, \eta)$ , is

$$\ln L_n(\gamma) = \sum_{i=1}^n \ln f(v_i(\theta), \eta) - \frac{n}{2} \ln(\sigma^2) + \ln |S_n(\lambda)| + \ln |R_n(\rho)|, \quad (2)$$

where  $\gamma = [\theta', \eta']'$ ,  $S_n(\lambda) = I_n - \lambda W_n$  with  $I_n$  being the  $n$ -dimensional identity matrix,  $R_n(\rho) = I_n - \rho M_n$ , and  $v_i(\theta) = \frac{1}{\sigma} e'_{ni} R_n(\rho) [S_n(\lambda) Y_n - X_n \beta]$ , with  $e_{ni}$  being the  $i$ th column of  $I_n$ . We may fix  $\eta$  at some particular value or estimate it jointly with  $\theta$ . We focus on the case where  $\eta$  is estimated jointly with  $\theta$ , as in Fiorentini and Sentana (2019). An NGPMLE of  $\gamma$  is derived by maximizing  $\ln L_n(\gamma)$  in (2).

We first introduce some regularity conditions for later analysis on model (1).

**Assumption 1** (Topological space). Let  $\mathbb{D} \subset \mathbb{R}^{c_d}$ ,  $c_d \geq 1$ , be a lattice of (possibly) unevenly placed locations in  $\mathbb{R}^{c_d}$ .  $\mathbb{D}$  is infinitely countable and the distance  $d(i, j)$  between any two elements  $i$  and  $j$  in  $\mathbb{D}$  is larger than or equal to a specific positive constant, say 1 without loss of generality.  $n$  individual units in an economy for model (1) are located or living in a region  $\mathbb{D}_n \subset \mathbb{D}$ , where the cardinality of  $\mathbb{D}_n$  is  $n$ .

<sup>9</sup>It is asymptotically equivalent to a GMM estimator with linear and quadratic moments, where the linear moments correspond to the instrumental variables estimation of the parameters in the equation on the dependent variable in a GS2SLS approach.

<sup>10</sup>This condition is the same as that for the consistency of the ordinary least-squares estimator (Lee, 2002).

Since the general density function  $f(x, \eta)$  can introduce nonlinearity into the pseudo log-likelihood function, we require a proper law of large numbers (LLN) for analysis. We use the LLN for near-epoch-dependent (NED) spatial processes, developed in Jenish and Prucha (2012). Assumption 1 maintains some conditions required for such an LLN. The assumption provides basic settings on individual units. The minimum distance assumption on individual units corresponds to increasing domain asymptotics in the spatial literature.<sup>11</sup>

Let  $\|\cdot\|_\infty$  and  $\|\cdot\|_1$  be, respectively, the row sum and column sum matrix norms.

**Assumption 2** (Basic conditions on model elements). (i)  $v_i$ 's are i.i.d. with mean zero and unit variance. (ii)  $W_n$  and  $M_n$  are nonstochastic matrices such that  $\sup_n \|W_n\|_\infty < \infty$  and  $\sup_n \|M_n\|_\infty < \infty$ . (iii)  $c_0 \equiv \max\{|\lambda_0| \sup_n \|W_n\|_\infty, |\rho_0| \sup_n \|M_n\|_\infty\} < 1$ . (iv) The elements of  $X_n$  are uniformly bounded constants.

We consider i.i.d. innovations as in many papers on spatial econometric models. The uniform boundedness condition on the spatial weights matrices in Assumption 2(ii), originated in Kelejian and Prucha (1998, 1999, 2001), limits the degree of spatial dependence to be manageable.<sup>12</sup> The elements of spatial weights matrices are often nonnegative in practice, but our theoretical analysis does not require such an assumption. Assumption 2(iii) implies the nonsingularity of  $R_n \equiv R_n(\rho_0)$  and  $S_n \equiv S_n(\lambda_0)$  for any  $n$ . In Assumption 2(iv), the elements of  $X_n$  are assumed to be constants for simplicity, as in Lee (2004).<sup>13</sup>

## 2.1. Consistency

Model (1) can be written as

$$R_n S_n Y_n = R_n X_n \beta_0 + \sigma_0 V_n. \quad (3)$$

Thus, for given  $\lambda_0$  and  $\rho_0$ , (3) is a linear regression model with  $R_n S_n Y_n$  being a vector of observations on the dependent variable and  $R_n X_n$  being the explanatory variable matrix. Newey and Steigerwald (1997) establish a set of results on the consistency of the NGPMLE for coefficients in a conditional heteroskedasticity model, which nests the linear regression model as a special case. These results depend on whether the model has an intercept term or whether model innovations are symmetric.<sup>14</sup> The regression (3) may not have an intercept term, but if  $M_n$  is

<sup>11</sup> Another commonly used asymptotic method is called infill asymptotics, for which the sample region is fixed and the growth of the sample size is achieved by sampling points arbitrarily dense in the given region. See Cressie (1993) and Conley (1999) for more explanations and examples. If  $f(x, \eta)$  is the density function of normal distributions, then  $\ln f(x, \eta)$  is a quadratic function of  $x$ . In this special case, asymptotic analysis can be based on the LLN for linear-quadratic forms (Kelejian and Prucha, 2001); therefore, Assumption 1 is not needed.

<sup>12</sup> In the spatial econometric literature, a spatial weights matrix is often assumed to be bounded in both the row- and column-sum norms. Later we introduce conditions that imply  $\sup_n \|W_n\|_1 < \infty$  and  $\sup_n \|M_n\|_1 < \infty$ ; therefore, Assumption 2(ii) only involves the row-sum norms of  $W_n$  and  $M_n$ .

<sup>13</sup> Alternatively,  $X_n$  can be allowed to be stochastic with the existence of certain moments.

<sup>14</sup> To gain some intuition on the results, consider the case that the assumed density  $f$  is symmetric and non-Gaussian. As  $f$  is not a Gaussian density, the mean of the dependent variable in a linear regression model is generally not a

row-normalized and  $X_n$  contains an intercept term such that  $X_n = [1_n, X_{2n}]$ , where  $1_n$  is an  $n \times 1$  vector of ones, then  $R_n X_n = [(1 - \rho_0) 1_n, R_n X_{2n}]$  contains an intercept term. Hence, for given  $\lambda_0$  and  $\rho_0$ , we expect the consistency of the NGPMLE of some parameters in (3) under some regularity conditions. However, we have to properly take into account that the spatial dependence parameters  $\lambda$  and  $\rho$  are also estimated. In the following, we provide sufficient conditions for the consistent NGPMLE of some parameters in (3).

Under regularity conditions,  $\frac{1}{n} \ln L_n(\gamma) - \frac{1}{n} E[\ln L_n(\gamma)]$  converges to zero uniformly on a compact parameter space of  $\gamma$ . Suppose that  $\lim_{n \rightarrow \infty} \frac{1}{n} E[\ln L_n(\gamma)]$  is uniquely maximized at some pseudo-true value of  $\gamma$ , then the NGPMLE of  $\gamma$  converges to the pseudo-true value in probability under regularity conditions. The following Assumptions 3 and 4 guarantee that  $E[\ln L_n(\gamma)]$  is uniquely maximized at the pseudo-true value, where some components of the pseudo-true value will be equal to their true values. Denote  $\beta = [\beta_1, \beta_2']'$  in the case that  $X_n$  contains an intercept term, where  $\beta_1$  is the parameter for  $1_n$ . Accordingly, let  $\beta_0 = [\beta_{10}, \beta_{20}']'$ . For a square matrix  $A$ , let  $\text{vec}_D(A)$  be a column vector formed by the diagonal elements of  $A$ . Denote  $A_{1n} = M_n R_n^{-1}$ ,  $A_{2n} = R_n W_n S_n^{-1} R_n^{-1}$ ,  $A_{3n} = M_n W_n S_n^{-1} R_n^{-1}$ , and  $T_n(\tau) = R_n(\rho) S_n(\lambda) S_n^{-1} R_n^{-1} = [t_{n,ij}(\tau)]$  with  $\tau = [\lambda, \rho]'$ .

**Assumption 3** (Identification A). (i)  $f(x, \eta) > 0$ , for any  $x$  and  $\eta$ , and  $E[\ln f(v_i(\theta), \eta)] < \infty$  for all  $\gamma$  in its parameter space. (ii)  $X_n' R_n' R_n X_n$  is nonsingular. (iii) For any  $(\alpha_1, \alpha_2)$ , every element of  $1_n + \alpha_1 \text{vec}_D(A_{1n}) + \alpha_2 \text{vec}_D(A_{2n}) + \alpha_1 \alpha_2 \text{vec}_D(A_{3n})$  is nonzero. (iv)  $g_n(\tau) > 0$ , for  $\tau \neq \tau_0$ , where  $g_n(\tau) = \sum_{i=1}^n \ln |t_{n,ii}(\tau)| - \ln |T_n(\tau)|$ .

Assumption 3(i) is a usual regularity condition. The nonsingularity of  $X_n' R_n' R_n X_n$  in Assumption 3(ii) is for the identification of  $\beta_0$ . Assumption 3(iii) implies that  $t_{n,ii}(\tau) \neq 0$  for any  $i$  and any  $\tau$ . Note that  $T_n(\tau_0) = I_n$ , whose diagonal elements are all equal to 1. Then the assumption is satisfied at least for  $\tau$  close to  $\tau_0$ .

Assumption 3(iv) is for the identification of  $\tau_0$ . It is a generalized version of Hadamard's inequality for positive semidefinite matrices. Lin and Sinnamon (2020) provide sufficient conditions for Assumption 3(iv), which require all principal minors of  $T_n(\tau)$  to be nonnegative and to satisfy a Fischer-type inequality. Alternatively, we could investigate conditions for Assumption 3(iv) in a neighborhood of  $\tau_0$ . Since  $g_n(\tau_0) = 0$  and  $\frac{\partial g_n(\tau_0)}{\partial \tau} = 0$ , we have  $g_n(\tau) > 0$  for  $\tau \neq \tau_0$  in a neighborhood of  $\tau_0$  if  $\frac{\partial^2 g_n(\tau_0)}{\partial \tau \partial \tau'}$  is positive-definite. Let  $T_{1n} = A_{1n} - \text{diag}(A_{1n})$  and  $T_{2n} = A_{2n} - \text{diag}(A_{2n})$ , where  $\text{diag}(A)$  for a square matrix  $A$  denotes a diagonal

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natural location parameter of the assumed density. Thus, if  $f$  differs from the true density, the consistency of the NGPMLE of the parameters for the mean is not guaranteed. When the true density is symmetric, the mean, median, and mode of the dependent variable are equal; thus, the mean and the natural location parameter are the same for  $f$ . It follows that the parameters for the mean can be consistently estimated by the non-Gaussian PML under regularity conditions. In the case that the true density is asymmetric, if there is no intercept term, the difference between the mean and the natural location parameter for  $f$  leads to the inconsistency of the NGPMLE of the parameters for the mean. The existence of an intercept in a linear regression model accounts for the difference, so other parameters for the mean can still be consistently estimated by the non-Gaussian PML.

matrix formed by the diagonal elements of  $A$ . Then  $\frac{\partial^2 g_n(\tau_0)}{\partial \tau \partial \tau'}$  is positive-definite when  $W_n$  and  $M_n$  are equal,  $T_{1n}$  and  $T_{2n}$  are linearly independent, and either  $W_n$  is symmetric or it is row-normalized from a symmetric matrix (see Lemma B.1 in Appendix B).

**Assumption 4** (Identification B). Either the following (i) or (ii) holds:

- (i) (a)  $M_n$  is row-normalized. (b)  $X_n$  contains an intercept term. (c)  $E[\ln f(\frac{\sigma_0 v_i - \alpha}{\sigma}, \eta)] - \ln(\sigma)$  has a unique maximum at  $[\sigma_\infty, \alpha_\infty, \eta'_\infty]'$ .
- (ii) (a)  $v_i$  is symmetrically distributed around zero with unimodal density  $k(v)$ , which satisfies that  $k(v_1) \leq k(v_2)$  for  $|v_1| \geq |v_2|$ . (b) For each  $\eta$ ,  $f(v, \eta) = f(-v, \eta)$  and  $f(v_1, \eta) < f(v_2, \eta)$  for  $|v_1| > |v_2|$ . (c)  $E[\ln f(\frac{\sigma_0 v_i}{\sigma}, \eta)] - \ln(\sigma)$  has a unique maximum at  $[\sigma_\infty, \eta'_\infty]'$ .

The spatial weights matrix  $M_n$  can be either row-normalized or not row-normalized, but a row-normalized  $M_n$  facilitates the interpretation of the spatial dependence parameter  $\rho$ , since it indicates that each element of  $M_n U_n$  is a weighted average of  $U_n$  for a nonnegative  $M_n$ . Thus, spatial weights matrices are often row-normalized in practice.<sup>15</sup> An intercept term is usually included in the SARAR model in empirical research.<sup>16</sup> Assumption 4(i)(c) and (ii)(c) is the same as Assumptions 2.4 and 2.6 in Newey and Steigerwald (1997), respectively. Assumption 4(i)(c) strengthens Assumption 4(ii)(c). With a row-normalized  $M_n$  and an intercept term in  $X_n$ , the term  $\frac{1}{\sigma}(\sigma_0 v_i - \alpha)$  in Assumption 4(i)(c) is equal to  $v_i(\theta)$  evaluated at  $\theta = [\lambda_0, \rho_0, \frac{\alpha}{1-\rho_0} + \beta'_{10}, \beta'_{20}, \sigma^2]'$ . Newey and Steigerwald (1997) provide some insights on Assumption 4(ii)(c). A necessary condition for it is that  $E[\ln f(\frac{\sigma_0 v_i}{\sigma}, \eta_\infty)] - \ln \sigma$  is uniquely maximized at  $\sigma = \sigma_\infty$ . Therefore,  $f(x, \eta)$  should be chosen such that  $\sigma_\infty$  minimizes the Kullback–Leibler distance between the true innovation density and the pseudo density  $\frac{\sigma_0}{\sigma} f(\frac{\sigma_0 x}{\sigma}, \eta_\infty)$ . Such an assumption holds for the Gaussian likelihood, the likelihood for a standardized Student's  $t$  distribution with more than two degrees of freedom, and a generalized Gaussian likelihood with  $\ln f(x, \eta) = -|x|^\eta [\Gamma(3/\eta)/\Gamma(1/\eta)]^{\eta/2} + c$ , where  $c$  is a constant and  $\Gamma(\cdot)$  denotes the gamma function (Fan et al., 2014). The assumption also implies that  $\sigma_\infty$  is generally different from  $\sigma_0$ , although it is straightforward to show that  $\sigma_\infty = \sigma_0$  if  $f(\cdot)$  is a Gaussian density.<sup>17</sup> For the case with symmetric innovations, Assumption 4(ii)(a) and (b) is the same as Assumption 2.3 in Newey

<sup>15</sup> Another reason is that it implies a simple interval of  $\rho$  for the nonsingularity of  $I_n - \rho M_n$ . See the discussions in, e.g., Kelejian and Prucha (2010). Some authors prefer not to row-normalize a spatial weights matrix (e.g., Baltagi, Egger, and Pfaffermayr, 2008).

<sup>16</sup> In some rare cases, an intercept term is not included, e.g., when  $Y_n$  and  $X_n$  are normalized to have mean zero. An example can be found in LeSage (1999, p. 72).

<sup>17</sup> Furthermore,  $\sigma_\infty/\sigma_0$  and  $\eta_\infty$  only depend on the true disturbance distribution and the chosen density function  $f(v, \eta)$ , but do not depend on model characteristics such as spatial weights matrices, exogenous variables, and parameter values. The  $\sigma_\infty/\sigma_0$  differs from 1 even when the true innovation distribution and the chosen density function  $f(v, \eta)$  are spherically symmetric. We report the values of  $\sigma_\infty/\sigma_0$  for some chosen disturbance distributions and a density function  $f(v, \eta)$  in the Supplementary Material.

and Steigerwald (1997). Both the true density function of  $v_i$  and the assumed density function  $f(v, \eta)$  are required to be unimodal.

**PROPOSITION 1.** (i) If Assumptions 1–3 and 4(i) are satisfied, then  $E[\ln L_n(\gamma)]$  is uniquely maximized at  $\gamma_* = [\lambda_0, \rho_0, \beta_{1\infty}, \beta'_{20}, \sigma^2_{\infty}, \eta'_{\infty}]'$ , where  $\beta_{1\infty} = \beta_{10} + \frac{\alpha_{\infty}}{1-\rho_0}$ . (ii) If Assumptions 1–3 and 4(ii) are satisfied, then  $E[\ln L_n(\gamma)]$  is uniquely maximized at  $\gamma_{\#} = [\lambda_0, \rho_0, \beta'_0, \sigma^2_{\infty}, \eta'_{\infty}]'$ .

In the case with a row-normalized  $M_n$ , the intercept term and the variance parameter are generally not consistently estimated, whereas other model parameters can be consistently estimated; in the case with symmetric innovations, only the variance parameter is inconsistently estimated.

**Remark 1.** For a SAR model with no SAR process of disturbances, i.e.,  $Y_n = \lambda_0 W_n Y_n + X_n \beta_0 + \sigma_0 V_n$ , a result similar to Proposition 1(i) holds, where Assumption 4(i) reduces to that  $X_n$  contains an intercept term and  $E[\ln f(\frac{\sigma_0 v_i - \alpha}{\sigma}, \eta)] - \ln(\sigma)$  has a unique maximum at  $[\sigma_{\infty}, \alpha_{\infty}, \eta'_{\infty}]'$ . As  $M_n$  does not appear in the model, the condition of a row-normalized  $M_n$  is irrelevant. For a given  $\lambda_0$ , the SAR model is a linear regression model with the dependent variable  $S_n Y_n$  and the exogenous variable matrix  $X_n$ . It can also be seen as a special case of the SARAR model with a row-normalized  $M_n$  and  $\rho_0 = 0$ ; therefore, it is not considered separately.<sup>18</sup>

In the case with neither row normalization of  $M_n$  nor the symmetry of innovations, we could add a location parameter  $\alpha$  to the non-Gaussian pseudo log-likelihood function to obtain the modified function<sup>19</sup>

$$\ln L_n(\delta) = \sum_{i=1}^n \ln f\left(v_i(\theta) - \frac{1}{\sigma} \alpha, \eta\right) - \frac{n}{2} \ln(\sigma^2) + \ln |S_n(\lambda)| + \ln |R_n(\rho)|, \quad (4)$$

where  $\delta = [\lambda, \rho, \beta', \sigma^2, \alpha, \eta']'$ . This function is formed as if we had the model  $Y_n = \lambda_0 W_n Y_n + X_n \beta_0 + U_n$ , where  $U_n = \alpha_0 1_n + \rho_0 M_n U_n + \sigma_0 V_n$ . This model can be rewritten as  $R_n S_n Y_n = R_n X_n \beta_0 + \alpha_0 1_n + \sigma_0 V_n$ , which has an intercept term. Thus, as the above analysis under Assumption 4(i), we could show that  $E[\ln L_n(\delta)]$  is uniquely maximized at  $\delta_{\#} = [\lambda_0, \rho_0, \beta'_0, \sigma^2_{\infty}, \alpha_{\infty}, \eta'_{\infty}]'$  under regularity conditions.

**PROPOSITION 2.** If Assumptions 1–3 and 4(i)(c) are satisfied and  $R_n X_n$  does not contain an intercept term, then  $E[\ln L_n(\delta)]$  is uniquely maximized at  $\delta = \delta_{\#}$ .

<sup>18</sup>See the Supplementary Material for formal analysis.

<sup>19</sup>When  $M_n$  is row-normalized and  $X_n$  contains an intercept term, since  $v_i(\theta) - \frac{\alpha}{\sigma} = \frac{1}{\sigma} e'_i R_n(\rho) [S_n(\lambda) Y_n - X_{2n} \beta_2] - \frac{(1-\rho)\beta_1 + \alpha}{\sigma}$ ,  $\ln L_n(\delta)$  is not uniquely maximized and thus should not be used. When  $v_i$  is symmetric,  $\ln L_n(\delta)$  can still be used to derive an NGPML, but there might be efficiency loss. Newey and Steigerwald (1997) study such efficiency loss for conditional heteroskedasticity models. We do not examine the issue theoretically for SAR models in this study, but we investigate it by Monte Carlo experiments.

The identification results in Propositions 1 and 2 are for a finite  $n$ . To prove the convergence of the NGPMLE, we need to strengthen the identification inequalities to the limit.<sup>20</sup>

**Assumption 5** (Identification for large samples). For the log-likelihood function  $\ln L_n(\gamma)$  in (2), assume that  $\limsup_{n \rightarrow \infty} \frac{1}{n} \{E[\ln L_n(\gamma)] - E[\ln L_n(\gamma_*)]\} < 0$ , for any  $\gamma \neq \gamma_*$ , if Assumption 4(i) holds, and assume that  $\limsup_{n \rightarrow \infty} \frac{1}{n} \{E[\ln L_n(\gamma)] - E[\ln L_n(\gamma_\#)]\} < 0$ , for any  $\gamma \neq \gamma_\#$ , if Assumption 4(ii) holds. For  $\ln L_n(\delta)$  in (4), assume that  $\limsup_{n \rightarrow \infty} \frac{1}{n} \{E[\ln L_n(\delta)] - E[\ln L_n(\delta_\#)]\} < 0$ , for any  $\delta \neq \delta_\#$ .

We introduce more regularity conditions for the analysis on the consistency of NGPMLEs.

**Assumption 6** (Consistency A). (i)  $S_n(\lambda)$  is invertible for any  $\lambda$  in its parameter space  $\Lambda$  and  $\{S_n^{-1}(\lambda)\}$  is bounded in either the row sum or column sum matrix norm uniformly on  $\Lambda$ . Similar conditions hold for  $R_n(\rho)$ . (ii) The parameter space  $\Gamma$  of  $\gamma$  is a compact subset of  $\mathbb{R}^{k_\gamma}$ , where  $k_\gamma$  is the length of  $\gamma$ . Similar conditions hold for  $\delta$  and  $\kappa$ .

Assumption 6(i) is required due to the nonlinearity involved in the log Jacobians  $\ln |S_n(\lambda)|$  and  $\ln |R_n(\rho)|$  in the pseudo log-likelihood functions. The compactness of parameter spaces in Assumption 6(ii) is a familiar assumption on extremum estimators.

**Assumption 7** (Consistency B). At least one of the following two conditions (i) and (ii) is satisfied:

(i) Only individuals whose distances are less than or equal to some specific constant  $\bar{d}_0$  may affect each other directly, i.e.,  $w_{n,jk}$  and  $m_{n,jk}$  can be nonzero only if  $d(j,k) \leq \bar{d}_0$  for any  $j, k$ , and  $n$ .

(ii) (a) For every  $n$ , the number of columns  $w_{n,j}$  of  $W_n$  with  $|\lambda_0| \sum_{i=1}^n |w_{n,ij}| > c_0$  is less than or equal to some fixed nonnegative integer that does not depend on  $n$ , denoted as  $N$ .<sup>21</sup> A similar condition holds for  $M_n$ . (b) There are constants  $\pi_1$  and  $\pi_2$  with  $\pi_2 > c_d$  such that  $|w_{n,jk}| \leq \pi_1 d(j,k)^{-\pi_2}$  and  $|m_{n,jk}| \leq \pi_1 d(j,k)^{-\pi_2}$ , where  $c_d$  is in Assumption 1.

**Assumption 8** (Consistency C). (i)  $f(x, \eta)$  is differentiable with respect to  $x$  and  $\eta$  such that  $|\frac{\partial \ln f(x, \eta)}{\partial x}| \leq c_f(|x|^{c_t} + 1)$  and  $\|\frac{\partial \ln f(x, \eta)}{\partial \eta}\| \leq c_f(|x|^{1+c_t} + 1)$  for some constant  $c_f$  and  $c_t = 0$  or  $1$ . (ii) For the  $c_t$  in (i),  $E(|v_i|^{2+2c_t+\iota}) < \infty$ , for some  $\iota > 0$ .

Assumptions 7 and 8 are maintained to show the NED properties of some relevant terms. Assumption 7 on the spatial weights matrices is the same as Assumption 3 in Xu and Lee (2015) for a SAR Tobit model. Assumption 7(i) does not allow direct interactions between individuals far from each other. While

<sup>20</sup>It is common to assume separate identification conditions for a finite  $n$  and for large samples in the spatial econometric literature. See, e.g., Assumption 8 in Xu and Lee (2015).

<sup>21</sup>The  $c_0$  here is some positive number smaller than 1, which can be different from that in Assumption 2(iii). We use  $c_0$  for simplicity as in Xu and Lee (2015).

Assumption 7(ii)(b) allows any off-diagonal element of spatial weights matrices to be nonzero, the interaction needs to decay fast enough. Assumption 7(ii)(a) corresponds to the existence of a limited number of spatial units that can have large aggregated effects on other spatial units.

Assumption 8(i) covers the case with a bounded  $\frac{\partial \ln f(x, \eta)}{\partial x}$  and the case where  $|\frac{\partial \ln f(x, \eta)}{\partial x}| \leq c_f(|x| + 1)$  for some constant  $c_f$ . The derivative  $\frac{\partial \ln f(x, \eta)}{\partial x}$  is bounded for a smooth enough  $f(x, \eta)$  whose tail behavior is proportional to  $|x|^{-a}$ , for  $a \geq 1$ , or  $e^{-b|x|^a}$ , for  $0 < a \leq 1$  and  $b > 0$ . Examples include Student's  $t$  and the logistic distributions. On the other hand,  $|\frac{\partial \ln f(x, \eta)}{\partial x}| \leq c_f(|x| + 1)$  for some constant  $c_f$  for a smooth enough  $f(x, \eta)$  whose tail behavior is proportional to  $e^{-b|x|^a}$ , for  $0 < a \leq 2$  and  $b > 0$ . An example is the normal distribution. The condition on  $\frac{\partial \ln f(x, \eta)}{\partial \eta}$  is also satisfied for Student's  $t$ , logistic, and normal distributions. Depending on whether  $\frac{\partial \ln f(x, \eta)}{\partial x}$  is bounded or  $|\frac{\partial \ln f(x, \eta)}{\partial x}| \leq c_f(|x| + 1)$ , Assumption 8(ii) requires different moment conditions on  $v_i$ . With a bounded  $\frac{\partial \ln f(x, \eta)}{\partial x}$ , we only need  $v_i$  to have a finite moment with the order  $2 + \iota$  for some  $\iota > 0$ .

Denote the NGPMLEs that maximize  $\ln L_n(\gamma)$  and  $\ln L_n(\delta)$  by, respectively,  $\hat{\gamma}$  and  $\hat{\delta}$ . The convergence of the NGPMLEs is summarized in the following theorem.

**THEOREM 1.** *Suppose that Assumptions 1–3 and 5–8 are satisfied.*

- (i) *For the case with a row-normalized  $M_n$ , if Assumption 4(i) is also satisfied, then  $\hat{\gamma} = \gamma_* + o_p(1)$ .*
- (ii) *For the case with symmetric  $v_i$ , if Assumption 4(ii) is also satisfied, then  $\hat{\gamma} = \gamma_* + o_p(1)$ .*
- (iii) *For the case with neither row-normalization of  $M_n$  nor the symmetry of  $v_i$ , if Assumption 4(i)(c) is also satisfied and  $R_n X_n$  does not contain an intercept term, then  $\hat{\delta} = \delta_* + o_p(1)$ .*

## 2.2. Asymptotic Distributions

The asymptotic distributions of the NGPMLEs can be derived by mean value theorem expansions of their first-order conditions at the pseudo-true values, and applying a proper central limit theorem (CLT).

As an example, consider the case with symmetric  $v_i$ . With the reduced form  $Y_n = S_n^{-1}(X_n \beta_0 + \sigma_0 R_n^{-1} V_n)$ , each element of  $\frac{\partial \ln L_n(\gamma_{\#})}{\partial \gamma}$  is a special case of the general form

$$\omega_n = \varepsilon_n' A_n V_n + b_n' \varepsilon_n + 1_n' \Psi_n - E(\varepsilon_n' A_n V_n), \quad (5)$$

where  $\varepsilon_n = \left[ \frac{\partial f(\frac{\sigma_0}{\sigma_{\infty}} v_1, \eta_{\infty})}{\partial v}, \dots, \frac{\partial f(\frac{\sigma_0}{\sigma_{\infty}} v_n, \eta_{\infty})}{\partial v} \right]' \equiv [\epsilon_i]$ ,  $\Psi_n = \left[ \frac{\partial f(\frac{\sigma_0}{\sigma_{\infty}} v_1, \eta_{\infty})}{\partial \eta}, \dots, \frac{\partial f(\frac{\sigma_0}{\sigma_{\infty}} v_n, \eta_{\infty})}{\partial \eta} \right]' c_{\eta} \equiv [\psi_i]$  with  $c_{\eta}$  being a  $k_{\eta} \times 1$  vector of constants,  $A_n = [a_{n,ij}]$  is an  $n \times n$  nonstochastic matrix,  $b_n = [b_{ni}]$  is an  $n \times 1$  vector of constants, and  $\varepsilon_n$ ,  $V_n$ , and  $\Psi_n$  have zero means (see the proof of Theorem 2). The  $\omega_n$  can be shown to be asymptotically normal by a CLT for martingale difference arrays, as the proof

for the asymptotic normality of linear-quadratic forms of innovations in Kelejian and Prucha (2001). Such a result is provided in Lemma 6 of Yang and Lee (2017).

We maintain the following assumption for the analysis on the asymptotic distributions.

**Assumption 9** (Asymptotic distributions). (i)  $\gamma_*$ ,  $\gamma_\#$ , and  $\delta_\#$  are in the interior of their respective parameter spaces. (ii)  $f(x, \eta)$  is thrice differentiable with respect to  $z = [x, \eta']'$ , such that  $\|\frac{\partial^2 \ln f(x, \eta)}{\partial z \partial z'}\| \leq c_f(|x|^{2c_t} + 1)$  and  $\|\frac{\partial^3 \ln f(x, \eta)}{\partial z \partial z' \partial z_i}\| \leq c_f(|x|^{3c_t} + 1)$  for each element  $z_i$  of  $z$ , where  $c_t = 0$  for the case with bounded  $\frac{\partial \ln f(x, \eta)}{\partial x}$ , and  $c_t = 1$  for the case with  $|\frac{\partial \ln f(x, \eta)}{\partial x}| \leq c_f(|x| + 1)$ , as stated in Assumption 8(ii). (iii)  $E(|v_i|^{3c_t+3}) < \infty$ . (iv) If Assumption 7(i) holds, assume that  $\sup_n \|S_n^{-1}\|_1 < \infty$  and  $\sup_n \|R_n^{-1}\|_1 < \infty$ .

Assumption 9(i) is a familiar condition required for the  $\sqrt{n}$ -convergence of extremum estimators. Assumption 9(ii) contains further smoothness conditions on  $f(x, \eta)$ . It is similar to Assumption 10 in Xu and Lee (2018), and it is satisfied with  $c_t = 0$  for Student's  $t$ , logistic, and normal distributions. With Assumption 9(ii), only a finite third moment of innovations is needed in Assumption 9(iii) for the case with bounded  $\frac{\partial \ln f(x, \eta)}{\partial x}$ .<sup>22</sup> As the GPMLE is shown to be  $\sqrt{n}$ -consistent only under the existence of moments of innovations with an order higher than 4, it is possible that it has a rate of convergence slower than  $\sqrt{n}$  when innovations only have a finite third moment. In such a situation, the NGPMLE is certainly more efficient than the GPMLE by Theorem 2. Assumption 9(ii) and (iii) is maintained to show the convergence of the Hessian matrices  $\frac{1}{n} \frac{\partial^2 \ln L_n(\hat{\gamma})}{\partial \gamma \partial \gamma'}$  and  $\frac{1}{n} \frac{\partial^2 \ln L_n(\hat{\delta})}{\partial \delta \partial \delta'}$ . Assumption 9(iv) of boundedness in the column-sum norm of  $S_n^{-1}$  and  $R_n^{-1}$  is required for asymptotic distributions as in Kelejian and Prucha (1998) and Lee (2004). It is not required in the situation of Assumption 7(ii) since it can be directly proved (see Lemma B.6).

**THEOREM 2.** *Suppose that Assumptions 1–3 and 5–9 are satisfied.*

- (i) *For the case with a row-normalized  $M_n$ , if Assumption 4(i) is also satisfied, then  $\sqrt{n}(\hat{\gamma} - \gamma_*) \xrightarrow{d} N(0, \lim_{n \rightarrow \infty} \mathcal{A}^{-1} \mathcal{B} \mathcal{A}^{-1})$ , where  $\mathcal{A} = -\frac{1}{n} E(\frac{\partial^2 \ln L_n(\gamma_*)}{\partial \gamma \partial \gamma'})$  and  $\mathcal{B} = \frac{1}{n} E(\frac{\partial \ln L_n(\gamma_*)}{\partial \gamma} \frac{\partial \ln L_n(\gamma_*)}{\partial \gamma'})$ .*
- (ii) *For the case with symmetric  $v_i$ , if Assumption 4(ii) is also satisfied, then  $\sqrt{n}(\hat{\gamma} - \gamma_\#) \xrightarrow{d} N(0, \lim_{n \rightarrow \infty} \mathcal{A}^{-1} \mathcal{B} \mathcal{A}^{-1})$ , where  $\mathcal{A} = -\frac{1}{n} E(\frac{\partial^2 \ln L_n(\gamma_\#)}{\partial \gamma \partial \gamma'})$  and  $\mathcal{B} = \frac{1}{n} E(\frac{\partial \ln L_n(\gamma_\#)}{\partial \gamma} \frac{\partial \ln L_n(\gamma_\#)}{\partial \gamma'})$ .*
- (iii) *For the case with neither row-normalization of  $M_n$  nor the symmetry of  $v_i$ , if Assumption 4(i)(c) is also satisfied and  $R_n X_n$  does not contain an intercept term, then  $\sqrt{n}(\hat{\delta} - \delta_\#) \xrightarrow{d} N(0, \lim_{n \rightarrow \infty} \mathcal{A}^{-1} \mathcal{B} \mathcal{A}^{-1})$ , where  $\mathcal{A} = -\frac{1}{n} E(\frac{\partial^2 \ln L_n(\delta_\#)}{\partial \delta \partial \delta'})$  and  $\mathcal{B} = \frac{1}{n} E(\frac{\partial \ln L_n(\delta_\#)}{\partial \delta} \frac{\partial \ln L_n(\delta_\#)}{\partial \delta'})$ .*

<sup>22</sup>It is possible to develop formal tests for finiteness of moments of innovations in the SARAR model, which is beyond the scope of this paper.

The specific expressions of  $\mathcal{A}$  and  $\mathcal{B}$  are in Appendix A.<sup>23</sup> For easy reference, denote the NGPMLE without added parameter by NGPMLE<sub>o</sub>, and that with an added parameter by NGPMLE<sub>a</sub>. For the special case of a spatial error model with symmetric innovations, i.e., model (1) with  $\lambda_0 W_n Y_n$  omitted and symmetric  $v_i$ , we could show that  $\mathcal{A}$  and  $\mathcal{B}$  for NGPMLE<sub>o</sub> are block diagonal and the NGPMLE<sub>o</sub> of  $\beta$  has a more explicit expression, as presented in the following corollary.

**COROLLARY 1.** *For the spatial error model with symmetric  $v_i$ , the NGPMLE<sub>o</sub> of  $\beta$  has the asymptotic variance  $\lim_{n \rightarrow \infty} \frac{\sigma_0^2 E(\xi_{1i}^2)}{[E(\xi_{2i})]^2} (\frac{1}{n} X_n' R_n' R_n X_n)^{-1}$ , where  $\xi_{1i} = \frac{\sigma_0}{\sigma_\infty} \frac{\partial \ln f(\frac{\sigma_0}{\sigma_\infty} v_i, \eta_\infty)}{\partial v}$  and  $\xi_{2i} = -\frac{\sigma_0^2}{\sigma_\infty^2} \frac{\partial^2 \ln f(\frac{\sigma_0}{\sigma_\infty} v_i, \eta_\infty)}{\partial v^2}$ , and the GPMLE and BGMME of  $\beta$  have the asymptotic variance  $\lim_{n \rightarrow \infty} \sigma_0^2 (\frac{1}{n} X_n' R_n' R_n X_n)^{-1}$ .*

By the above corollary, for the spatial error model with symmetric  $v_i$ , the BGMME of  $\beta$  has no efficiency improvement over the GPMLE, and the efficiency of NGPMLE<sub>o</sub> relative to the GPMLE is determined by the scalar  $\frac{E(\xi_{1i}^2)}{[E(\xi_{2i})]^2}$ . For the general SARAR model,  $\mathcal{A}$  and  $\mathcal{B}$  are not block diagonal and the estimation of  $\eta$  may affect the asymptotic efficiency of the NGPMLE of model parameters. Thus, it is not easy to compare analytically the efficiencies of the NGPMLE and other estimators.

### 2.3. Efficiency Comparisons

In this subsection, we compare the estimation efficiency of our NGPMLE with those of the GPMLE and BGMME.<sup>24</sup> For the asymptotic variance of the NGPMLE, as the closed form is not available, we compute the asymptotic variance in Theorem 2 for a given sample size with numerical integration. Student's  $t$  distribution with unknown degrees of freedom is used in deriving the NGPMLE.<sup>25</sup>

<sup>23</sup>One may estimate  $\mathcal{A}$  and  $\mathcal{B}$  using the expressions in Appendix A for inference purposes. Alternatively,  $\mathcal{A}$  can be estimated using  $-\frac{1}{n} \frac{\partial^2 \ln L_n(\hat{\gamma})}{\partial \gamma \partial \gamma'}$  or  $-\frac{1}{n} \frac{\partial^2 \ln L_n(\hat{\delta})}{\partial \delta \partial \delta'}$ , and  $\mathcal{B}$  can be estimated according to the martingale structure of the non-Gaussian score.

<sup>24</sup>Various impacts arising from a change in an exogenous explanatory variable, as defined in, e.g., LeSage and Pace (2009), are functions of the spatial lag parameter  $\lambda_0$  and the coefficient on the variable. Then by the delta method, if the NGPMLE is asymptotically more efficient than other estimators, so are the impact estimators computed with the NGPMLE than those computed with other estimators. Some efficiency comparisons for impact estimators based on numerical integration and Monte Carlo experiments are provided in the Supplementary Material. The patterns are the same as those for estimators. We thank an anonymous referee for the suggestion of considering impact estimators.

<sup>25</sup>In this study, we have not theoretically considered the choice of distributions used to derive the NGPMLEs. As suggested in Fan et al. (2014), the distributions can be chosen to minimize the asymptotic variance of the NGPMLE in Theorem 2. In addition, the NGPMLE and the GPMLE can be aggregated to derive an estimator that is more efficient than both. A more practical method can be based on diagnostic tests. In the Supplementary Material, we derive some diagnostic tests such as the normality and excess kurtosis tests of innovations in the SARAR model. Nonnormal innovations imply that a proper NGPMLE can be more efficient than the GPMLE. If the excess kurtosis test suggests a positive excess kurtosis, then we can use a leptokurtic distribution such as Student's  $t$  distribution; otherwise, a platykurtic distribution such as the raised cosine distribution can be used. Our applications imply leptokurtic distributions of innovations; therefore, we use Student's  $t$  distribution with one parameter, which is relatively simple

TABLE 1. Models considered for efficiency comparisons.

	Row-normalized $M_n$	Non-row-normalized $M_n$
Spatial error model:	Symmetric and asymmetric $v_i$	—
SARAR model:	Asymmetric $v_i$	Symmetric and asymmetric $v_i$

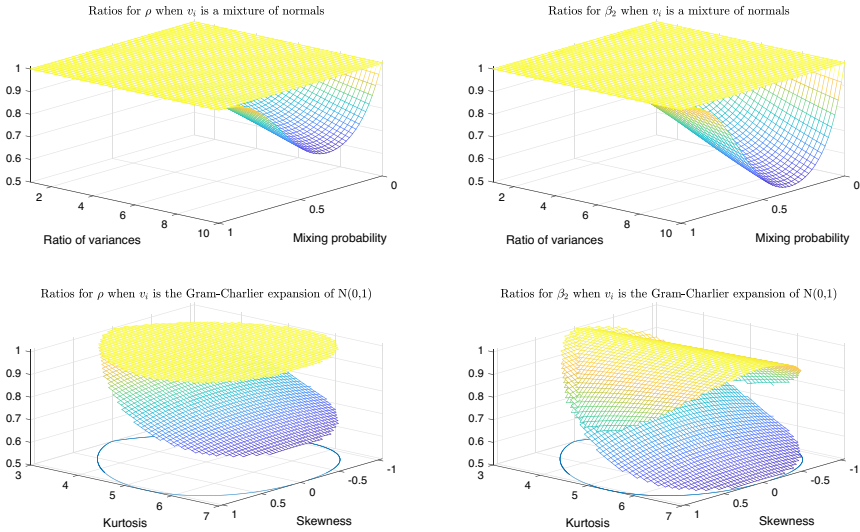
The considered models are listed in Table 1. For the SARAR model, the spatial weights matrix  $M_n$  is block-diagonal and each diagonal block is based on the matrix for the study of crimes across 49 districts in Columbus, Ohio, in Anselin (1988);  $M_n$  is either row-normalized or normalized by its spectral radius;  $W_n$  is set to be equal to  $M_n$ ; the exogenous variable matrix  $X_n$  contains an intercept term and a standard normal random variable; the spatial dependence parameters  $\lambda_0$  and  $\rho_0$  are equal to 0.4 and 0.2, respectively; the coefficients for  $X_n$  are set to 1; the true variance parameter  $\sigma_0^2$  is 0.25; and the sample size is 147. For the case with symmetric innovations,  $v_i$  is set to be a mixture of two normal distributions with mean zero, and for the case with asymmetric innovations,  $v_i$  is an admissible fourth-order Gram–Charlier expansion of the standard normal distribution as a function of the skewness and kurtosis coefficients.<sup>26</sup> The settings for the spatial error model are the same as for the SARAR model, except for the omission of  $\lambda_0 W_n Y_n$ .

**2.3.1. Spatial Error Model with a Row-Normalized  $M_n$ .** We first consider the spatial error model with a row-normalized  $M_n$ . Figure 1 reports the results for both symmetric and asymmetric innovations. We observe that NGPMLE<sub>0</sub> improves upon GPMLE in all cases with a nonnormal true disturbance distribution, and the efficiency improvement can be up to about 50%. In the case with symmetric innovations, BGMME shows almost no efficiency improvement over GPMLE; in the case with asymmetric innovations, BGMME shows some efficiency improvement over GPMLE but usually much less than NGPMLE<sub>0</sub>. Only in the case with asymmetric innovations and for the parameter  $\beta_2$ , BGMME can be slightly more efficient than NGPMLE<sub>0</sub>, which occurs when the skewness coefficient is relatively large and the kurtosis coefficient is small. For the case with asymmetric innovations, the efficiency of NGPMLE<sub>0</sub> relative to GPMLE increases with kurtosis, whereas it is almost not affected by skewness.

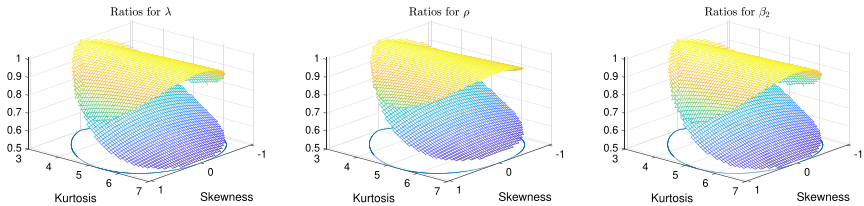
**2.3.2. SARAR Model with a Row-Normalized  $M_n$  and Asymmetric  $v_i$ .** Figure 2 reports the efficiency comparison results for the SARAR model with a row-normalized  $M_n$  and asymmetric innovations. Similar to the results for the spatial

and can have various degrees of excess kurtosis. As pointed out by the Co-Editor and an anonymous referee, using a sufficiently general family of distributions can lead to efficiency loss because many more parameters are estimated alongside other model parameters, whereas using diagnostic tests to choose distributions can suffer from the pretesting issue (e.g., Giles and Giles, 1993). We leave those issues to future study.

<sup>26</sup>The admissible combinations of the skewness and kurtosis coefficients can be seen from, e.g., Spiring (2011).



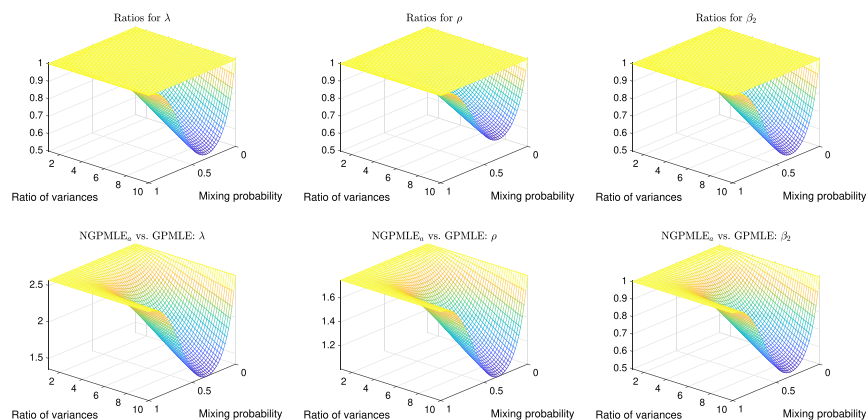
**FIGURE 1.** Efficiency comparisons of different estimators for the spatial error model with a row-normalized  $M_n$ . The lower mesh in each subfigure shows the ratios of the asymptotic variance of NGPMLE<sub>0</sub> to that of GPMLE, whereas the upper mesh shows the ratios of the asymptotic variance of BGMME to that of GPMLE.



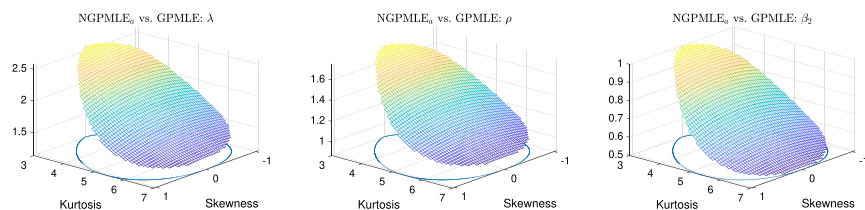
**FIGURE 2.** Efficiency comparisons of different estimators for the SARAR model with a row-normalized  $M_n$  and asymmetric innovations. The  $v_i$  is an admissible fourth-order Gram–Charlier expansion of the standard normal distribution as a function of the skewness and kurtosis coefficients. The lower mesh in each subfigure shows the ratios of the asymptotic variance of NGPMLE<sub>0</sub> to that of GPMLE, whereas the upper mesh shows the ratios of the asymptotic variance of BGMME to that of GPMLE.

error model, NGPMLE<sub>0</sub> shows a significant efficiency improvement over GPMLE, and the improvement is much larger than that of BGMME in most cases.

**2.3.3. SARAR Model with a Non-Row-Normalized  $M_n$ .** We next consider the SARAR model with a non-row-normalized  $M_n$ . When the innovations are symmetric, we consider NGPMLE<sub>0</sub> as well as NGPMLE<sub>a</sub> since both estimators of  $\lambda$ ,  $\rho$ , and  $\beta_2$  are consistent. Figure 3 shows the results. NGPMLE<sub>0</sub> is still observed



**FIGURE 3.** Efficiency comparisons of different estimators for the SARAR model with a non-row-normalized  $M_n$  and symmetric innovations. The  $v_i$  is a mixture of two normal distributions with mean zero. For the first three subfigures, the lower mesh in each subfigure shows the ratios of the asymptotic variance of NGPMLE<sub>0</sub> to that of GPMLE, whereas the upper mesh shows the ratios of the asymptotic variance of BGMME to that of GPMLE. For the fourth to sixth subfigures, the mesh in each subfigure shows the ratios of the asymptotic variance of NGPMLE<sub>a</sub> to that of GPMLE.



**FIGURE 4.** Efficiency comparisons of different estimators for the SARAR model with a non-row-normalized  $M_n$  and asymmetric innovations. The  $v_i$  is an admissible fourth-order Gram–Charlier expansion of the standard normal distribution as a function of the skewness and kurtosis coefficients. The mesh in each subfigure shows the ratios of the asymptotic variance of NGPMLE<sub>a</sub> to that of GPMLE.

to have a significant efficiency improvement over GPMLE, but NGPMLE<sub>a</sub> only has smaller variance than that of GPMLE for  $\beta_2$ , and its variances for the spatial dependence parameters  $\lambda$  and  $\rho$  are typically much larger than those of GPMLE. Figure 4 further demonstrates the efficiency loss of NGPMLE<sub>a</sub> due to an added parameter, for the case with asymmetric innovations.

To summarize, our experiments based on Student's  $t$  distribution in Sections 2.3.1–2.3.3 show that NGPMLE<sub>0</sub> has a uniform efficiency improvement upon GPMLE, which is usually much larger than the efficiency improvement of BGMME, but NGPMLE<sub>a</sub>, the NGPMLE with an added parameter, can be less efficient than GPMLE.

### 3. NON-GAUSSIAN SCORE TEST FOR SPATIAL DEPENDENCE

In this section, we propose a score test for spatial dependence based on the non-Gaussian pseudo log-likelihood function  $\ln L_n(\gamma)$  in (2).<sup>27</sup>

Consider a test of the null hypothesis that  $\tau_0 = 0$ . Let  $\check{\gamma} = [0, 0, \check{\beta}', \check{\sigma}^2, \check{\eta}']'$  be the restricted NGPML of  $\gamma$ , which is derived by maximizing  $\ln L_n(\gamma)$  in (2) with the restriction  $\tau = 0$  imposed. The non-Gaussian score test is based on the asymptotic distribution of  $\frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\check{\gamma})}{\partial \tau}$ . Note that

$$\frac{\partial \ln L_n(\check{\gamma})}{\partial \tau} = \left[ -\frac{1}{\check{\sigma}} \sum_{i=1}^n \frac{\partial \ln f(v_i(\check{\theta}), \check{\eta})}{\partial v} e'_{ni} W_n Y_n, -\sum_{i=1}^n \frac{\partial \ln f(v_i(\check{\theta}), \check{\eta})}{\partial v} e'_{ni} M_n V_n(\check{\theta}) \right]',$$

where  $\check{\theta} = [0, 0, \check{\beta}', \check{\sigma}^2]'$ ,  $v_i(\check{\theta}) = \frac{1}{\check{\sigma}} e'_{ni} (Y_n - X_n \check{\beta})$ , and  $V_n(\check{\theta}) = [v_1(\check{\theta}), \dots, v_n(\check{\theta})]'$ . A special case of interest is the test for spatial dependence in the spatial error model. In this case, the test is based on the asymptotic distribution of  $-\sum_{i=1}^n \frac{\partial \ln L_n(v_i(\check{\theta}), \check{\eta})}{\partial v} e'_{ni} M_n V_n(\check{\theta})$  for a spatial weights matrix  $M_n$ . The statistic  $-\sum_{i=1}^n \frac{\partial \ln L_n(v_i(\check{\theta}), \check{\eta})}{\partial v} e'_{ni} M_n V_n(\check{\theta})$  generalizes the quadratic form  $V'_n(\check{\theta}) M_n V_n(\check{\theta})$  for Moran's  $I$  test for spatial dependence, where  $V'_n(\check{\theta}) M_n V_n(\check{\theta})$  can also be derived from the Gaussian score (Burrige, 1980).

We may apply the mean value theorem to derive the asymptotic distribution of  $\frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\check{\gamma})}{\partial \tau}$  under the null hypothesis. Let  $\mathcal{A} = -\frac{1}{n} E\left(\frac{\partial^2 \ln L_n(\gamma_\infty)}{\partial \gamma \partial \gamma'}\right)$  and  $\mathcal{B} = \frac{1}{n} E\left(\frac{\partial \ln L_n(\gamma_\infty)}{\partial \gamma} \frac{\partial \ln L_n(\gamma_\infty)}{\partial \gamma'}\right)$ .<sup>28</sup> For any two subvectors  $\gamma_1$  and  $\gamma_2$  of  $\gamma$ , denote  $\mathcal{A}_{\gamma_1 \gamma_2} = -\frac{1}{n} E\left(\frac{\partial^2 \ln L_n(\gamma_\infty)}{\partial \gamma_1 \partial \gamma_2'}\right)$  and  $\mathcal{B}_{\gamma_1 \gamma_2} = \frac{1}{n} E\left(\frac{\partial \ln L_n(\gamma_\infty)}{\partial \gamma_1} \frac{\partial \ln L_n(\gamma_\infty)}{\partial \gamma_2'}\right)$ . Under the null hypothesis and regularity conditions,

$$\begin{aligned} \frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\check{\gamma})}{\partial \tau} &= \frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_\infty)}{\partial \tau} + \frac{1}{n} E\left(\frac{\partial^2 \ln L_n(\gamma_\infty)}{\partial \tau \partial \gamma'_u}\right) \sqrt{n}(\check{\gamma}_u - \gamma_{u\infty}) + o_p(1) \\ &= \Delta \frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_\infty)}{\partial \gamma} + o_p(1) \xrightarrow{d} N\left(0, \lim_{n \rightarrow \infty} \Delta \mathcal{B} \Delta'\right), \end{aligned}$$

where  $\gamma_u = [\beta', \sigma^2, \eta']'$ ,  $\gamma_{u\infty}$  is the pseudo-true value of  $\gamma_u$ , and  $\Delta = [I_2, -\mathcal{A}_{\tau \gamma_u} \mathcal{A}_{\gamma_u \gamma_u}^{-1}]$ . Let  $\hat{\Delta}$  and  $\hat{\mathcal{B}}$  be estimators of, respectively,  $\Delta$  and  $\mathcal{B}$ , such that  $\hat{\Delta} = \Delta + o_p(1)$  and  $\hat{\mathcal{B}} = \mathcal{B} + o_p(1)$  under the null hypothesis. The test statistic has the form

$$t_n = \frac{1}{n} \frac{\partial \ln L_n(\check{\gamma})}{\partial \tau'} (\hat{\Delta} \hat{\mathcal{B}} \hat{\Delta}')^{-1} \frac{\partial \ln L_n(\check{\gamma})}{\partial \tau}, \quad (6)$$

which is asymptotically chi-square distributed with two degrees of freedom under the null hypothesis.

<sup>27</sup> A test based on  $\ln L_n(\delta)$  in (4) is omitted since the last section shows that the resulting NGPML can be less efficient than the GPML, and the efficiency of an estimator relates to the power of related tests, as shown in Theorem 3.

<sup>28</sup> We use  $\gamma_\infty$  here for simplicity. By  $\tau_0 = 0$  and Theorem 1,  $\gamma_\infty = [0, 0, \beta'_{1\infty}, \beta'_{20}, \sigma^2_{\infty}, \eta'_{\infty}]'$  in the case with a row-normalized  $M_n$ , and  $\gamma_\infty = [0, 0, \beta'_0, \sigma^2_{\infty}, \eta'_{\infty}]'$  in the case with symmetric  $v_i$ .

For the asymptotic analysis on  $t_n$ , we assume that the true  $\tau$  in the data generating process follows the Pitman drift in the following assumption.

**Assumption 10** (Pitman drift).  $\tau_n = \frac{1}{\sqrt{n}}c_\tau$ , where  $c_\tau$  is a  $2 \times 1$  vector of constants.

**THEOREM 3.** *If Assumptions 1–4 and 5–10 are satisfied, then  $t_n \xrightarrow{d} \chi^2_2(\lim_{n \rightarrow \infty} c'_\tau \Lambda (\Delta \mathcal{B} \Delta')^{-1} \Lambda c_\tau)$ , where  $\Lambda = \mathcal{A}_{\tau\tau} - \mathcal{A}_{\tau\gamma u} \mathcal{A}_{\gamma u \gamma u}^{-1} \mathcal{A}_{\gamma u \tau}$  and  $\chi^2_{a_1}(a_2)$  denotes a noncentral chi-square distribution with  $a_1$  degrees of freedom and noncentrality parameter  $a_2$ .*

By Theorem 2 and the partitioned matrix inverse formula, the asymptotic variance of the NGPMLE  $\hat{\tau}$  has the form  $\Upsilon = \lim_{n \rightarrow \infty} \Lambda^{-1} \Delta \mathcal{B} \Delta' \Lambda^{-1}$ . The noncentrality parameter for the asymptotic noncentral chi-square distribution of  $t_n$  is equal to  $c'_\tau \Upsilon^{-1} c_\tau$ . Thus, if the NGPMLE of  $\tau$  is asymptotically more efficient than the GPMLE, then the non-Gaussian score test is locally more powerful than the Gaussian score test.

## 4. MONTE CARLO

In this section, we implement some Monte Carlo experiments to investigate the finite-sample performance of the NGPMLE and non-Gaussian score test. As in Section 2.3, the NGPMLE is derived by assuming Student's  $t$  distribution with unknown degrees of freedom.

### 4.1. Estimators

We consider three cases: the SARAR model with a row-normalized  $M_n$  and asymmetric  $v_i$ , the SARAR model with a non-row-normalized  $M_n$  and symmetric  $v_i$ , and the SAR model with symmetric  $v_i$ .<sup>29</sup> For the SAR model, we also consider the adaptive estimators proposed in Robinson (2010).<sup>30</sup> Parameters for the innovations correspond to cases where the NGPMLE and BGMME show different levels of efficiency improvements in Section 2.3. The number of Monte Carlo repetitions is 5,000. Other settings are the same as those in Section 2.3.

Table 2 reports the biases, SDs, and root-mean-squared errors (RMSE) of various estimators for the SARAR model with a row-normalized  $M_n$  and asymmetric innovations. The biases of GPMLE, BGMME, and NGPMLE<sub>0</sub> are similar in magnitude. Since the biases are small compared with the SDs, the RMSEs

<sup>29</sup>For the three cases considered, the identification conditions in Assumption 3 are satisfied. In the Supplementary Material, we report some Monte Carlo results for the case when Assumption 3 fails. We observe that the NGPMLE for the spatial dependence parameters and the coefficients on nonintercept exogenous variables still has similar bias as the GPMLE. Thus, it is possible that the NGPMLE for some model parameters is consistent even when Assumption 3 fails. We leave this question to future research.

<sup>30</sup>The adaptive estimators do not apply to the SARAR model (Remark 3 on page 9 of Robinson, 2010).

**TABLE 2.** Performance of various estimators for the SARAR model with a row-normalized  $M_n$  and asymmetric  $v_i$ .

Kurtosis	Skewness		$\lambda$			$\rho$			$\beta_2$		
			Bias	SD	RMSE	Bias	SD	RMSE	Bias	SD	RMSE
Panel A: $n = 147$											
6	0.8	GPMLE	−0.008	0.076	0.077	−0.025	0.142	0.144	−0.001	0.042	0.042
		BGMME	−0.006	0.073	0.073	−0.013	0.143	0.144	0.000	0.040	0.040
		NGPMLE <sub>o</sub>	−0.006	0.059	0.060	−0.018	0.119	0.120	−0.001	0.032	0.032
6	0.05	GPMLE	−0.006	0.076	0.076	−0.032	0.141	0.145	−0.002	0.041	0.041
		BGMME	−0.005	0.078	0.079	−0.017	0.146	0.147	−0.002	0.042	0.042
		NGPMLE <sub>o</sub>	−0.004	0.060	0.060	−0.026	0.120	0.123	−0.001	0.033	0.033
4	0.4	GPMLE	−0.006	0.076	0.076	−0.029	0.143	0.146	−0.002	0.042	0.042
		BGMME	−0.005	0.077	0.077	−0.016	0.148	0.149	−0.002	0.042	0.042
		NGPMLE <sub>o</sub>	−0.005	0.073	0.074	−0.028	0.140	0.143	−0.002	0.040	0.041
4	0.05	GPMLE	−0.008	0.077	0.078	−0.027	0.143	0.145	−0.001	0.042	0.042
		BGMME	−0.007	0.080	0.081	−0.012	0.148	0.149	−0.001	0.043	0.043
		NGPMLE <sub>o</sub>	−0.008	0.075	0.075	−0.026	0.141	0.143	−0.001	0.041	0.041
3.05	0.05	GPMLE	−0.008	0.076	0.077	−0.027	0.142	0.144	−0.002	0.042	0.042
		BGMME	−0.007	0.080	0.080	−0.011	0.147	0.147	−0.002	0.043	0.043
		NGPMLE <sub>o</sub>	−0.009	0.079	0.080	−0.026	0.146	0.148	−0.002	0.043	0.043

(continued)

TABLE 2. (continued)

Kurtosis	Skewness		$\lambda$			$\rho$			$\beta_2$		
			Bias	SD	RMSE	Bias	SD	RMSE	Bias	SD	RMSE
Panel B: $n = 294$											
6	0.8	GPMLE	−0.003	0.052	0.052	−0.012	0.098	0.099	−0.001	0.030	0.030
		BGMME	−0.003	0.049	0.049	−0.007	0.098	0.098	−0.001	0.028	0.028
		NGPMLE <sub>o</sub>	−0.002	0.040	0.040	−0.009	0.082	0.083	−0.001	0.023	0.023
6	0.05	GPMLE	−0.003	0.052	0.053	−0.014	0.099	0.100	−0.001	0.029	0.029
		BGMME	−0.002	0.053	0.053	−0.006	0.101	0.101	−0.001	0.030	0.030
		NGPMLE <sub>o</sub>	−0.001	0.041	0.041	−0.012	0.082	0.083	−0.001	0.023	0.023
4	0.4	GPMLE	−0.003	0.052	0.052	−0.013	0.098	0.099	−0.001	0.029	0.029
		BGMME	−0.003	0.052	0.052	−0.007	0.099	0.099	0.000	0.029	0.029
		NGPMLE <sub>o</sub>	−0.003	0.050	0.050	−0.013	0.096	0.097	−0.001	0.028	0.028
4	0.05	GPMLE	−0.004	0.051	0.052	−0.013	0.099	0.100	−0.001	0.029	0.029
		BGMME	−0.003	0.052	0.052	−0.006	0.101	0.101	−0.001	0.029	0.029
		NGPMLE <sub>o</sub>	−0.003	0.050	0.050	−0.012	0.097	0.098	−0.001	0.028	0.028
3.05	0.05	GPMLE	−0.004	0.052	0.052	−0.014	0.098	0.099	0.000	0.029	0.029
		BGMME	−0.003	0.053	0.053	−0.005	0.099	0.099	−0.001	0.030	0.030
		NGPMLE <sub>o</sub>	−0.004	0.055	0.055	−0.013	0.101	0.102	−0.001	0.030	0.030

Notes: The true disturbance distribution is a fourth-order Gram–Charlier expansion of the standard normal distribution as a function of the skewness and kurtosis coefficients.  $\beta_2$  is the coefficient on the nonintercept variable in  $X_n$ .  $\lambda_0 = 0.4$ ,  $\rho_0 = 0.2$ ,  $\beta_{10} = 1$ ,  $\beta_{20} = 1$ , and  $\sigma_0^2 = 0.25$ .

are similar to the SDs. NGPMLE has a smaller SD than GPMLE when the kurtosis coefficient of innovations is equal to 4 or 6, whereas BGMME only has a smaller SD for  $\lambda$  and  $\beta_2$  when the kurtosis and skewness coefficients are both the largest, i.e., the kurtosis coefficient is 6 and the skewness coefficient is 0.8. When the kurtosis coefficient is 3.05 and the skewness coefficient is 0.05 so that the distribution of innovations is close to the normal distribution, NGPMLE and BGMME have slightly larger SDs than GPMLE. For NGPMLE, a larger kurtosis leads to a smaller SD, whereas skewness does not have much impact on the SD.

Table 3 reports the estimation results for the SARAR model with a non-row-normalized  $M_n$  and symmetric innovations. In addition to NGPMLE<sub>o</sub>, GPMLE, and BGMME, we also consider NGPMLE<sub>a</sub> to investigate its efficiency loss due to an added parameter. The patterns for the relative efficiencies of GPMLE, NGPMLE<sub>o</sub> and BGMME are similar to those in Table 2. When the disturbance distribution is a mixture of two normal distributions with mean zero and the ratio of the variances for the two distributions being close to 1, or when the innovations follow the normal distribution, NGPMLE and BGMME have slightly larger SDs than that of GPMLE. While the NGPMLE<sub>a</sub> of  $\beta_2$  has a smaller SD than that of GPMLE in some cases, the NGPMLE<sub>a</sub> of  $\lambda$  and  $\rho$  has a significantly larger SD than that of GPMLE in most cases, which is consistent with the efficiency comparisons based on numerical integration in Section 2.3.

Estimation results for the SAR model with symmetric innovations are presented in Table 4. The disturbance distribution is a mixture of two normal distributions with mean zero. The ratio of variances for the two normal distributions is 10, and the mixing probability is 0.3.<sup>31</sup> We consider two adaptive estimators (AE) proposed in Robinson (2010): AE<sub>a</sub> and AE<sub>b</sub>, where AE<sub>b</sub> is a bias-corrected version of AE<sub>a</sub>. As in Robinson (2010), we use the polynomial functions  $(x, \dots, x^L)$  or the bounded functions  $(\frac{x}{(1+x^2)^{1/2}}, \dots, \frac{x^L}{(1+x^2)^{L/2}})$  to estimate the score function for the AEs. An AE<sub>a</sub> with  $(x, \dots, x^L)$  is denoted by AE<sub>a</sub>( $p, L$ ), and that with  $(\frac{x}{(1+x^2)^{1/2}}, \dots, \frac{x^L}{(1+x^2)^{L/2}})$  is denoted by AE<sub>a</sub>( $b, L$ ). AE<sub>b</sub> is similarly denoted. We set  $L$  to 1, 2, or 4, as in Robinson (2010). The initial estimate for the AEs is either the NGPMLE or ordinary least-squares estimate (OLSE). Table 4 shows that, while the biases of GPMLE, BGMME, and NGPMLE are relatively small, those of AEs can be large. Some versions of AEs can have smaller SDs than GPMLE, but all AEs have uniformly larger SDs and RMSEs than NGPMLE.

## 4.2. Non-Gaussian and Gaussian Score Tests

Tables 5 and 6 report, respectively, the empirical sizes and powers of score tests for spatial dependence in the SARAR model with a row-normalized  $M_n$  and asymmetric innovations. With a nominal size of 5%, the size distortions of the non-Gaussian and Gaussian score tests are all within 0.5 percentage point. Neither the

<sup>31</sup>Results for some other parameter settings are reported in the Supplementary Material. The patterns are similar.

**TABLE 3.** Performance of various estimators for the SARAR model with a non-row-normalized  $M_n$  and symmetric  $v_i$ .

RV		$\lambda$			$\rho$			$\beta_2$		
		Bias	SD	RMSE	Bias	SD	RMSE	Bias	SD	RMSE
Panel A: $n = 147$										
9	GPMLE	−0.004	0.064	0.064	−0.046	0.172	0.178	−0.002	0.042	0.042
	BGMME	−0.003	0.067	0.067	−0.023	0.186	0.188	−0.002	0.042	0.042
	NGPMLE <sub><i>o</i></sub>	−0.003	0.049	0.049	−0.035	0.146	0.150	−0.001	0.032	0.032
	NGPMLE <sub><i>a</i></sub>	−0.007	0.080	0.080	−0.038	0.157	0.161	−0.002	0.032	0.032
6	GPMLE	−0.004	0.064	0.064	−0.044	0.171	0.176	−0.001	0.042	0.042
	BGMME	−0.004	0.069	0.069	−0.022	0.187	0.188	−0.001	0.043	0.043
	NGPMLE <sub><i>o</i></sub>	−0.003	0.055	0.055	−0.038	0.158	0.162	−0.001	0.036	0.036
	NGPMLE <sub><i>a</i></sub>	−0.010	0.092	0.093	−0.040	0.175	0.179	−0.002	0.037	0.037
3	GPMLE	−0.002	0.064	0.064	−0.044	0.168	0.173	−0.002	0.042	0.042
	BGMME	−0.002	0.068	0.068	−0.020	0.181	0.182	−0.002	0.043	0.043
	NGPMLE <sub><i>o</i></sub>	−0.003	0.062	0.062	−0.041	0.166	0.171	−0.002	0.041	0.041
	NGPMLE <sub><i>a</i></sub>	−0.010	0.104	0.105	−0.048	0.184	0.191	−0.003	0.041	0.042
1.1	GPMLE	−0.005	0.064	0.064	−0.047	0.171	0.178	−0.001	0.042	0.042
	BGMME	−0.004	0.068	0.068	−0.021	0.184	0.185	−0.001	0.043	0.043
	NGPMLE <sub><i>o</i></sub>	−0.006	0.068	0.069	−0.045	0.175	0.181	−0.001	0.043	0.043
	NGPMLE <sub><i>a</i></sub>	−0.009	0.115	0.116	−0.069	0.194	0.206	−0.002	0.047	0.047
Panel B: $n = 294$										
9	GPMLE	−0.002	0.044	0.044	−0.024	0.115	0.118	0.000	0.030	0.030
	BGMME	−0.001	0.045	0.045	−0.013	0.119	0.119	0.000	0.030	0.030
	NGPMLE <sub><i>o</i></sub>	−0.001	0.034	0.034	−0.018	0.097	0.099	0.000	0.023	0.023
	NGPMLE <sub><i>a</i></sub>	−0.004	0.055	0.055	−0.018	0.109	0.110	−0.001	0.023	0.023
6	GPMLE	−0.002	0.044	0.044	−0.021	0.117	0.119	0.000	0.030	0.030
	BGMME	−0.002	0.045	0.045	−0.009	0.120	0.121	0.000	0.030	0.030
	NGPMLE <sub><i>o</i></sub>	−0.002	0.038	0.038	−0.017	0.108	0.109	0.000	0.025	0.025
	NGPMLE <sub><i>a</i></sub>	−0.005	0.061	0.061	−0.018	0.119	0.121	−0.001	0.026	0.026
3	GPMLE	−0.002	0.044	0.044	−0.021	0.118	0.119	0.000	0.029	0.029
	BGMME	−0.002	0.045	0.045	−0.009	0.121	0.121	0.000	0.030	0.030
	NGPMLE <sub><i>o</i></sub>	−0.002	0.042	0.042	−0.021	0.117	0.119	0.000	0.028	0.028
	NGPMLE <sub><i>a</i></sub>	−0.006	0.071	0.071	−0.022	0.132	0.133	−0.001	0.029	0.029
1.1	GPMLE	−0.001	0.044	0.044	−0.022	0.115	0.117	0.000	0.029	0.029
	BGMME	−0.001	0.045	0.045	−0.009	0.118	0.118	0.000	0.029	0.029
	NGPMLE <sub><i>o</i></sub>	−0.002	0.044	0.044	−0.022	0.117	0.119	0.000	0.029	0.029
	NGPMLE <sub><i>a</i></sub>	0.000	0.084	0.084	−0.043	0.152	0.158	−0.001	0.032	0.032

(continued)

TABLE 3. (continued)

RV	$\lambda$			$\rho$			$\beta_2$		
	Bias	SD	RMSE	Bias	SD	RMSE	Bias	SD	RMSE
Panel C: Normal innovations, $n = 147$									
GPMLE	-0.004	0.063	0.064	-0.046	0.172	0.179	-0.001	0.042	0.042
BGMME	-0.004	0.068	0.068	-0.021	0.186	0.187	-0.001	0.044	0.044
NGPMLE <sub>o</sub>	-0.005	0.071	0.071	-0.045	0.179	0.184	-0.001	0.043	0.043
NGPMLE <sub>a</sub>	-0.008	0.113	0.113	-0.069	0.197	0.208	-0.002	0.051	0.051
Panel D: Normal innovations, $n = 294$									
GPMLE	-0.001	0.044	0.044	-0.024	0.117	0.119	0.000	0.029	0.029
BGMME	-0.001	0.045	0.045	-0.010	0.120	0.120	0.000	0.030	0.030
NGPMLE <sub>o</sub>	-0.002	0.046	0.046	-0.023	0.118	0.121	0.000	0.030	0.030
NGPMLE <sub>a</sub>	-0.003	0.088	0.088	-0.041	0.151	0.157	-0.001	0.033	0.033

Notes: For Panels A and B, the true disturbance distribution is a mixture of two normal distributions with mean zero. The mixing probability of the two normal distributions is set to 0.3. "RV" denotes the ratio of variances of the two distributions.  $\beta_2$  is the coefficient on the nonintercept variable in  $X_n$ .  $\lambda_0 = 0.4$ ,  $\rho_0 = 0.2$ ,  $\beta_{10} = 1$ ,  $\beta_{20} = 1$ , and  $\sigma_0^2 = 0.25$ .

Gaussian score test nor the non-Gaussian score test dominates each other in terms of size distortions. For the empirical powers, we observe that the non-Gaussian score test is uniformly more powerful than the Gaussian score test, except for the case when the innovations are very close to be normally distributed. The power of each test increases as  $\lambda_0$  or  $\rho_0$  increases.

## 5. EMPIRICAL APPLICATION

In this section, we apply our NGPMLE to the well-known Harrison and Rubinfeld (1978) hedonic pricing data from the Boston Standard Metropolitan Statistical Area with 506 observations.<sup>32</sup> This dataset is popular in the spatial econometric literature. It has been used in textbooks such as LeSage (1999), LeSage and Pace (2009), and Arbia (2014).

Following LeSage (1999, p. 78), we estimate an SARAR model, where the dependent variable is the log median value of owner-occupied homes in \$1,000's, and the explanatory variables include crime rate (CRIM), proportion of area zoned with large lots (ZN), proportion of nonretail business areas (INDUS), location contiguous to the Charles River (CHAS), squared levels of nitrogen oxides (NOX<sup>2</sup>), squared average number of rooms (RM<sup>2</sup>), proportion of structures built before 1940 (AGE), weighted distances to the employment centers (DIS), an index

<sup>32</sup> Available at <http://lib.stat.cmu.edu/datasets/>. Gilley and Pace (1996) corrected several miscoded observations and Pace and Gilley (1997) added the location of each tract in latitude and longitude. In the Supplementary Material, we also apply our NGPMLE to the crime dataset with 49 observations in Anselin (1988) and to the presidential election dataset with 3,107 observations in Pace and Barry (1997).

**TABLE 4.** Performance of various estimators for the SAR model with symmetric  $v_i$ .

	$\lambda$			$\beta_2$		
	Bias	SD	RMSE	Bias	SD	RMSE
GPMLE	−0.010	0.055	0.055	0.000	0.042	0.042
BGMME	−0.008	0.055	0.056	−0.002	0.042	0.042
NGPMLE <sub>o</sub>	−0.007	0.043	0.044	0.000	0.032	0.032
AEs with GPMLE as the initial estimate						
AE <sub>a</sub> ( $p, 1$ )	0.100	0.066	0.120	−0.010	0.043	0.044
AE <sub>a</sub> ( $b, 1$ )	0.078	0.053	0.094	−0.008	0.034	0.035
AE <sub>b</sub> ( $p, 1$ )	0.327	0.105	0.343	−0.031	0.050	0.059
AE <sub>b</sub> ( $b, 1$ )	0.229	0.080	0.243	−0.023	0.038	0.044
AE <sub>a</sub> ( $p, 2$ )	0.095	0.067	0.116	−0.010	0.044	0.045
AE <sub>a</sub> ( $b, 2$ )	0.075	0.054	0.092	−0.008	0.035	0.036
AE <sub>b</sub> ( $p, 2$ )	0.308	0.103	0.325	−0.030	0.050	0.059
AE <sub>b</sub> ( $b, 2$ )	0.222	0.080	0.236	−0.022	0.039	0.045
AE <sub>a</sub> ( $p, 4$ )	0.074	0.062	0.096	−0.008	0.042	0.042
AE <sub>a</sub> ( $b, 4$ )	0.060	0.063	0.087	−0.006	0.042	0.042
AE <sub>b</sub> ( $p, 4$ )	0.235	0.088	0.251	−0.023	0.045	0.051
AE <sub>b</sub> ( $b, 4$ )	0.189	0.086	0.207	−0.018	0.044	0.048
AEs with OLSE as the initial estimate						
AE <sub>a</sub> ( $p, 1$ )	0.044	0.060	0.074	−0.005	0.042	0.042
AE <sub>a</sub> ( $b, 1$ )	0.022	0.048	0.053	−0.003	0.033	0.034
AE <sub>b</sub> ( $p, 1$ )	0.318	0.109	0.336	−0.030	0.050	0.058
AE <sub>b</sub> ( $b, 1$ )	0.206	0.081	0.222	−0.020	0.037	0.043
AE <sub>a</sub> ( $p, 2$ )	0.041	0.061	0.074	−0.005	0.043	0.044
AE <sub>a</sub> ( $b, 2$ )	0.021	0.050	0.054	−0.003	0.035	0.035
AE <sub>b</sub> ( $p, 2$ )	0.298	0.108	0.317	−0.029	0.050	0.058
AE <sub>b</sub> ( $b, 2$ )	0.199	0.081	0.215	−0.020	0.039	0.043
AE <sub>a</sub> ( $p, 4$ )	0.018	0.059	0.061	−0.003	0.041	0.041
AE <sub>a</sub> ( $b, 4$ )	0.013	0.060	0.061	−0.002	0.042	0.042
AE <sub>b</sub> ( $p, 4$ )	0.214	0.089	0.232	−0.021	0.044	0.049
AE <sub>b</sub> ( $b, 4$ )	0.170	0.085	0.190	−0.017	0.044	0.047

*Notes:* The true disturbance distribution is a mixture of two normal distributions with mean zero. The ratio of variances for the two normal distributions is 10, and the mixing probability is 0.3.  $\beta_2$  is the coefficient on the nonintercept variable in  $X_n$ .  $\lambda_0 = 0.4$ ,  $\rho_0 = 0.2$ ,  $\beta_{10} = 1$ ,  $\beta_{20} = 1$ , and  $\sigma_0^2 = 0.25$ . The sample size is 147.

**TABLE 5.** Empirical sizes of score tests for spatial dependence in the SARAR model with a row-normalized  $M_n$  and asymmetric  $v_i$ .

Kurtosis	Skewness	$n = 147$		$n = 294$	
		GPMLE	NGPMLE <sub>o</sub>	GPMLE	NGPMLE <sub>o</sub>
6	0.8	0.051	0.051	0.048	0.049
6	0.05	0.047	0.044	0.052	0.051
4	0.4	0.048	0.047	0.045	0.046
4	0.05	0.056	0.051	0.049	0.047
3.05	0.05	0.046	0.049	0.045	0.046

*Notes:* The nominal size is 5%. The true disturbance distribution is a fourth-order Gram–Charlier expansion of the standard normal distribution as a function of the skewness and kurtosis coefficients.  $\beta_{10} = 1$ ,  $\beta_{20} = 1$ , and  $\sigma_0^2 = 0.25$ .

of accessibility (RAD), property tax rate (TAX), pupil–teacher ratio (PTRATIO), black population proportion (B), and lower status population proportion (LSTAT). All variables are normalized to have mean zero and unit variance as in LeSage (1999). The spatial weights matrix  $W_n$  is a first-order continuity matrix and row-normalized. The  $M_n$  is set to equal  $W_n$ .

Table 7 reports the empirical results. We carry out several diagnostic tests. First, a normality test of innovations rejects the null of normal innovations at the 1% level. With nonnormal innovations, the GPMLE will lose efficiency compared to a true ML estimator. We further test skewness and excess kurtosis of innovations.<sup>33</sup> While the null hypothesis of zero skewness is not rejected at any usual significance level, the null hypothesis of zero excess kurtosis is rejected at the 1% level. The estimated kurtosis coefficient is 5.751. These results show some evidence of symmetric and leptokurtic innovations for this dataset. GPMLE, BGMME, and NGPMLE have the same sign for each model parameter except the coefficient on INDUS, but their differences in magnitude can be relatively large.<sup>34</sup> For example, for the variable AGE, BGMME is about 60% larger in magnitude than GPMLE, whereas NGPMLE is more than three times that of GPMLE. The standard errors (SEs) of BGMME are very close to those of GPMLE, whereas the SEs of NGPMLE are about 30% smaller than those of GPMLE. Due to the differences in the estimates and SEs, for the variables NOX<sup>2</sup> and AGE, we observe different results on coefficient significance from different estimation methods. For the coefficient on NOX<sup>2</sup>, GPMLE and BGMME are significant at the 1% level, whereas NGPMLE is significant only at the 10% level; for the coefficient on AGE, GPMLE is not significant at any usual significance level, BGMME is significant

<sup>33</sup>All the test statistics are derived in the Supplementary Material. The normality test is a special case of that for the SARAR model with parametric heteroskedasticity in Jin, Lee, and Yang (2022), which follows the Lagrange multiplier principle as in Jarque and Bera (1980). We present it in the Supplementary Material for completeness. The skewness and excess-kurtosis tests are on the basis of the delta method, as in Godfrey and Orme (1991).

<sup>34</sup>We only consider the NGPMLE with no added parameter, since the NGPMLE with an added parameter does not perform well in Monte Carlo experiments.

**TABLE 6.** Empirical powers of score tests for spatial dependence in the SARAR model with a row-normalized  $M_n$  and asymmetric  $v_i$ .

Kurtosis		Skewness		$\lambda_0 = 0$			$\rho_0 = 0$		
				$\rho_0 = 0.1$	$\rho_0 = 0.2$	$\rho_0 = 0.3$	$\lambda_0 = 0.1$	$\lambda_0 = 0.2$	$\lambda_0 = 0.3$
Panel A: $n = 147$									
6	0.8	GPMLE	0.114	0.330	0.663	0.259	0.784	0.982	
		NGPMLE <sub>o</sub>	0.144	0.436	0.794	0.369	0.918	0.998	
6	0.05	GPMLE	0.106	0.330	0.673	0.255	0.777	0.983	
		NGPMLE <sub>o</sub>	0.128	0.420	0.789	0.355	0.909	0.996	
4	0.4	GPMLE	0.111	0.321	0.656	0.254	0.779	0.985	
		NGPMLE <sub>o</sub>	0.119	0.340	0.679	0.272	0.800	0.990	
4	0.05	GPMLE	0.112	0.340	0.666	0.261	0.789	0.982	
		NGPMLE <sub>o</sub>	0.119	0.352	0.684	0.275	0.798	0.985	
3.05	0.05	GPMLE	0.103	0.328	0.653	0.259	0.772	0.984	
		NGPMLE <sub>o</sub>	0.104	0.328	0.652	0.260	0.772	0.983	
Panel B: $n = 294$									
6	0.8	GPMLE	0.171	0.598	0.929	0.473	0.974	1.000	
		NGPMLE <sub>o</sub>	0.237	0.738	0.978	0.661	0.998	1.000	
6	0.05	GPMLE	0.182	0.589	0.928	0.478	0.980	1.000	
		NGPMLE <sub>o</sub>	0.237	0.714	0.977	0.660	0.998	1.000	
4	0.4	GPMLE	0.182	0.601	0.933	0.470	0.972	1.000	
		NGPMLE <sub>o</sub>	0.190	0.623	0.941	0.498	0.977	1.000	
4	0.05	GPMLE	0.176	0.587	0.931	0.463	0.975	1.000	
		NGPMLE <sub>o</sub>	0.188	0.608	0.940	0.491	0.982	1.000	
3.05	0.05	GPMLE	0.174	0.591	0.927	0.479	0.977	1.000	
		NGPMLE <sub>o</sub>	0.175	0.589	0.925	0.478	0.976	1.000	

*Notes:* The true disturbance distribution is a fourth-order Gram–Charlier expansion of the standard normal distribution as a function of the skewness and kurtosis coefficients.  $\beta_{10} = 1$ ,  $\beta_{20} = 1$ , and  $\sigma_0^2 = 0.25$ .

at the 5% level, whereas NGPMLE is significant at the 1% level. These differences in coefficient significance also carry over to impact measures such as the average total, direct, and indirect impacts, which we report in the Supplementary Material. Overall, the application shows that more efficient estimation methods for the SARAR model can be valuable in practice.

6. CONCLUSIONS

This study considers the non-Gaussian PML estimation of the SARAR model. If the spatial weights matrix  $M_n$  in the SAR process of disturbances is row-normalized or the model reduces to the SAR model with no SAR process of disturbances, the NGPMLE for model parameters except the intercept term and the

**TABLE 7.** Empirical results for the hedonic pricing data.

	GPMLE		BGMME		NGPMLE	
	Estimate	SE	Estimate	SE	Estimate	SE
$\lambda$	0.188***	0.060	0.267***	0.055	0.121***	0.044
$\rho$	0.626***	0.061	0.612***	0.062	0.673***	0.048
CRIM	-0.187***	0.023	-0.177***	0.023	-0.166***	0.015
ZN	0.065**	0.031	0.063**	0.031	0.046**	0.021
INDUS	0.016	0.046	-0.001	0.046	0.001	0.031
CHAS	-0.007	0.021	-0.010	0.021	-0.014	0.014
NOX <sup>2</sup>	-0.191***	0.055	-0.310***	0.055	-0.071*	0.038
RM <sup>2</sup>	0.199***	0.024	0.193***	0.024	0.415***	0.016
AGE	-0.046	0.036	-0.074**	0.037	-0.161***	0.025
DIS	-0.256***	0.055	-0.207***	0.055	-0.172***	0.039
RAD	0.342***	0.061	0.387***	0.061	0.202***	0.041
TAX	-0.259***	0.057	-0.234***	0.057	-0.213***	0.038
PTRATIO	-0.127***	0.030	-0.103***	0.030	-0.079***	0.021
B	0.119***	0.026	0.131***	0.026	0.152***	0.018
LSTAT	-0.378***	0.035	-0.373***	0.035	-0.155***	0.023

Test for normality of innovations:

Test statistic: 67.742;  $p$ -value: 0.000.

Test for skewness of innovations:

Test statistic: 0.755;  $p$ -value: 0.450; estimated skewness coefficient = 0.267.

Test for excess kurtosis of innovations:

Test statistic: 4.115;  $p$ -value: 0.000; estimated kurtosis coefficient = 5.751.

Notes: \*, \*\*, and \*\*\* denote significance at, respectively, the 10%, 5%, and 1% levels.

variance parameter  $\sigma^2$  is consistent. If  $M_n$  is not row-normalized but innovations are symmetric, the NGPMLE for model parameters except  $\sigma^2$  is consistent. With neither row-normalization of  $M_n$  nor the symmetry of innovations, a location parameter can be added to the non-Gaussian pseudo log-likelihood function to obtain consistent estimation of model parameters except  $\sigma^2$ . We formally prove the convergence and asymptotic normality of the NGPMLE. An advantage of the NGPMLE is that it can have a significant efficiency improvement upon the GPMLE and BGMME. We also propose a non-Gaussian score test for spatial dependence, which is locally more powerful than the Gaussian score test when the NGPMLE is more efficient than the GPMLE. Using Student's  $t$  distribution to formulate the non-Gaussian likelihood function, our numerical integration and Monte Carlo results show that the NGPMLE with no added parameter can have a significant efficiency improvement upon the GPMLE and BGMME, but the NGPMLE with an added parameter can be less efficient than the GPMLE. The

non-Gaussian score test based on the NGPMLE with no added parameter is more powerful than the Gaussian score test in finite samples. Therefore, we recommend the use of the NGPMLE with no added parameter and the non-Gaussian score test based on it when they are applicable.

## APPENDIX A. Expressions for Asymptotic Variances

In this appendix, we present the expressions for asymptotic variances of NGPMLEs in Theorem 2.

### A.1. Row-Normalized $M_n$

For model (1) with a row-normalized  $M_n$ , the NGPMLE maximizes  $\ln L_n(\gamma)$  in (2). Note that  $v_i(\theta_*) = \frac{\sigma_0}{\sigma_\infty} v_i + \frac{\sigma_0}{\sigma_\infty} c_v$ , where  $c_v = -\frac{1}{\sigma_0}(1 - \rho_0)(\beta_{1\infty} - \beta_{10})$ . Denote  $\zeta_{1i} = \frac{\sigma_0}{\sigma_\infty} \frac{\partial \ln f(v_i(\theta_*), \eta_\infty)}{\partial v}$ ,  $\zeta_{2i} = \zeta_{1i} v_i + 1$ ,  $\zeta_{3i} = -\frac{\partial \ln f(v_i(\theta_*), \eta_\infty)}{\partial \eta}$ ,  $\zeta_{4i} = -\frac{\sigma_0^2}{\sigma_\infty^2} \frac{\partial^2 \ln f(v_i(\theta_*), \eta_\infty)}{\partial v^2}$ ,  $\zeta_{5i} = \frac{\sigma_0}{\sigma_\infty} \frac{\partial^2 \ln f(v_i(\theta_*), \eta_\infty)}{\partial v \partial \eta}$ ,  $\zeta_{6i} = -\frac{\partial^2 \ln f(v_i(\theta_*), \eta_\infty)}{\partial \eta \partial \eta'}$ ,  $D_n = R_n W_n S_n^{-1} R_n^{-1} = [d_{n,ij}]$ ,  $Z_n = M_n R_n^{-1} = [z_{n,ij}]$ ,  $Q_n = \frac{1}{\sigma_0} R_n W_n S_n^{-1} X_n \beta_0 = [q_{ni}]$ , and  $c_\beta = -\frac{1}{\sigma_0}(\beta_{1\infty} - \beta_{10})$ . By Assumption 4(i)(c),  $E(\zeta_{ji}) = 0$ , for  $j = 1, 2, 3$ . For any two subvectors  $\delta_1$  and  $\delta_2$  of  $\delta$ , let  $\mathcal{B}_{\delta_1 \delta_2} = \frac{1}{n} E(\frac{\partial \ln L_n(\gamma_*)}{\partial \delta_1} \frac{\partial \ln L_n(\gamma_*)}{\partial \delta_2})$  and  $\mathcal{A}_{\delta_1 \delta_2} = -\frac{1}{n} E(\frac{\partial^2 \ln L_n(\gamma_*)}{\partial \delta_1 \partial \delta_2})$ .

For the expression of  $\mathcal{B}$ , using the reduced form of  $Y_n$ , we have  $\frac{\partial \ln L_n(\gamma_*)}{\partial \lambda} = -\sum_{i=1}^n q_{ni} \zeta_{1i} - \sum_{i=1}^n d_{n,ii} \zeta_{2i} - \sum_{i=1}^n \zeta_{1i} \sum_{j \neq i} d_{n,ij} v_j$ ,  $\frac{\partial \ln L_n(\gamma_*)}{\partial \rho} = -\sum_{i=1}^n c_\beta \zeta_{1i} - \sum_{i=1}^n z_{n,ii} \zeta_{2i} - \sum_{i=1}^n \zeta_{1i} \sum_{j \neq i} z_{n,ij} v_j$ ,  $\frac{\partial \ln L_n(\gamma_*)}{\partial \beta} = -\frac{1}{\sigma_0} \sum_{i=1}^n \zeta_{1i} X_n' R_n' e_i$ ,  $\frac{\partial \ln L_n(\gamma_*)}{\partial \sigma^2} = -\frac{1}{2\sigma_\infty^2} \sum_{i=1}^n (c_v \zeta_{1i} + \zeta_{2i})$ , and  $\frac{\partial \ln L_n(\gamma_*)}{\partial \eta} = -\sum_{i=1}^n \zeta_{3i}$ . Then,

$$\begin{aligned} \mathcal{B}_{\lambda\lambda} &= \frac{1}{n} E(\zeta_{1i}^2) \sum_{i=1}^n q_{ni}^2 + \frac{2}{n} E(\zeta_{1i} \zeta_{2i}) \sum_{i=1}^n q_{ni} d_{n,ii} + \frac{1}{n} E(\zeta_{2i}^2) \sum_{i=1}^n d_{n,ii}^2 + \frac{1}{n} E(\zeta_{1i}^2) \sum_{i=1}^n \sum_{j \neq i} d_{n,ij}^2 \\ &\quad + \frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} d_{n,ij} d_{n,ji} \\ &= \frac{1}{n} E(\zeta_{1i}^2) Q_n' Q_n + \frac{2}{n} E(\zeta_{1i} \zeta_{2i}) Q_n' \text{vec}_D(D_n) \\ &\quad + \frac{1}{n} [E(\zeta_{2i}^2) - E(\zeta_{1i}^2) - 1] \text{vec}_D'(D_n) \text{vec}_D(D_n) + \frac{1}{n} E(\zeta_{1i}^2) \text{tr}(D_n' D_n) + \frac{1}{n} \text{tr}(D_n^2), \\ \mathcal{B}_{\lambda\rho} &= \frac{c_\beta}{n} E(\zeta_{1i}^2) Q_n' 1_n + \frac{1}{n} E(\zeta_{1i} \zeta_{2i}) [Q_n' \text{vec}_D(Z_n) + c_\beta \text{tr}(D_n)] \\ &\quad + \frac{1}{n} [E(\zeta_{2i}^2) - E(\zeta_{1i}^2) - 1] \text{vec}_D'(D_n) \text{vec}_D(Z_n) + \frac{1}{n} E(\zeta_{1i}^2) \text{tr}(D_n' Z_n) + \frac{1}{n} \text{tr}(D_n Z_n), \\ \mathcal{B}_{\lambda\beta} &= \frac{1}{n\sigma_0} E(\zeta_{1i}^2) Q_n' R_n X_n + \frac{1}{n\sigma_0} E(\zeta_{1i} \zeta_{2i}) \text{vec}_D'(D_n) R_n X_n, \quad \mathcal{B}_{\lambda\sigma^2} = \frac{1}{2n\sigma_\infty^2} [c_v E(\zeta_{1i}^2) + \\ &\quad E(\zeta_{1i} \zeta_{2i})] Q_n' 1_n + \frac{1}{2n\sigma_\infty^2} [c_v E(\zeta_{1i} \zeta_{2i}) + E(\zeta_{2i}^2)] \text{tr}(D_n), \quad \mathcal{B}_{\lambda\eta} = \frac{1}{n} E(\zeta_{1i} \zeta_{3i}') Q_n' 1_n \\ &\quad + \frac{1}{n} E(\zeta_{2i} \zeta_{3i}') \text{tr}(D_n), \end{aligned}$$

$$\begin{aligned}
 \mathcal{B}_{\rho\rho} &= c_\beta^2 E(\zeta_{1i}^2) + \frac{2c_\beta}{n} E(\zeta_{1i}\zeta_{2i}) \text{tr}(Z_n) + \frac{1}{n} [E(\zeta_{2i}^2) - E(\zeta_{1i}^2) - 1] \text{vec}_D'(Z_n) \text{vec}_D(Z_n) \\
 &\quad + \frac{1}{n} E(\zeta_{1i}^2) \text{tr}(Z_n' Z_n) + \frac{1}{n} \text{tr}(Z_n^2), \\
 \mathcal{B}_{\rho\beta} &= \frac{c_\beta}{n\sigma_0} E(\zeta_{1i}^2) 1_n' R_n X_n + \frac{1}{n\sigma_0} E(\zeta_{1i}\zeta_{2i}) \text{vec}_D'(Z_n) R_n X_n, \quad \mathcal{B}_{\rho\sigma^2} = \frac{c_\beta}{2\sigma_\infty^2} [c_v E(\zeta_{1i}^2) + \\
 &\quad E(\zeta_{1i}\zeta_{2i})] + \frac{1}{2n\sigma_\infty^2} [c_v E(\zeta_{1i}\zeta_{2i}) + E(\zeta_{2i}^2)] \text{tr}(Z_n), \quad \mathcal{B}_{\rho\eta} = c_\beta E(\zeta_{1i}\zeta_{3i}') + \frac{1}{n} E(\zeta_{2i}\zeta_{3i}') \text{tr}(Z_n), \\
 \mathcal{B}_{\beta\beta} &= \frac{1}{n\sigma_0^2} E(\zeta_{1i}^2) X_n' R_n' R_n X_n, \quad \mathcal{B}_{\beta\sigma^2} = \frac{1}{2n\sigma_\infty^2 \sigma_0} [c_v E(\zeta_{1i}^2) + E(\zeta_{1i}\zeta_{2i})] X_n' R_n' 1_n, \quad \mathcal{B}_{\beta\eta} = \\
 &\quad \frac{1}{n\sigma_0} X_n' R_n' 1_n E(\zeta_{1i}\zeta_{3i}'), \quad \mathcal{B}_{\sigma^2\sigma^2} = \frac{1}{4\sigma_\infty^4} [c_v^2 E(\zeta_{1i}^2) + 2c_v E(\zeta_{1i}\zeta_{2i}) + E(\zeta_{2i}^2)], \quad \mathcal{B}_{\sigma^2\eta} = \\
 &\quad \frac{1}{2\sigma_\infty^2} [c_v E(\zeta_{1i}\zeta_{3i}') + E(\zeta_{2i}\zeta_{3i}')] \text{ and } \mathcal{B}_{\eta\eta} = E(\zeta_{3i}\zeta_{3i}').
 \end{aligned}$$

For the expression of  $\mathcal{A}$ , using the explicit form of  $\frac{\partial \ln L_n(\gamma)}{\partial \gamma \partial \gamma'}$  in the Supplementary Material and the reduced form of  $Y_n$ , we have

$$\begin{aligned}
 \mathcal{A}_{\lambda\lambda} &= \frac{1}{n} E(\zeta_{4i}) Q_n' Q_n + \frac{2}{n} E(\zeta_{4i} v_i) Q_n' \text{vec}_D(D_n) \\
 &\quad + \frac{1}{n} [E(\zeta_{4i} v_i^2) - E(\zeta_{4i})] \text{vec}_D'(D_n) \text{vec}_D(D_n) + \frac{1}{n} E(\zeta_{4i}) \text{tr}(D_n' D_n) + \frac{1}{n} \text{tr}(D_n^2), \\
 \mathcal{A}_{\lambda\rho} &= \frac{c_\beta}{n} E(\zeta_{4i}) Q_n' 1_n + \frac{1}{n} E(\zeta_{4i} v_i) [Q_n' \text{vec}_D(Z_n) + c_\beta \text{tr}(D_n)] \\
 &\quad + \frac{1}{n} [E(\zeta_{4i} v_i^2) - E(\zeta_{4i})] \text{vec}_D'(D_n) \text{vec}_D(Z_n) + \frac{1}{n} E(\zeta_{4i}) \text{tr}(D_n' Z_n) + \frac{1}{n} \text{tr}(D_n Z_n), \\
 \mathcal{A}_{\lambda\beta} &= \frac{1}{n\sigma_0} E(\zeta_{4i}) Q_n' R_n X_n + \frac{1}{n\sigma_0} E(\zeta_{4i} v_i) \text{vec}_D'(D_n) R_n X_n, \\
 \mathcal{A}_{\lambda\sigma^2} &= \frac{1}{2n\sigma_\infty^2} [c_v E(\zeta_{4i}) + E(\zeta_{4i} v_i)] Q_n' 1_n + \frac{1}{2n\sigma_\infty^2} [c_v E(\zeta_{4i} v_i) + E(\zeta_{4i} v_i^2) + 1] \text{tr}(D_n), \\
 \mathcal{A}_{\lambda\eta} &= \frac{1}{n} E(\zeta_{5i}') Q_n' 1_n + \frac{1}{n} E(v_i \zeta_{5i}') \text{tr}(D_n), \\
 \mathcal{A}_{\rho\rho} &= c_\beta^2 E(\zeta_{4i}) + \frac{2c_\beta}{n} E(\zeta_{4i} v_i) \text{tr}(Z_n) \\
 &\quad + \frac{1}{n} [E(\zeta_{4i} v_i^2) - E(\zeta_{4i})] \text{vec}_D'(Z_n) \text{vec}_D(Z_n) + \frac{1}{n} E(\zeta_{4i}) \text{tr}(Z_n' Z_n) + \frac{1}{n} \text{tr}(Z_n^2), \\
 \mathcal{A}_{\rho\beta} &= \frac{c_\beta}{n\sigma_0} E(\zeta_{4i}) 1_n' R_n X_n + \frac{1}{n\sigma_0} E(\zeta_{4i} v_i) \text{vec}_D'(Z_n) R_n X_n, \quad \mathcal{A}_{\rho\sigma^2} = \frac{c_\beta}{2\sigma_\infty^2} [c_v E(\zeta_{4i}) + \\
 &\quad E(\zeta_{4i} v_i)] + \frac{1}{2n\sigma_\infty^2} [c_v E(\zeta_{4i} v_i) + E(\zeta_{4i} v_i^2) + 1] \text{tr}(Z_n), \quad \mathcal{A}_{\rho\eta} = c_\beta E(\zeta_{5i}') + \frac{1}{n} E(v_i \zeta_{5i}') \text{tr}(Z_n), \\
 \mathcal{A}_{\beta\beta} &= \frac{1}{n\sigma_0^2} E(\zeta_{4i}) X_n' R_n' R_n X_n, \quad \mathcal{A}_{\beta\sigma^2} = \frac{1}{2n\sigma_\infty^2 \sigma_0} [c_v E(\zeta_{4i}) + E(\zeta_{4i} v_i)] X_n' R_n' 1_n, \quad \mathcal{A}_{\beta\eta} = \\
 &\quad \frac{1}{n\sigma_0} X_n' R_n' 1_n E(\zeta_{5i}'), \quad \mathcal{A}_{\sigma^2\sigma^2} = \frac{1}{4\sigma_\infty^4} [c_v^2 E(\zeta_{4i}) + 2c_v E(\zeta_{4i} v_i) + E(\zeta_{4i} v_i^2) + 1], \quad \mathcal{A}_{\sigma^2\eta} = \\
 &\quad \frac{1}{2\sigma_\infty^2} [c_v E(\zeta_{5i}') + E(v_i \zeta_{5i}')], \text{ and } \mathcal{A}_{\eta\eta} = E(\zeta_{6i}).
 \end{aligned}$$

## A.2. Symmetric $v_i$

As in the last subsection, the NGPML in this case maximizes  $\ln L_n(\gamma)$  in (2). In this and the next subsections, let  $\zeta_{1i}$  to  $\zeta_{6i}$  be as defined in the last subsection except that  $v_i(\theta^*)$  is replaced by  $\frac{\sigma_0}{\sigma_\infty} v_i$ . It is shown in the proof of Theorem 2 that  $E(\zeta_{ji}) = 0$ , for  $j = 1, 2, 3$ . Then

the expressions of  $\mathcal{A}$  and  $\mathcal{B}$  are the same as those in the last subsection, except the additional restrictions  $c_v = 0$  and  $c_\beta = 0$ .

With symmetric  $v_i$ , it is shown in the proof of Corollary 1 that  $E(\zeta_{1i}\zeta_{2i}) = 0$ ,  $E(\zeta_{1i}\zeta_{3i}) = 0$ ,  $E(\zeta_{4i}v_i) = 0$ , and  $E(\zeta_{5i}) = 0$ . Then  $\mathcal{A}_{\beta\rho} = 0$ ,  $\mathcal{A}_{\beta\sigma^2} = 0$ ,  $\mathcal{A}_{\beta\eta} = 0$ ,  $\mathcal{B}_{\beta\rho} = 0$ ,  $\mathcal{B}_{\beta\sigma^2} = 0$ , and  $\mathcal{B}_{\beta\eta} = 0$ .

In the case that  $\tau_0 = 0$ ,  $D_n = W_n$  and  $Z_n = M_n$ . As  $W_n$  and  $M_n$  have zero diagonals,  $\text{vec}_D(D_n) = 0$ ,  $\text{vec}_D(Z_n) = 0$ ,  $\text{tr}(D_n) = 0$ , and  $\text{tr}(T_n) = 0$ . Then some components of  $\mathcal{A}$  and  $\mathcal{B}$  can be simplified accordingly. In particular,  $\mathcal{A}_{\rho\sigma^2} = 0$ ,  $\mathcal{A}_{\rho\eta} = 0$ ,  $\mathcal{B}_{\rho\sigma^2} = 0$ , and  $\mathcal{B}_{\rho\eta} = 0$ .

### A.3. Non-Row-Normalized $M_n$ and Asymmetric $v_i$

In this case, the NGPMLM maximizes  $\ln L_n(\delta)$  in (4). By Assumption 4(i)(c),  $E(\zeta_{ji}) = 0$ , for  $j = 1, 2, 3$ . The expressions of  $\mathcal{B}_{\delta_1\delta_2} = \frac{1}{n} E(\frac{\partial \ln L_n(\delta_\#)}{\partial \delta_1} \frac{\partial \ln L_n(\delta_\#)}{\partial \delta_2})$  and  $\mathcal{A}_{\delta_1\delta_2} = -\frac{1}{n} E(\frac{\partial^2 \ln L_n(\delta_\#)}{\partial \delta_1 \partial \delta_2})$  for  $\delta_1$  and  $\delta_2$  not containing  $\alpha$  can be derived by imposing  $c_v = -\frac{\alpha_\infty}{\sigma_0}$  and  $c_\beta = 0$  in the corresponding expressions in Appendix A.1. The remaining components of  $\mathcal{B}$  are  $\mathcal{B}_{\alpha\lambda} = \frac{1}{n\sigma_0} E(\zeta_{1i}^2 Q'_n 1_n + \frac{1}{n\sigma_0} E(\zeta_{1i}\zeta_{2i}) \text{tr}(D_n))$ ,  $\mathcal{B}_{\alpha\rho} = \frac{1}{n\sigma_0} E(\zeta_{1i}\zeta_{2i}) \text{tr}(Z_n)$ ,  $\mathcal{B}_{\alpha\beta} = \frac{1}{n\sigma_0^2} E(\zeta_{1i}^2) 1'_n R_n X_n$ ,  $\mathcal{B}_{\alpha\sigma^2} = \frac{1}{2\sigma_\infty^2 \sigma_0} [-\frac{\alpha_\infty}{\sigma_0} E(\zeta_{1i}^2) + E(\zeta_{1i}\zeta_{2i})]$ ,  $\mathcal{B}_{\alpha\alpha} = \frac{1}{\sigma_0^2} E(\zeta_{1i}^2)$ , and  $\mathcal{B}_{\alpha\eta} = \frac{1}{\sigma_0} E(\zeta_{1i}\zeta_{3i})$ . The remaining components of  $\mathcal{A}$  are  $\mathcal{A}_{\alpha\lambda} = \frac{1}{n\sigma_0} E(\zeta_{4i}) Q'_n 1_n + \frac{1}{n\sigma_0} E(\zeta_{4i}v_i) \text{tr}(D_n)$ ,  $\mathcal{A}_{\alpha\rho} = \frac{1}{n\sigma_0} E(\zeta_{4i}v_i) \text{tr}(Z_n)$ ,  $\mathcal{A}_{\alpha\beta} = \frac{1}{n\sigma_0^2} E(\zeta_{4i}) 1'_n R_n X_n$ ,  $\mathcal{A}_{\alpha\sigma^2} = \frac{1}{2\sigma_\infty^2 \sigma_0} [-\frac{\alpha_\infty}{\sigma_0} E(\zeta_{4i}) + E(\zeta_{4i}v_i)]$ ,  $\mathcal{A}_{\alpha\alpha} = \frac{1}{\sigma_0^2} E(\zeta_{4i})$ , and  $\mathcal{A}_{\alpha\eta} = \frac{1}{\sigma_0} E(\zeta'_{5i})$ .

## APPENDIX B. Lemmas

The following Lemma B.1 provides more primitive conditions for  $g_n(\tau) > 0$  at  $\tau \neq \tau_0$  in a neighborhood of  $\tau_0$ , where  $g_n(\tau)$  is in Assumption 3. The matrices  $T_{1n}$  and  $T_{2n}$  below are defined after Assumption 3.

**LEMMA B.1.** *Suppose that  $W_n = M_n$  and that  $T_{1n}$  and  $T_{2n}$  are linearly independent. If  $W_n$  is symmetric or is row-normalized from a symmetric matrix, then  $g_n(\tau) > 0$  at  $\tau \neq \tau_0$  in a neighborhood of  $\tau_0$ .*

**Proof.** As explained below Assumption 3, we need to show that  $\frac{\partial^2 g_n(\tau_0)}{\partial \tau \partial \tau'}$  is positive-definite, which requires that  $\text{tr}(T_{1n}^2) > 0$ ,  $\text{tr}(T_{2n}^2) > 0$ , and  $\text{tr}(T_{1n}^2) \text{tr}(T_{2n}^2) > \text{tr}^2(T_{1n}T_{2n})$ , by some calculation.

If  $W_n$  is symmetric, with  $W_n = M_n$ , it is obvious that  $T_{1n}$  and  $T_{2n}$  are symmetric. Then  $\text{tr}(T_{jn}^2) = \text{tr}(T'_{jn}T_{jn}) \geq 0$ , for  $j = 1, 2$ . By the Cauchy–Schwarz inequality,  $\text{tr}(T_{1n}^2) \text{tr}(T_{2n}^2) = \text{tr}(T'_{1n}T_{1n}) \text{tr}(T'_{2n}T_{2n}) \geq \text{tr}^2(T'_{1n}T_{2n}) = \text{tr}^2(T_{1n}T_{2n})$ . The inequality is strict when  $T_{1n}$  and  $T_{2n}$  are linearly independent, which also implies that  $\text{tr}(T_{jn}^2) > 0$ , for  $j = 1, 2$ .

If  $W_n$  is row-normalized from a symmetric matrix such that  $W_n = H_n A_n$ , where  $H_n = \text{diag}(1/(e'_{n1}A_n 1_n), \dots, 1/(e'_{nm}A_n 1_n))$  and  $A_n$  is symmetric, let  $B_n = H_n^{1/2} A_n H_n^{1/2}$

and  $C_n(\lambda) = I_n - \lambda B_n$ . Then  $B_n$  and  $C_n(\lambda)$  are symmetric and satisfy  $B_n C_n(\lambda) = C_n(\lambda) B_n$ . We have  $S_n(\lambda) = H_n^{1/2} C_n(\lambda) H_n^{-1/2}$ ,  $A_{1n} = H_n A_n \cdot H_n^{1/2} C_n^{-1}(\rho_0) H_n^{-1/2} = H_n^{1/2} B_n C_n^{-1}(\rho_0) H_n^{-1/2}$ , and  $A_{2n} = H_n^{1/2} C_n(\rho_0) H_n^{-1/2} \cdot H_n A_n \cdot H_n^{1/2} C_n^{-1}(\lambda_0) H_n^{-1/2}$ .  $H_n^{1/2} C_n^{-1}(\rho_0) H_n^{-1/2} = H_n^{1/2} B_n C_n^{-1}(\lambda_0) H_n^{-1/2}$ . For  $n \times n$  matrices  $E_{1n}$  and  $E_{2n}$ , if  $E_{2n}$  is diagonal, then  $\text{diag}(E_{1n} E_{2n}) = \text{diag}(E_{1n}) E_{2n}$ . Thus,  $T_{1n} = H_n^{1/2} D_{1n} H_n^{-1/2}$ ,  $T_{2n} = H_n^{1/2} D_{2n} H_n^{-1/2}$ , and  $T_{1n} T_{2n} = H_n^{1/2} D_{1n} D_{2n} H_n^{-1/2}$ , where  $D_{1n} = B_n C_n^{-1}(\rho_0) - \text{diag}(B_n C_n^{-1}(\rho_0))$  and  $D_{2n} = B_n C_n^{-1}(\lambda_0) - \text{diag}(B_n C_n^{-1}(\lambda_0))$  are symmetric. Thus,  $\text{tr}(T_{jn}^2) = \text{tr}(D_{jn}^2) = \text{tr}(D'_{jn} D_{jn}) \geq 0$ , for  $j = 1, 2$ . Furthermore,  $\text{tr}(T_{1n}^2) \text{tr}(T_{2n}^2) = \text{tr}(D_{1n}^2) \text{tr}(D_{2n}^2) = \text{tr}(D'_{1n} D_{1n}) \text{tr}(D'_{2n} D_{2n}) \geq \text{tr}^2(D'_{1n} D_{2n}) = \text{tr}^2(T_{1n} T_{2n})$  by the Cauchy–Schwarz inequality. The inequality is strict when  $D_{1n}$  and  $D_{2n}$  are linearly independent, i.e.,  $T_{1n}$  and  $T_{2n}$  are linearly independent, which also implies that  $\text{tr}(T_{jn}^2) > 0$ , for  $j = 1, 2$ .  $\square$

**LEMMA B.2.** For  $j = 1, \dots, l$ , let  $A_{jn}$  be  $n \times n$  nonstochastic matrices that are bounded in the row-sum norm, and let  $U_{jn} = [u_{jn,1}, \dots, u_{jn,n}]'$  be  $n \times 1$  vectors such that  $\sup_{i,j,n} E(|u_{jn,i}|^{a_j}) < \infty$ , for  $a_j > 1$ . Then  $\sup_{i,n} E[(\prod_{j=1}^l |e'_{ni} A_{jn} U_{jn}|)^{1/\sum_{j=1}^l \frac{1}{a_j}}] < \infty$ .

**Proof.** This is a special case of Lemma 1(ii) in Jin and Lee (2019).  $\square$

**LEMMA B.3.** Suppose that  $h(x)$  is a scalar function,  $v_i$ 's in  $V_n = [v_1, \dots, v_n]'$  are i.i.d. with mean zero and variance  $\sigma_0^2$ ,  $A_n = [a_{ni}]$  and  $B_n = [b_{ni}]$  are  $n \times n$  nonstochastic matrices that are bounded in both the row- and column-sum norms, and  $E(|v_i|^{c_v}) < \infty$  and  $E(|h(v_i)|^{c_h}) < \infty$ , for some  $c_v > 0$  and  $c_h > 0$ . Then  $c_{1n} - E(c_{1n}) = o_p(1)$  if  $\frac{1}{c_h} + \frac{2}{c_v} < 1$ , and  $c_{2n} - E(c_{2n}) = o_p(1)$  if  $\frac{1}{c_h} + \frac{1}{c_v} < 1$ , where  $c_{1n} = \frac{1}{n} \sum_{i=1}^n h(v_i) (\sum_{j=1}^n a_{ni} v_j v_j) (\sum_{k=1}^n b_{ni} v_k v_k)$  and  $c_{2n} = \frac{1}{n} \sum_{i=1}^n h(v_i) (\sum_{j=1}^n a_{ni} v_j v_j)$ .

**Proof.** This lemma is proved by an LLN for martingale differences. The details are in the Supplementary Material.  $\square$

**LEMMA B.4.** Suppose that  $A_n = [a_{ni}]$  is an  $n \times n$  nonstochastic matrix that is bounded in both the row- and column-sum norms;  $b_n = [b_{ni}]$  is an  $n \times 1$  vector of uniformly bounded constants;  $\epsilon_n = [\epsilon_{ni}]$ ,  $V_n = [v_{ni}]$ , and  $\Psi_n = [\psi_{ni}]$  are  $n \times 1$  random vectors with mean zero;  $[\epsilon_{ni}, v_{ni}, \psi_{ni}]$ , for  $i = 1, \dots, n$ , are independent; and  $\sup_{i,n} E(|\epsilon_{ni} v_{ni}|^{2+\iota}) + \sup_{i,n} E(|\epsilon_{ni}|^{2+\iota}) + \sup_{i,n} E(|v_{ni}|^{2+\iota}) + \sup_{i,n} E(|\psi_{ni}|^{2+\iota}) < \infty$ , for some  $\iota > 0$ . Let  $\omega_n = \epsilon'_n A_n V_n + b'_n \Psi_n - E(\epsilon'_n A_n V_n)$  and  $\sigma_{\omega_n}^2 = \text{var}(\omega_n)$ . If  $\inf_n \frac{1}{n} \sigma_{\omega_n}^2 > 0$ , then  $\frac{\omega_n}{\sigma_{\omega_n}} \xrightarrow{d} N(0, 1)$ .

**Proof.** This lemma is a special case of Lemma 6 in Yang and Lee (2017).  $\square$

**LEMMA B.5.** Suppose that Assumption 1 holds. Let each of  $A_n = [a_{ni}]$  and  $B_n = [b_{ni}]$  be one of the matrices  $W_n$ ,  $M_n$ ,  $R_n$ , and  $S_n$ . Denote  $C_n = A_n B_n = [c_{ni}]$ . If  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |a_{ni}| = 0$ ,  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |b_{ni}| = 0$ , and  $\sup_n \|A_n\|_\infty + \sup_n \|B_n\|_\infty < \infty$ , then  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |c_{ni}| = 0$ .

**Proof.** As  $c_{n,ij} = \sum_{k=1}^n a_{n,ik} b_{n,kj}$ ,

$$\begin{aligned} \sup_{i,n} \sum_{j: d(i,j) > r} |c_{n,ij}| &\leq \sup_{i,n} \sum_{j: d(i,j) > r} \sum_{k: d(j,k) > r/2} |a_{n,ik} b_{n,kj}| \\ &\quad + \sup_{i,n} \sum_{j: d(i,j) > r} \sum_{k: d(j,k) \leq r/2} |a_{n,ik} b_{n,kj}| \\ &\leq \sup_{i,n} \sum_{k=1}^n |a_{n,ik}| \sum_{j: d(j,k) > r/2} |b_{n,kj}| + \sup_{i,n} \sum_{k: d(i,k) > r/2} |a_{n,ik}| \sum_{j=1}^n |b_{n,kj}| \\ &\leq \sup_n \|A_n\|_\infty \sup_{k,n} \sum_{j: d(j,k) > r/2} |b_{n,kj}| + \sup_{i,n} \sum_{k: d(i,k) > r/2} |a_{n,ik}| \cdot \sup_n \|B_n\|_\infty, \end{aligned}$$

where the second inequality holds because  $d(i,j) > r$  and  $d(j,k) \leq r/2$  imply that  $d(i,k) > r/2$ . Thus,  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |c_{n,ij}| = 0$ .  $\square$

For any matrix  $A = [a_{ij}]$ , denote  $\text{abs}(A) = [|a_{ij}|]$ .

**LEMMA B.6.** (i) If Assumptions 1 and 2(iii) hold, then  $\sup_n \|S_n^{-1}\|_\infty < \infty$  and  $\sup_n \|R_n^{-1}\|_\infty < \infty$ . (ii) If Assumptions 1, 2(ii), and 7 hold, then  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |w_{n,ij}| = 0$ ,  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |m_{n,ij}| = 0$ ,  $\sup_n \|W_n\|_1 < \infty$ , and  $\sup_n \|M_n\|_1 < \infty$ . (iii) If Assumptions 1, 2(ii) and (iii), and 7(ii) hold, then  $\sup_n \|S_n^{-1}\|_1 < \infty$  and  $\sup_n \|R_n^{-1}\|_1 < \infty$ .

**Proof.** (i) As  $\|\lambda_0 W_n\|_\infty \leq c_0 < 1$ ,  $S_n^{-1} = \sum_{k=0}^\infty (\lambda_0 W_n)^k$ . Thus, by the triangle inequality,  $\sup_n \|S_n^{-1}\|_\infty \leq \sup_n \sum_{k=0}^\infty (\|\lambda_0 W_n\|_\infty)^k \leq \sum_{k=0}^\infty c_0^k = \frac{1}{1-c_0} < \infty$ . Similarly,  $\sup_n \|R_n^{-1}\|_\infty < \infty$ .

(ii) Under Assumption 7(i),  $w_{n,ij} = 0$  if  $d(i,j) > \bar{d}_0$ . Then  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |w_{n,ij}| = 0$ . By Lemma A.1 in Jenish and Prucha (2009),  $|\{j: k \leq d(i,j) < k+1\}| \leq ck^{cd-1}$ , for  $k \geq 1$ , and some constant  $c > 0$ , where  $|A|$  for a set  $A$  denotes its cardinality. Then  $\sup_n \|W_n\|_1 = \sup_{j,n} \sum_{i: d(i,j) \leq \bar{d}_0} |w_{n,ij}| \leq \sup_{j,n} \sum_{k=1}^{[\bar{d}_0]+1} \sum_{i: k \leq d(i,j) < k+1} c w = cc_w \sum_{k=1}^{[\bar{d}_0]+1} k^{cd-1} < \infty$ , where  $c_w = \sup_n \|W_n\|_\infty < \infty$  under Assumption 2(ii) and  $[\bar{d}_0]$  is the smallest integer that is nongreater than  $\bar{d}_0$ .

Under Assumption 7(ii),  $\sup_{i,n} \sum_{j: d(i,j) > r} |w_{n,ij}| \leq \sup_{i,n} \sum_{k=[r]}^\infty \sum_{j: k \leq d(i,j) < k+1} |w_{n,ij}| \leq \sup_{i,n} \sum_{k=[r]}^\infty \sum_{j: k \leq d(i,j) < k+1} \pi_1 k^{-\pi_2} \leq \sum_{k=[r]}^\infty c \pi_1 k^{cd-\pi_2-1}$ . As  $\pi_2 > cd$ ,  $\sum_{k=1}^\infty c \pi_1 k^{cd-\pi_2-1} < \infty$ . Then  $\lim_{r \rightarrow \infty} \sum_{k=[r]}^\infty c \pi_1 k^{cd-\pi_2-1} = 0$ . It follows that  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |w_{n,ij}| = 0$ . Similarly,  $\sup_n \|W_n\|_1 = \sup_{j,n} \sum_{i: d(i,j) \geq 1} |w_{n,ij}| \leq \sum_{k=1}^\infty c \pi_1 k^{cd-\pi_2-1} < \infty$ .

The results on  $M_n$  can be similarly proved.

(iii) Under the maintained assumptions, we have  $\|\lambda_0^l [\text{abs}(W_n)]^l\|_1 \leq \max\{lN, 1\} \omega c_0^{l-1}$ , where  $\omega = |\lambda_0| \sup_n \|W_n\|_1 < \infty$ , as in the proof of Lemma 1 of Xu and Lee (2015). The only difference is that our upper bound  $\max\{lN, 1\} \omega c_0^{l-1}$  has  $c_0^{l-1}$  instead of  $\zeta^{l-1}$ , where  $\zeta$  is the upper bound of the compact parameter space of  $\lambda$ . Since we have a linear SAR process, there is no need to introduce  $\zeta$  and the proof is similar. Then

$\sup_n \|S_n^{-1}\|_1 \leq \sup_n \sum_{k=0}^{\infty} (\|\lambda_0 W_n\|_1)^k \leq c(1 + \sum_{k=1}^{\infty} k c_0^{k-1}) < \infty$  for some constant  $c$ . Similarly,  $\sup_n \|R_n^{-1}\|_1 < \infty$ .  $\square$

LEMMA B.7. Under Assumptions 1, 2(i)–(iii), and 7,  $\{e'_{ni} A_n V_n\}$  is  $L_2$ -NED on  $\{v_1, \dots, v_n\}$ , where  $A_n$  is either  $S_n^{-1} R_n^{-1}$ ,  $W_n S_n^{-1} R_n^{-1}$ ,  $M_n S_n^{-1} R_n^{-1}$ , or  $W_n M_n S_n^{-1} R_n^{-1}$ .

**Proof.** As  $\|\lambda_0 W_n\|_{\infty} \leq c_0 < 1$ ,  $S_n^{-1} = \sum_{k=0}^{\infty} (\lambda_0 W_n)^k$ . Then  $\text{abs}(S_n^{-1}) \leq^* \sum_{k=0}^{\infty} [\text{abs}(\lambda_0 W_n)]^k \leq^* [I_n - \text{abs}(\lambda_0 W_n)]^{-1}$ , where  $A_n \leq^* B_n$  for two  $n \times n$  matrices  $A_n = [a_{n,ij}]$  and  $B_n = [b_{n,ij}]$  means that  $a_{n,ij} \leq b_{n,ij}$  for any  $i, j$ . Since the proof of Proposition 1 in Xu and Lee (2015, p. 274) shows that  $[I_n - \text{abs}(\lambda_0 W_n)]^{-1}$  satisfies  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |e'_{ni} [I_n - \text{abs}(\lambda_0 W_n)]^{-1} e_{nj}| = 0$  under Assumptions 1, 2(iii), and 7, we also have  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |e'_{ni} S_n^{-1} e_{nj}| = 0$ . Similarly,  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |e'_{ni} R_n^{-1} e_{nj}| = 0$ . By Lemma B.6,  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |w_{n,ij}| = 0$ ,  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} |m_{n,ij}| = 0$ ,  $\sup_n \|S_n^{-1}\|_{\infty} < \infty$ , and  $\sup_n \|R_n^{-1}\|_{\infty} < \infty$ . Thus, by Lemma B.5,  $\lim_{r \rightarrow \infty} \sup_{i,n} \sum_{j: d(i,j) > r} [\text{abs}(A_n)]_{ij} = 0$ , where  $A_n$  is either  $S_n^{-1} R_n^{-1}$ ,  $W_n S_n^{-1} R_n^{-1}$ ,  $M_n S_n^{-1} R_n^{-1}$ , or  $W_n M_n S_n^{-1} R_n^{-1}$ . Hence, by Proposition 1 in Jenish and Prucha (2012),  $\{e'_{ni} A_n V_n\}$  is  $L_2$ -NED on  $\{v_1, \dots, v_n\}$ .  $\square$

## APPENDIX C. Proofs

For the following proofs of Propositions 1 and 2, denote  $\Psi_{ni}(\theta) = \sigma_0 e'_{ni} T_n(\tau) V_n - \sigma_0 v_i t_{n,ii}(\tau) + e'_{ni} R_n(\rho) [S_n(\lambda) S_n^{-1} X_n \beta_0 - X_n \beta]$ , which does not depend on  $v_i$ . As  $Y_n = S_n^{-1} (X_n \beta_0 + \sigma_0 R_n^{-1} V_n)$ ,  $v_i(\theta) = \frac{1}{\sigma} \Psi_{ni}(\theta) + \frac{1}{\sigma} \sigma_0 v_i t_{n,ii}(\tau)$ .

**Proof of Proposition 1.** (i) We first prove the result under Assumption 4(i). As  $T_n(\tau) = I_n + (\rho_0 - \rho) A_{1n} + (\lambda_0 - \lambda) A_{2n} + (\rho_0 - \rho)(\lambda_0 - \lambda) A_{3n}$ , under Assumption 3(iii),  $t_{n,ii}(\tau) \neq 0$  for any  $i$  and  $\tau$ . Since  $M_n$  is row-normalized,  $R_n 1_n = (1 - \rho_0) 1_n$ . Then the nonsingularity of  $R_n$  implies that  $\rho_0 \neq 1$ . Denote  $\mathcal{Q}(\sigma, \beta_1, \eta) = E[\ln f(\frac{\sigma_0 v_i - (1 - \rho_0)(\beta_1 - \beta_{10})}{\sigma}, \eta)] - \frac{1}{2} \ln(\sigma^2)$ ,  $\sigma_{ni} = \frac{\sigma}{t_{n,ii}(\tau)}$ , and  $\beta_{1,ni} = \beta_{10} - \frac{1}{(1 - \rho_0)t_{n,ii}(\tau)} \Psi_{ni}(\theta)$ . Since  $E[\ln f(\frac{\sigma_0 v_i - \alpha}{\sigma}, \eta)] - \frac{1}{2} \ln(\sigma^2)$  is uniquely maximized at  $(\sigma_{\infty}, \alpha_{\infty}, \eta_{\infty})$ ,  $\mathcal{Q}(\sigma, \beta_1, \eta)$  is uniquely maximized at  $(\sigma_{\infty}, \beta_{1\infty}, \eta_{\infty})$ , where  $\beta_{1\infty} = \beta_{10} + \frac{\alpha_{\infty}}{1 - \rho_0}$ . Let  $E_{-i}(\cdot)$  be the conditional expectation given  $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$ . Then,

$$\begin{aligned} E[\ln L_n(\gamma)] &= \sum_{i=1}^n E\{E_{-i}[\ln f(v_i(\theta), \eta)]\} - \frac{n}{2} \ln(\sigma^2) + \ln |S_n(\lambda)| + \ln |R_n(\rho)| \\ &= \sum_{i=1}^n E[\mathcal{Q}(\sigma_{ni}, \beta_{1,ni}, \eta)] - \sum_{i=1}^n \ln |t_{n,ii}(\tau)| + \ln |S_n(\lambda)| + \ln |R_n(\rho)| \\ &\leq n \mathcal{Q}(\sigma_{\infty}, \beta_{1\infty}, \eta_{\infty}) - \sum_{i=1}^n \ln |t_{n,ii}(\tau)| + \ln |S_n(\lambda)| + \ln |R_n(\rho)| \end{aligned} \quad (\text{C.1})$$

$$\begin{aligned} &= n \mathcal{Q}(\sigma_{\infty}, \beta_{1\infty}, \eta_{\infty}) - \sum_{i=1}^n \ln |t_{n,ii}(\tau)| + \ln |T_n(\tau)| + \ln |S_n| + \ln |R_n| \\ &\leq E[\ln L_n(\gamma_*)], \end{aligned} \quad (\text{C.2})$$

where (C.1) uses the property that  $\mathcal{Q}(\sigma, \beta_1, \eta)$  is uniquely maximized at  $(\sigma_\infty, \beta_{1\infty}, \eta_\infty)$  and (C.2) uses the assumption that  $\ln|T_n(\tau)| \leq \sum_{i=1}^n \ln|t_{n,ii}(\tau)|$ . The inequality in (C.2) is strict if  $\tau \neq \tau_0$ . With  $\tau = \tau_0$ , we have  $T_n(\tau) = I_n$ ,  $t_{n,ii}(\tau) = 1$ ,  $\sigma_{ni} = \sigma$ , and  $\beta_{1,ni} = \beta_{10} - \frac{1}{1-\rho_0} e'_{ni} R_n X_n (\beta_0 - \beta) = \beta_1 - \frac{1}{1-\rho_0} e'_{ni} R_n X_{2n} (\beta_{20} - \beta_2)$ . Since  $R_n X_n$  has full column rank,  $\beta_{1,ni} \neq \beta_{1\infty}$  for some  $i$  if  $\beta_2 \neq \beta_{20}$ . Thus, with  $\tau = \tau_0$ , the inequality in (C.1) is strict if  $(\beta_2, \sigma, \eta) \neq (\beta_{20}, \sigma_\infty, \eta_\infty)$ . It follows that  $E[\ln L_n(\gamma)]$  is uniquely maximized at  $\gamma = \gamma_*$ . (ii) We next prove the result under Assumption 4(ii). Because  $v_i$ 's are symmetrically distributed around zero with unimodal density, by Lemma A in Newey and Steigerwald (1997),  $E[\ln f(v_i(\theta), \eta)] = E[E_{-i}[\ln f(v_i(\theta), \eta)]] \leq E[E_{-i}[\ln f(\frac{\sigma_0}{\sigma} v_{i,ii}(\tau), \eta)]] = E[\ln f(\frac{\sigma_0}{\sigma} v_{i,ii}(\tau), \eta)]$ , where the inequality is strict if  $\Psi_{ni}(\theta) \neq 0$ . Denote  $\mathcal{Q}(\sigma, \eta) = E[\ln f(\frac{\sigma_0 v_i}{\sigma}, \eta)] - \frac{1}{2} \ln(\sigma^2)$ . Then,

$$E[\ln L_n(\gamma)] \leq \sum_{i=1}^n \mathcal{Q}(\sigma_{ni}, \eta) - \sum_{i=1}^n \ln|t_{n,ii}(\tau)| + \ln|S_n(\lambda)| + \ln|R_n(\rho)| \quad (\text{C.3})$$

$$\leq n\mathcal{Q}(\sigma_\infty, \eta_\infty) - \sum_{i=1}^n \ln|t_{n,ii}(\tau)| + \ln|S_n(\lambda)| + \ln|R_n(\rho)| \quad (\text{C.4})$$

$$\leq E[\ln L_n(\gamma_\#)], \quad (\text{C.5})$$

where (C.4) uses the assumption that  $\mathcal{Q}(\sigma, \eta)$  is uniquely maximized at  $(\sigma_\infty, \eta_\infty)$ , and (C.5) uses the assumption that  $\ln|T_n(\tau)| \leq \sum_{i=1}^n \ln|t_{n,ii}(\tau)|$  as in the proof for (i) above. Furthermore, the inequality in (C.5) is strict if  $\tau \neq \tau_0$ . With  $\tau = \tau_0$ , the inequality in (C.4) is strict if  $(\sigma, \eta) \neq (\sigma_\infty, \eta_\infty)$ . With  $(\tau, \sigma, \eta) = (\tau_0, \sigma_\infty, \eta_\infty)$ , we have  $T_n(\tau) = I_n$  and  $\Psi_{ni}(\theta) = e'_{ni} R_n X_n (\beta_0 - \beta)$ . Since  $R_n X_n$  has full column rank, with  $(\tau, \sigma, \eta) = (\tau_0, \sigma_\infty, \eta_\infty)$ , the inequality in (C.3) is strict if  $\beta \neq \beta_0$ . Hence,  $E[\ln L_n(\gamma)]$  is uniquely maximized at  $\gamma_\#$ .  $\square$

**Proof of Proposition 2.** Denote  $\mathcal{Q}(\sigma, \alpha, \eta) = E[\ln f(\frac{\sigma_0 v_i - \alpha}{\sigma}, \eta)] - \frac{1}{2} \ln(\sigma^2)$ ,  $\sigma_{ni} = \frac{\sigma}{t_{n,ii}(\tau)}$ , and  $\alpha_{ni} = \frac{\alpha - \Psi_{ni}(\theta)}{t_{n,ii}(\tau)}$ . Then,

$$\begin{aligned} E[\ln L_n(\delta)] &= \sum_{i=1}^n E\left\{E_{-i}\left[\ln f\left(v_i(\theta) - \frac{\alpha}{\sigma}, \eta\right)\right]\right\} - \frac{n}{2} \ln(\sigma^2) + \ln|S_n(\lambda)| + \ln|R_n(\rho)| \\ &= \sum_{i=1}^n E[\mathcal{Q}(\sigma_{ni}, \alpha_{ni}, \eta)] - \sum_{i=1}^n \ln|t_{n,ii}(\tau)| + \ln|S_n(\lambda)| + \ln|R_n(\rho)| \\ &\leq n\mathcal{Q}(\sigma_\infty, \alpha_\infty, \eta_\infty) - \sum_{i=1}^n \ln|t_{n,ii}(\tau)| + \ln|S_n(\lambda)| + \ln|R_n(\rho)| \end{aligned} \quad (\text{C.6})$$

$$\begin{aligned} &= n\mathcal{Q}(\sigma_\infty, \alpha_\infty, \eta_\infty) - \sum_{i=1}^n \ln|t_{n,ii}(\tau)| + \ln|T_n(\tau)| + \ln|S_n| + \ln|R_n| \\ &\leq E[\ln L_n(\delta_\#)], \end{aligned} \quad (\text{C.7})$$

where (C.6) uses the property that  $\mathcal{Q}(\sigma, \alpha, \eta)$  is uniquely maximized at  $(\sigma_\infty, \alpha_\infty, \eta_\infty)$  and (C.7) uses the assumption that  $\ln|T_n(\tau)| \leq \sum_{i=1}^n \ln|t_{n,ii}(\tau)|$ . The inequality in (C.7) is strict if  $\tau \neq \tau_0$ . With  $\tau = \tau_0$ , we have  $T_n(\tau) = I_n$ ,  $t_{n,ii}(\tau) = 1$ ,  $\sigma_{ni} = \sigma$ , and  $\alpha_{ni} = \alpha - e'_{ni} R_n X_n (\beta_0 - \beta)$ . Since  $R_n X_n$  has full column rank and does not contain an intercept term,  $\alpha_{ni} \neq \alpha_\infty$  for some  $i$  if  $\beta \neq \beta_0$ . Thus, with  $\tau = \tau_0$ , the inequality in (C.6) is strict

if  $(\beta, \sigma, \alpha, \eta) \neq (\beta_0, \sigma_\infty, \alpha_\infty, \eta_\infty)$ . It follows that  $E[\ln L_n(\delta)]$  is uniquely maximized at  $\delta = \delta_\#$ .  $\square$

**Proof of Theorem 1.** We only prove the convergence of  $\hat{\gamma}$  in the case with symmetric  $v_i$ , since the proofs for other cases are similar. As  $Y_n = S_n^{-1}(X_n\beta_0 + \sigma_0 R_n^{-1}V_n)$ ,  $R_n(\rho) = R_n + (\rho_0 - \rho)M_n$ , and  $S_n(\lambda) = S_n + (\lambda_0 - \lambda)W_n$ , we have

$$\begin{aligned} R_n(\rho)[S_n(\lambda)Y_n - X_n\beta] &= \sigma_0 V_n + (\lambda_0 - \lambda)R_n W_n S_n^{-1} X_n \beta_0 + \sigma_0 (\lambda_0 - \lambda)R_n W_n S_n^{-1} R_n^{-1} V_n + R_n X_n (\beta_0 - \beta) \\ &\quad + \sigma_0 (\rho_0 - \rho)M_n R_n^{-1} V_n + (\rho_0 - \rho)(\lambda_0 - \lambda)M_n W_n S_n^{-1} X_n \beta_0 \\ &\quad + \sigma_0 (\rho_0 - \rho)(\lambda_0 - \lambda)M_n W_n S_n^{-1} R_n^{-1} V_n + (\rho_0 - \rho)M_n X_n (\beta_0 - \beta). \end{aligned}$$

Under Assumption 2(iii), by Lemma B.6,  $R_n^{-1}$  and  $S_n^{-1}$  are bounded in the row-sum norm. As  $W_n$  and  $M_n$  are also bounded in the row-sum norm, so are the products of  $W_n, M_n, R_n^{-1}$ , and  $S_n^{-1}$ . With  $\sup_i E(|v_i|^{2+2c_t+t}) < \infty$  in Assumption 8(ii),  $v_i(\theta) = \frac{1}{\sigma} e'_{ni} R_n(\rho) [S_n(\lambda)Y_n - X_n\beta]$  is uniformly  $L_{(2+2c_t+t)}$  bounded by Lemma B.2. Furthermore, by Lemma B.7,  $\{v_i(\theta)\}$  is  $L_2$ -NED on  $\{v_1, \dots, v_n\}$ . With  $|\frac{\partial \ln f(x, \eta)}{\partial x}| \leq c_f(|x|^{c_t} + 1)$  for  $c_t = 0$  in Assumption 8(i), i.e.,  $\frac{\partial \ln f(x, \eta)}{\partial x}$  is bounded, by Proposition 2 of Jenish and Prucha (2012),  $\ln f(v_i(\theta), \eta)$  is  $L_2$ -NED on  $\{v_1, \dots, v_n\}$ ; on the other hand, with  $|\frac{\partial \ln f(x, \eta)}{\partial x}| \leq c_f(|x|^{c_t} + 1)$  for  $c_t = 1$  in Assumption 8(i), by Lemma A.4 in Xu and Lee (2015),  $\ln f(v_i(\theta), \eta)$  is uniformly  $L_2$ -NED on  $\{v_1, \dots, v_n\}$ . By the mean value theorem,  $\ln f(v_i(\theta), \eta) = \ln f(0, \eta) + \frac{\partial \ln f(c v_i(\theta), \eta)}{\partial v} v_i(\theta)$ , where  $c$  is some constant between 0 and 1. Thus, with  $\sup_i E(|v_i|^{2+2c_t+t}) < \infty$ ,  $\ln f(v_i(\theta), \eta)$  is uniformly  $L_2$  bounded by Lemma B.2. It follows by the LLN in Theorem 1 of Jenish and Prucha (2012) that  $\frac{1}{n} \ln L_n(\gamma) - \frac{1}{n} E[\ln L_n(\gamma)] = o_p(1)$ .

We next prove that  $\frac{1}{n} \ln L_n(\gamma)$  is stochastically equicontinuous (SE) and  $\frac{1}{n} E[\ln L_n(\gamma)]$  is equicontinuous. With  $|\frac{\partial \ln f(x, \eta)}{\partial x}| \leq c_f(|x|^{c_t} + 1)$ ,

$$\frac{1}{n} E \left| \frac{\partial \ln L_n(\gamma)}{\partial \lambda} \right| \leq \frac{c_f}{n\sigma} E \sum_{i=1}^n [|v_i(\theta)|^{c_t} + 1] \cdot |e'_{ni} R_n(\rho) W_n Y_n| + \frac{1}{n} |\text{tr}[W_n S_n^{-1}(\lambda)]|, \quad (\text{C.8})$$

where  $\frac{c_f}{n\sigma} E \sum_{i=1}^n [|v_i(\theta)|^{c_t} + 1] \cdot |e'_{ni} R_n(\rho) W_n Y_n| = O(1)$  by  $Y_n = S_n^{-1}(X_n\beta_0 + \sigma_0 R_n^{-1}V_n)$  and Lemma B.2, and  $\frac{1}{n} |\text{tr}[W_n S_n^{-1}(\lambda)]| = O(1)$  since  $\sup_n \|W_n\|_\infty < \infty$  by Assumption 2(ii),  $\sup_n \|W_n\|_1 < \infty$  by Lemma B.6 and  $S_n^{-1}(\lambda)$  is bounded in either the row- or column-sum norm. Thus,  $\frac{1}{n} E \left| \frac{\partial \ln L_n(\gamma)}{\partial \lambda} \right| = O(1)$  and  $\frac{1}{n} \frac{\partial \ln L_n(\gamma)}{\partial \lambda} = O_p(1)$ . As  $\sigma v_i(\theta)$  is linear in every element of  $\theta$  and the parameter space of  $\gamma$  is compact, by (C.8),  $E \sup_{\gamma \in \Gamma} \left| \frac{1}{n} \frac{\partial \ln L_n(\gamma)}{\partial \lambda} \right| = O(1)$  and  $\sup_{\gamma \in \Gamma} \left| \frac{1}{n} \frac{\partial \ln L_n(\gamma)}{\partial \lambda} \right| = O_p(1)$ . Similarly, for other elements  $\gamma_j$  of  $\gamma$ ,  $E \sup_{\gamma \in \Gamma} \left| \frac{1}{n} \frac{\partial \ln L_n(\gamma)}{\partial \gamma_j} \right| = O(1)$  and  $\sup_{\gamma \in \Gamma} \left| \frac{1}{n} \frac{\partial \ln L_n(\gamma)}{\partial \gamma_j} \right| = O_p(1)$ . Hence,  $E \sup_{\gamma \in \Gamma} \left\| \frac{1}{n} \frac{\partial \ln L_n(\gamma)}{\partial \gamma} \right\| = O(1)$  and  $\sup_{\gamma \in \Gamma} \left\| \frac{1}{n} \frac{\partial \ln L_n(\gamma)}{\partial \gamma} \right\| = O_p(1)$ . By Lemma 3.6 in Newey and McFadden (1994),  $E \sup_{\gamma \in \Gamma} \left\| \frac{1}{n} \frac{\partial \ln L_n(\gamma)}{\partial \gamma} \right\| = O(1)$  implies that  $\frac{1}{n} \frac{\partial E[\ln L_n(\gamma)]}{\partial \gamma} = \frac{1}{n} E \left( \frac{\partial \ln L_n(\gamma)}{\partial \gamma} \right)$ . Therefore, by the mean value theorem and Theorem 21.10 in Davidson (1994),  $\frac{1}{n} \ln L_n(\gamma)$  is SE, and  $\frac{1}{n} E[\ln L_n(\gamma)]$  is equicontinuous.

The pointwise convergence  $\frac{1}{n} \ln L_n(\gamma) - \frac{1}{n} E[\ln L_n(\gamma)] = o_p(1)$  and the SE of  $\frac{1}{n} \ln L_n(\gamma)$  imply that  $\sup_{\gamma \in \Gamma} \left| \frac{1}{n} \ln L_n(\gamma) - \frac{1}{n} E[\ln L_n(\gamma)] \right| = o_p(1)$ . As  $\frac{1}{n} E[\ln L_n(\gamma)]$  is equicontinuous

and  $\lim_{n \rightarrow \infty} \frac{1}{n} E[\ln L_n(\gamma)]$  is uniquely maximized at  $\gamma = \gamma_{\#}$ , we have  $\hat{\gamma} = \gamma_{\#} + o_p(1)$  (White, 1994, Theorem 3.4).  $\square$

**Proof of Theorem 2.** We only prove the asymptotic distribution of  $\hat{\gamma}$  in the case with symmetric  $v_i$ , and omit similar proofs for other cases. By the mean value theorem,  $0 = \frac{\partial \ln L_n(\hat{\gamma})}{\partial \gamma} = \frac{\partial \ln L_n(\gamma_{\#})}{\partial \gamma} + \frac{\partial^2 \ln L_n(\tilde{\gamma})}{\partial \gamma \partial \gamma'} (\hat{\gamma} - \gamma_{\#})$ , where  $\tilde{\gamma}$  lies between  $\hat{\gamma}$  and  $\gamma_{\#}$ . Then,

$$\sqrt{n}(\hat{\gamma} - \gamma_{\#}) = - \left( \frac{1}{n} \frac{\partial^2 \ln L_n(\tilde{\gamma})}{\partial \gamma \partial \gamma'} \right)^{-1} \frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_{\#})}{\partial \gamma}. \quad (\text{C.9})$$

We prove that (i)  $\frac{1}{n} \frac{\partial^2 \ln L_n(\tilde{\gamma})}{\partial \gamma \partial \gamma'} = \frac{1}{n} \frac{\partial^2 \ln L_n(\gamma_{\#})}{\partial \gamma \partial \gamma'} + o_p(1)$  and (ii)  $\frac{1}{n} \frac{\partial^2 \ln L_n(\gamma_{\#})}{\partial \gamma \partial \gamma'} = \frac{1}{n} E \left( \frac{\partial^2 \ln L_n(\gamma_{\#})}{\partial \gamma \partial \gamma'} \right) + o_p(1)$  so that  $\frac{1}{n} \frac{\partial^2 \ln L_n(\tilde{\gamma})}{\partial \gamma \partial \gamma'} = \frac{1}{n} E \left( \frac{\partial^2 \ln L_n(\gamma_{\#})}{\partial \gamma \partial \gamma'} \right) + o_p(1)$ .

For (i), we prove that every element of  $\frac{1}{n} \frac{\partial^2 \ln L_n(\gamma)}{\partial \gamma \partial \gamma'}$  is SE under Assumption 9(ii) and (iii). With  $\left\| \frac{\partial^3 \ln f(v_i(\theta), \eta)}{\partial z \partial z' \partial z_i} \right\| \leq c_f(|v_i(\theta)|^{3c_t} + 1)$  in Assumption 9(ii), we could show that  $\sup_{\gamma \in \Gamma} \left\| \frac{\partial^3 \ln L_n(\gamma)}{\partial \gamma \partial \gamma' \partial \gamma_j} \right\| = O_p(1)$ , where  $\gamma_j$  is the  $j$ th element of  $\gamma$ . As an example, consider

$$\frac{\partial^3 \ln L_n(\gamma)}{\partial \lambda^3} = - \frac{1}{\sigma^3} \sum_{i=1}^n \frac{\partial^3 \ln f(v_i(\theta), \eta)}{\partial v^3} [e'_{ni} R_n(\rho) W_n Y_n]^3 - 2 \text{tr}\{[W_n S_n^{-1}(\lambda)]^3\},$$

where  $\left| \frac{\partial^3 \ln f(v_i(\theta), \eta)}{\partial v^3} \right| \leq c[|v_i(\theta)|^{3c_t} + 1]$ . With the reduced form  $Y_n = S_n^{-1}(X_n \beta_0 + \sigma_0 R_n^{-1} V_n)$  and  $E(|v_i|^{3+3c_t}) < \infty$ ,  $\frac{1}{n} \frac{\partial^3 \ln L_n(\gamma)}{\partial \lambda^3} = O_p(1)$  by Lemma B.2. As  $v_i(\theta) = \frac{1}{\sigma} e'_{ni} R_n(\rho) [S_n(\lambda) Y_n - X_n \beta]$  is linear in each element of  $[\lambda, \rho, \beta']'$ ,  $\{S_n^{-1}(\lambda)\}$  is bounded in either the row-sum or column-sum norm uniformly on the parameter space of  $\lambda$  and  $\Gamma$  is compact,  $\sup_{\gamma \in \Gamma} \left| \frac{1}{n} \frac{\partial^3 \ln L_n(\gamma)}{\partial \lambda^3} \right| = O_p(1)$ . Hence, (i) holds by the mean value theorem.

We prove (ii) by Lemma B.3. As an example, consider

$$\frac{1}{n} \frac{\partial^2 \ln L_n(\gamma_{\#})}{\partial \lambda^2} = \frac{1}{n \sigma^2 \infty} \sum_{i=1}^n \frac{\partial^2 \ln f(\frac{\sigma_0}{\sigma \infty} v_i, \eta_{\infty})}{\partial v^2} (e'_{ni} R_n W_n Y_n)^2 - \frac{1}{n} \text{tr}[(W_n S_n^{-1})^2].$$

Under Assumption 9(ii),  $\frac{\partial^2 \ln f(\frac{\sigma_0}{\sigma \infty} v_i, \eta_{\infty})}{\partial v^2}$  is either bounded or  $\left| \frac{\partial^2 \ln f(\frac{\sigma_0}{\sigma \infty} v_i, \eta_{\infty})}{\partial v^2} \right| \leq c_f(\frac{\sigma_0^2}{\sigma \infty^2} |v_i|^2 + 1)$ . In the latter case, as  $\sup_i E(|v_i|^{4+t}) < \infty$ ,  $E\left[\left| \frac{\partial^2 \ln f(\frac{\sigma_0}{\sigma \infty} v_i, \eta_{\infty})}{\partial v^2} \right|^{2+t/2}\right] < \infty$ . Then, using  $Y_n = S_n^{-1}(X_n \beta_0 + \sigma_0 R_n^{-1} V_n)$  and  $\sup_i E(|v_i|^{2+2c_t+t}) < \infty$ , where  $c_t = 0$  for the case with bounded  $\frac{\partial^2 \ln f(\frac{\sigma_0}{\sigma \infty} v_i, \eta_{\infty})}{\partial v^2}$  and  $c_t = 1$  for the case with  $\left| \frac{\partial^2 \ln f(\frac{\sigma_0}{\sigma \infty} v_i, \eta_{\infty})}{\partial v^2} \right| \leq c_f(\frac{\sigma_0^2}{\sigma \infty^2} |v_i|^2 + 1)$ , we have  $\frac{1}{n} \frac{\partial^2 \ln L_n(\gamma_{\#})}{\partial \lambda^2} - E\left(\frac{1}{n} \frac{\partial^2 \ln L_n(\gamma_{\#})}{\partial \lambda^2}\right) = o_p(1)$  by Lemma B.3.

With (i) and (ii), by (C.9),  $\sqrt{n}(\hat{\gamma} - \gamma_{\#}) = - \left( \frac{1}{n} E \frac{\partial^2 \ln L_n(\gamma_{\#})}{\partial \gamma \partial \gamma'} \right)^{-1} \frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_{\#})}{\partial \gamma} + o_p(1)$ . Under Assumption 4(ii),  $E[\ln f(\frac{\sigma_0}{\sigma} v_i + c, \eta)]$  is uniquely maximized at  $c = 0$  for any  $\sigma$  and  $\eta$ , by Lemma A in Newey and Steigerwald (1997). Then  $E(\zeta_{1i}) = 0$ , where  $\zeta_{1i} = \frac{\sigma_0}{\sigma \infty} \frac{\partial \ln f(\frac{\sigma_0}{\sigma \infty} v_i, \eta_{\infty})}{\partial v}$ . By Assumption 4(ii)(c),  $E(\zeta_{2i}) = 0$  and  $E(\zeta_{3i}) = 0$ , where  $\zeta_{2i} = \zeta_{1i} v_i + 1$  and  $\zeta_{3i} = - \frac{\partial \ln f(\frac{\sigma_0}{\sigma \infty} v_i, \eta_{\infty})}{\partial \eta}$ . Hence, every element of  $\frac{\partial \ln L_n(\gamma_{\#})}{\partial \gamma}$  is a special case of the

general linear-quadratic form  $\omega_n$  in Lemma B.4. By Assumptions 2(ii) and 9(iv) and Lemma B.6, the involved matrices  $S_n^{-1}R_n^{-1}$ ,  $W_nS_n^{-1}R_n^{-1}$ ,  $M_nS_n^{-1}R_n^{-1}$ , and  $W_nM_nS_n^{-1}R_n^{-1}$  in  $\omega_n$  are bounded in both the row- and column-sum norms. As  $|\frac{\partial \ln f(\frac{\sigma_0}{\sigma_\infty}v_i, \eta_\infty)}{\partial v}| \leq c_f(|\frac{\sigma_0}{\sigma_\infty}v_i|^{c_t} + 1)$  and  $\sup_i E(|v_i|^{2+2c_t+l}) < \infty$ , we have  $E[|\frac{\partial \ln f(\frac{\sigma_0}{\sigma_\infty}v_i, \eta_\infty)}{\partial v}|^{2+l/(1+c_t)}] < \infty$  and  $E[|\frac{\partial \ln f(\frac{\sigma_0}{\sigma_\infty}v_i, \eta_\infty)}{\partial v}|^{2+2c_t+l}] < \infty$  for  $c_t = 0$  or  $1$ . As  $\|\frac{\partial \ln f(\frac{\sigma_0}{\sigma_\infty}v_i, \eta_\infty)}{\partial \eta}\| \leq c_f(|\frac{\sigma_0}{\sigma_\infty}v_i|^{1+c_t} + 1)$  and  $\sup_i E(|v_i|^{2+2c_t+l}) < \infty$ ,  $E[\|\frac{\partial \ln f(\frac{\sigma_0}{\sigma_\infty}v_i, \eta_\infty)}{\partial \eta}\|^{2+l/(1+c_t)}] < \infty$ . Then Lemma B.4 implies that  $\frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_\#)}{\partial \gamma} \xrightarrow{d} N(0, \lim_{n \rightarrow \infty} \mathcal{B})$ , where  $\mathcal{B} = \frac{1}{n} E(\frac{\partial \ln L_n(\gamma_\#)}{\partial \gamma} \frac{\partial \ln L_n(\gamma_\#)}{\partial \gamma'})$ . Hence,  $\sqrt{n}(\hat{\gamma} - \gamma_\#) \xrightarrow{d} N(0, \lim_{n \rightarrow \infty} \mathcal{A}^{-1} \mathcal{B} \mathcal{A}^{-1})$ , where  $\mathcal{A} = -\frac{1}{n} E(\frac{\partial^2 \ln L_n(\gamma_\#)}{\partial \gamma \partial \gamma'})$ .  $\square$

**Proof of Corollary 1.** We first prove that: (i)  $E(\zeta_{1i}\zeta_{2i}) = 0$ , (ii)  $E(\zeta_{1i}\zeta_{3i}) = 0$ , (iii)  $E(\zeta_{4i}v_i) = 0$ ; and (iv)  $E(\zeta_{5i}) = 0$ , where  $\zeta_{1i}$  to  $\zeta_{5i}$  are defined in Appendix A.2 and they satisfy  $E(\zeta_{1i}) = 0$ ,  $E(\zeta_{2i}) = 0$ , and  $E(\zeta_{3i}) = 0$ , as shown in the proof of Theorem 2.

(i) Note that for any even function  $h_1(v)$  of  $v$ ,  $h_1(v) = h_1(|v|) = h_2(v^2)$ , where  $h_2(z) \equiv h_1(z^{1/2})$ , for  $z \geq 0$ . Then a symmetrically distributed  $v_i$  is also spherically symmetric (Fang, Kotz, and Ng, 1990, p. 35). Define  $g(\zeta, \eta) = f(\zeta^{1/2}, \eta)$ , for  $\zeta \geq 0$ , so that  $f(v, \eta) = f(|v|, \eta) =$

$g(v^2, \eta)$ . Then  $\frac{\partial \ln f(v, \eta)}{\partial v} = 2 \frac{\partial \ln g(v^2, \eta)}{\partial \zeta} v$  and  $E(\frac{\partial \ln f(\frac{\sigma_0}{\sigma_\infty}v_i, \eta_\infty)}{\partial v}) = \frac{2\sigma_0}{\sigma_\infty} E(\frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta} v_i)$ . Let  $v_i = |v_i| \cdot \varpi_i$ . It follows that  $|v_i|$  and  $\varpi_i$  are independent (Fang et al., 1990, p.

30). Then  $E(\zeta_{1i}\zeta_{2i}) = E[\zeta_{1i}(\zeta_{1i}v_i + 1)] = E(\zeta_{1i}^2v_i) = \frac{4\sigma_0^4}{\sigma_\infty^4} E[(\frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta})^2 v_i^3] = \frac{4\sigma_0^4}{\sigma_\infty^4} E[(\frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta})^2 |v_i|^3 \cdot \varpi_i^3] = \frac{4\sigma_0^4}{\sigma_\infty^4} E[(\frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta})^2 |v_i|^3] E(\varpi_i^3)$ . Since  $0 = E(v_i^3) = E(|v_i|^3 \cdot \varpi_i^3) = E(|v_i|^3) E(\varpi_i^3)$ ,  $E(\varpi_i^3) = 0$ . Thus,  $E(\zeta_{1i}\zeta_{2i}) = 0$ .

(ii)  $E(\zeta_{1i}\zeta_{3i}) = E(\frac{2\sigma_0}{\sigma_\infty} \frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta} v_i \cdot \frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \eta}) = \frac{2\sigma_0}{\sigma_\infty} E(\frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta} \frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \eta} |v_i| \cdot \varpi_i) = \frac{2\sigma_0}{\sigma_\infty} E(\frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta} \frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \eta} |v_i|) E(\varpi_i) = 0$ , where we use  $E(\varpi_i) = 0$  implied by  $0 = E(v_i) = E(|v_i|) E(\varpi_i)$ .

(iii) As  $\frac{\partial \ln f(v, \eta)}{\partial v} = 2 \frac{\partial \ln g(v^2, \eta)}{\partial \zeta} v$ ,  $\frac{\partial^2 \ln f(v, \eta)}{\partial v^2} = 4 \frac{\partial^2 \ln g(v^2, \eta)}{\partial \zeta^2} v^2 + 2 \frac{\partial \ln g(v^2, \eta)}{\partial \zeta}$ . Then  $E(\zeta_{4i}v_i) = -\frac{4\sigma_0^4}{\sigma_\infty^4} E(\frac{\partial^2 \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta^2} v_i^3) - \frac{2\sigma_0^2}{\sigma_\infty^2} E(\frac{\partial \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta} v_i) = 0$ .

(iv) As  $\frac{\partial \ln f(v, \eta)}{\partial v} = 2 \frac{\partial \ln g(v^2, \eta)}{\partial \zeta} v$ ,  $\frac{\partial^2 \ln f(v, \eta)}{\partial v \partial \eta} = 2 \frac{\partial^2 \ln g(v^2, \eta)}{\partial \zeta \partial \eta} v$ . Then  $E(\zeta_{5i}) = \frac{2\sigma_0}{\sigma_\infty} E(\frac{\partial^2 \ln g(\frac{\sigma_0^2}{\sigma_\infty^2}v_i^2, \eta_\infty)}{\partial \zeta \partial \eta} v_i) = 0$ .

By (i)–(iv) and Appendix A, we have  $\mathcal{A}_{\beta\rho} = 0$ ,  $\mathcal{A}_{\beta\sigma^2} = 0$ ,  $\mathcal{A}_{\beta\eta} = 0$ ,  $\mathcal{B}_{\beta\rho} = 0$ ,  $\mathcal{B}_{\beta\sigma^2} = 0$ , and  $\mathcal{B}_{\beta\eta} = 0$ . Hence, for the spatial error model, by Theorem 1, the asymptotic variance of the NGPML  $\hat{\beta}$  is  $\lim_{n \rightarrow \infty} \mathcal{A}_{\beta\beta}^{-1} \mathcal{B}_{\beta\beta} \mathcal{A}_{\beta\beta}^{-1} = \lim_{n \rightarrow \infty} [\frac{1}{n\sigma_0^2} E(\zeta_{4i}X_n'R_n'R_nX_n)]^{-1} \cdot$

$\frac{1}{n\sigma_0^2} E(\zeta_{1i}^2X_n'R_n'R_nX_n) \cdot [\frac{1}{n\sigma_0^2} E(\zeta_{4i}X_n'R_n'R_nX_n)]^{-1} = \lim_{n \rightarrow \infty} \frac{\sigma_0^2 E(\zeta_{1i}^2)}{[E(\zeta_{4i})]^2} (\frac{1}{n} X_n'R_n'R_nX_n)^{-1}$ .

The GPMLE is a special case of the NGPMLE with  $f(v, \eta) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$  and  $\sigma_\infty^2 = \sigma_0^2$ . Then, for the GPMLE,  $\zeta_{1i} = -v_i$ ,  $\zeta_{4i} = 1$ , and the asymptotic variance for  $\beta$  is  $\lim_{n \rightarrow \infty} \sigma_0^2 (\frac{1}{n} X_n' R_n' R_n X_n)^{-1}$ . The BGMME of  $\beta$  has the same asymptotic variance as the GPMLE, by Corollary 3 in Liu et al. (2010).  $\square$

**Proof of Theorem 3.** We could show that  $\check{\gamma} = \gamma_\infty + o_p(1)$  as the proof of Theorem 2. By the mean value theorem,  $0 = \frac{\partial \ln L_n(\check{\gamma})}{\partial \gamma_u} = \frac{\partial \ln L_n(\gamma_n)}{\partial \gamma_u} - \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \gamma_u \partial \tau'} \tau_n + \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \gamma_u \partial \gamma_u'} (\check{\gamma}_u - \gamma_{u\infty})$ , where  $\gamma_n = [\tau_n', \gamma_{u\infty}']'$  and  $\bar{\gamma}$  lies between  $\check{\gamma}$  and  $\gamma_\infty$ . Thus,  $\sqrt{n}(\check{\gamma}_u - \gamma_{u\infty}) = -(\frac{1}{n} \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \gamma_u \partial \gamma_u'})^{-1} (\frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_n)}{\partial \gamma_u} - \frac{1}{n} \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \gamma_u \partial \tau'} \cdot \sqrt{n} \tau_n)$ . As in the proof of Theorem 2, we could show that  $\frac{1}{n} \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \gamma_u \partial \gamma_u'} = \frac{1}{n} E(\frac{\partial^2 \ln L_n(\gamma_n)}{\partial \gamma_u \partial \gamma_u'}) + o_p(1)$  and  $\frac{1}{n} \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \gamma_u \partial \tau'} = \frac{1}{n} E(\frac{\partial^2 \ln L_n(\gamma_n)}{\partial \gamma_u \partial \tau'}) + o_p(1)$ . Hence,

$$\begin{aligned} \sqrt{n}(\check{\gamma}_u - \gamma_{u\infty}) &= -\left(\frac{1}{n} E \frac{\partial^2 \ln L_n(\gamma_n)}{\partial \gamma_u \partial \gamma_u'}\right)^{-1} \\ &\quad \times \left[\frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_n)}{\partial \gamma_u} - \frac{1}{n} E\left(\frac{\partial^2 \ln L_n(\gamma_n)}{\partial \gamma_u \partial \tau'}\right) \cdot \sqrt{n} \tau_n\right] + o_p(1). \end{aligned} \quad (C.10)$$

Similarly,

$$\frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\check{\gamma})}{\partial \tau} = \frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_n)}{\partial \tau} - \frac{1}{n} \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \tau \partial \tau'} \cdot \sqrt{n} \tau_n + \frac{1}{n} \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \tau \partial \gamma_u'} \cdot \sqrt{n}(\check{\gamma}_u - \gamma_{u\infty}), \quad (C.11)$$

where  $\frac{1}{n} \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \tau \partial \tau'} = \frac{1}{n} E(\frac{\partial^2 \ln L_n(\gamma_n)}{\partial \tau \partial \tau'}) + o_p(1)$  and  $\frac{1}{n} \frac{\partial^2 \ln L_n(\bar{\gamma})}{\partial \tau \partial \gamma_u'} = \frac{1}{n} E(\frac{\partial^2 \ln L_n(\gamma_n)}{\partial \tau \partial \gamma_u'}) + o_p(1)$ .

Plugging (C.10) into (C.11) yields  $\frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\check{\gamma})}{\partial \tau} = \Delta \cdot \frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_n)}{\partial \tau} + \frac{1}{n} \Lambda \cdot \sqrt{n} \tau_n + o_p(1)$ .

Since  $\frac{1}{\sqrt{n}} \frac{\partial \ln L_n(\gamma_n)}{\partial \gamma} \xrightarrow{d} N(0, \lim_{n \rightarrow \infty} \mathcal{B})$ , the result in the proposition follows.  $\square$

## SUPPLEMENTARY MATERIAL

Fei Jin and Yuqin Wang (2023): Supplement to “Consistent non-Gaussian pseudo maximum likelihood estimators of spatial autoregressive models,” *Econometric Theory* Supplementary Material. To view, please visit: <https://doi.org/10.1017/S0266466623000026>

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