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APPLICATIONS OF AN INTERSECTION FORMULA TO DUAL CONES

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Abstract

We give a succinct proof of a duality theorem obtained by Révész [‘Some trigonometric extremal problems and duality’, *J. Aust. Math. Soc. Ser.* A **50** (1991), 384–390] which concerns extremal quantities related to trigonometric polynomials. The key tool of our new proof is an intersection formula on dual cones in real Banach spaces. We show another application of this intersection formula which is related to integral estimates of nonnegative positive-definite functions.

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1. Introduction

Let X be a real Banach space and let X' denote its topological dual space endowed with the weak-* topology. For any set $D \subseteq X$, the *dual cone*, D^+, of D is defined by

$$D^+ = \{\varphi \in X' \mid \varphi(x) \geq 0 \ \forall x \in D\}$$

(see, for example, [2, Section 2]). The *polar cone*, D^-, is $D^- := -D^+$. Note that both D^+ and D^- are weak-* closed convex cones in X', no matter what the set D is. Moreover, by [2, Lemma 2.1], if C and $\mathcal{P}$ are convex sets in X such that $0 \in C \cap \mathcal{P}$ and $C \cap \operatorname{int} \mathcal{P} \neq \emptyset$, then

$$(C \cap \mathcal{P})^+ = C^+ + \mathcal{P}^+. \tag{1.1}$$

Consequently, in this case, $(C \cap \mathcal{P})^- = C^- + \mathcal{P}^-$.

In this short note we give two applications of the formula (1.1) that describes the structure of the dual cone of the intersection of cones. Both applications are of a Fourier-analytic nature. We first collect some basic facts and notation on this topic.

(i) For a locally compact abelian group G, the symbol $M(G)$ denotes the set of all complex-valued regular Borel measures on G with finite total variation. The set $M(G)$

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is a commutative, unital Banach algebra, where the norm is defined as $\|\mu\| = |\mu|(G)$ and the multiplication is defined by convolution [6, Corollary 1.3.2].

(ii) The symbol $L^1(G)$ stands for the set of all integrable functions on G (with respect to the Haar measure, denoted by λ). We may consider $L^1(G)$ as a subset of $M(G)$ by the embedding

$$\mu_{(\cdot)} : L^1(G) \to M(G), \quad f \mapsto \mu_f; \quad \mu_f(E) = \int_E f\,d\lambda \text{ for any Borel set } E \subseteq G.$$

In fact, $L^1(G)$ is a Banach subalgebra of $M(G)$ [6, Theorem 1.3.5]. Moreover, $L^1(G)$ is unital if and only if $L^1(G) = M(G)$ if and only if G is discrete [6, Theorem 1.7.3].

(iii) $L^\infty(G)$ stands for the set of all essentially bounded measurable functions on G.

(iv) A continuous group homomorphism from the locally compact abelian group G into the multiplicative group $\mathbb{T} = \{z \in C \mid |z| = 1\}$ is called a *character* of G. The set of all characters of G forms a group (with pointwise multiplication), which is called the *dual group* of G and denoted by $\hat{G}$.

(v) For any $\mu \in M(G)$ (or $f \in L^1(G)$), the symbol $\hat{\mu}$ (or $\hat{f}$) denotes the Fourier transform of μ (or f), that is,

$$\hat{\mu} \in \mathbb{C}^{\hat{G}}; \quad \hat{\mu}(\gamma) = \int_G \overline{\gamma}\,d\mu \quad (\gamma \in \hat{G})$$

and

$$\hat{f} \in \mathbb{C}^{\hat{G}}; \quad \hat{f}(\gamma) = \int_G f\overline{\gamma}\,d\lambda \quad (\gamma \in \hat{G}).$$

The Fourier transform is a continuous linear transformation from $L^1(G)$ into $C_0(\hat{G})$, where $C_0(\hat{G})$ denotes the set of all functions on $\hat{G}$ vanishing at infinity (the topology on $\hat{G}$ is the weak topology induced by the set of all functions $\hat{f}$ obtained as Fourier transforms of L^1 functions on G). Moreover, it is a contraction as $\|\hat{f}\|_\infty \le \|f\|_1$. (For details, see [6, Theorem 1.2.4].)

(vi) The following useful formula is an easy consequence of Fubini's theorem. If $\mu \in M(G), \nu \in M(\hat{G})$ and $\phi(x) = \int_{\hat{G}} \gamma(x)\,d\nu(\gamma)$ $(x \in G)$, then

$$\int_G \overline{\phi}\,d\mu = \int_{\hat{G}} \hat{\mu}\,d\overline{\nu}. \tag{1.2}$$

Now we turn to the detailed descriptions of the two applications of the intersection formula (1.1). Section 2 is devoted to the first and Section 3 contains the second.

2. A new proof of a duality theorem

In 1991, Révész proved a duality theorem on certain extremal quantities related to multivariable trigonometric polynomials [4]. That theorem is general enough to cover the duality statements appearing in [3, 5, 7]. The setting of the theorem is as follows.

Let d be a positive integer. Let us use the notation $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$ and

$$\mathbb{Z}_+^d = \{\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d \mid \exists j \in \{1, 2, \ldots, d\} \text{ such that } n_k = 0 \text{ for any } k < j \text{ and } n_j > 0\}.$$

Let $M \subseteq \mathbb{Z}_+^d$, $L \subseteq \mathbb{Z}_+^d$ and set $M^c := \mathbb{Z}_+^d \setminus M$, $L^c := \mathbb{Z}_+^d \setminus L$.

Consider the real Banach space $L^1_{\mathbb{R},s}(\mathbb{Z}^d)$ of all symmetric real-valued absolutely summable functions on $\mathbb{Z}^d$ with its topological dual space $L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$. Set

$$C := \{f \in L^1_{\mathbb{R},s}(\mathbb{Z}^d) \mid f \text{ has finite support, } \operatorname{supp}(f_+) \subseteq \{\mathbf{0}\} \cup M \cup -M \text{ and } \operatorname{supp}(f_-) \subseteq \{\mathbf{0}\} \cup L \cup -L\}.$$

The set $C \subseteq L^1_{\mathbb{R},s}(\mathbb{Z}^d)$ is a convex set. It is easy to see that the dual cone of C is

$$C^+ = \{t \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d) \mid \operatorname{supp}(t_+) \subseteq L^c \cup -L^c \text{ and } \operatorname{supp}(t_-) \subseteq M^c \cup -M^c\}.$$

Therefore, the polar cone of C is

$$C^- = \{t \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d) \mid \operatorname{supp}(t_+) \subseteq M^c \cup -M^c \text{ and } \operatorname{supp}(t_-) \subseteq L^c \cup -L^c\}.$$

Set

$$\mathcal{P} := \Big\{f \in L^1_{\mathbb{R},s}(\mathbb{Z}^d) \,\Big|\, \hat{f}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f(\mathbf{n}) e^{-i\mathbf{n}\cdot\mathbf{x}} = f(\mathbf{0}) + 2 \sum_{\mathbf{n} \in \mathbb{Z}_+^d} f(\mathbf{n}) \cos(\mathbf{n}\cdot\mathbf{x}) \geq 0 \; \forall \mathbf{x} \in \mathbb{T}^d\Big\}.$$

Clearly, $\mathcal{P}$ is a convex set. The following lemma describes its dual cone.

Lemma 2.1. *With the notation as above,*

$$\mathcal{P}^+ = \{h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d) \mid h \gg 0, \text{ that is, } h \text{ is positive definite}\}.$$

Proof. Recall that a function $h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ is said to be positive definite if

$$\sum_{i,j=1}^m z_i \overline{z_j} g(\mathbf{n}_i - \mathbf{n}_j) \geq 0$$

holds for any $\mathbf{n}_1, \ldots, \mathbf{n}_m \in \mathbb{Z}^d$ and $z_1, \ldots, z_m \in \mathbb{C}$. However, for symmetric real functions, positive definiteness is equivalent to the *a priori* weaker condition

$$\sum_{i,j=1}^m c_i c_j g(\mathbf{n}_i - \mathbf{n}_j) \geq 0 \quad \text{for } \mathbf{n}_1, \ldots, \mathbf{n}_m \in \mathbb{Z}^d,\ c_1, \ldots, c_m \in \mathbb{R}.$$

Let us recall Bochner's theorem [6, Theorem 1.4.3], which says that a function $h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ is positive definite if and only if there is a nonnegative symmetric measure $\nu \in M_{\mathbb{R},s}(\mathbb{T}^d)$ such that

$$h(\mathbf{n}) = \int_{\mathbb{T}^d} e^{i\mathbf{n}\cdot\mathbf{x}} \, d\nu(\mathbf{x}) \quad (\mathbf{n} \in \mathbb{Z}^d).$$

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Therefore, all the positive-definite functions are in $\mathcal{P}^+$ because any positive-definite $h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ can be written in the form $h(\mathbf{n}) = \int_{\mathbb{T}^d} e^{i\mathbf{n}\cdot\mathbf{x}}\, d\nu(\mathbf{x})$ ($\mathbf{n} \in \mathbb{Z}^d$) for some ν with $0 \le \nu \in M_{\mathbb{R},s}(\mathbb{T}^d)$ and hence, by (1.2), the inequality

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})h(\mathbf{n}) = \int_{\mathbb{T}^d} \hat{f}(\mathbf{x})\, d\nu(\mathbf{x}) \ge 0$$

holds for any $f \in \mathcal{P} \subset L^1_{\mathbb{R},s}(\mathbb{Z}^d)$.

Conversely, if $g \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ and g is not positive definite, that is,

$$\sum_{i,j=1}^{m} c_i c_j g(\mathbf{n}_i - \mathbf{n}_j) < 0$$

for some $\{\mathbf{n}_1, \dots, \mathbf{n}_m\} \subset \mathbb{Z}^d$ and $\{c_1, \dots, c_m\} \subset \mathbb{R}$, then

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} (x * \tilde{x})(\mathbf{n})g(\mathbf{n}) < 0,$$

where $x = \sum_{i=1}^m c_i \chi_{\{\mathbf{n}_i\}}$ and $\tilde{x}$ is defined by $\tilde{x}(\mathbf{n}) = x(-\mathbf{n})$ for $\mathbf{n} \in \mathbb{Z}^d$. But, clearly, $x * \tilde{x} \in \mathcal{P} \subset L^1_{\mathbb{R},s}(\mathbb{Z}^d)$, which means that $g \notin \mathcal{P}^+$. □

Now let $r \in L^\infty_{\mathbb{R},s}(\mathbb{Z}^d)$ with $r(\mathbf{0}) = 1$ be fixed and let us define the affine subspace

$$\mathcal{H} := \{f \in L^1_{\mathbb{R},s}(\mathbb{Z}^d) \mid f(\mathbf{0}) = 1\}.$$

According to [4, Equations (5) and (12)], let us define the extremal quantities

$$\alpha := \inf\Big\{\sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})r(\mathbf{n}) \,\Big|\, f \in C \cap \mathcal{P} \cap \mathcal{H}\Big\}$$

and

$$\begin{aligned}
\omega &:= \sup\{\delta \in \mathbb{R} \mid \exists t \in C^- \text{ such that } r + t - \delta\chi_{\{\mathbf{0}\}} \in \mathcal{P}^+\} \\
&= \sup\{\delta \in \mathbb{R} \mid \exists t \in C^- \text{ such that } \delta\chi_{\{\mathbf{0}\}} - r - t \in \mathcal{P}^-\} \\
&= \sup\{\delta \in \mathbb{R} \mid \delta\chi_{\{\mathbf{0}\}} - r \in C^- + \mathcal{P}^-\}.
\end{aligned}$$

(It is clear that the definition of α coincides with the definition in [4, Equation (5)]. It is less obvious that the definition of ω is the same as the one given in [4, Equation (12)]. However, the fact that the nonnegative symmetric measures on $\mathbb{T}^d$ are in one-to-one correspondence with the real positive-definite functions on $\mathbb{Z}^d$ by the Fourier transform ensures that the definition of ω is also correct.)

We mentioned before that C and $\mathcal{P}$ are convex sets in the real Banach space $L^1_{\mathbb{R},s}(\mathbb{Z}^d)$. It is clear that $0 \in C \cap \mathcal{P}$ and $C \cap \operatorname{int}\mathcal{P} \ne \emptyset$, as $\chi_{\{\mathbf{0}\}} \in C \cap \operatorname{int}\mathcal{P}$. (To see that $\chi_{\{\mathbf{0}\}} \in \operatorname{int}\mathcal{P}$, note that the Fourier transform is a contraction from $L^1_{\mathbb{R},s}(\mathbb{Z}^d)$ into $C_{\mathbb{R},s}(\mathbb{T}^d)$, and $\widehat{\chi_{\{\mathbf{0}\}}} = 1$.) Therefore, by [2, Lemma 2.1], the intersection formula

$$(C \cap \mathcal{P})^+ = C^+ + \mathcal{P}^+$$

holds. Consequently, we have $(C \cap \mathcal{P})^- = C^- + \mathcal{P}^-$. So, by this intersection formula, ω can be rewritten as

$$\omega = \sup\{\delta \in \mathbb{R} \,|\, \delta\chi_{\{\mathbf{0}\}} - r \in (C \cap \mathcal{P})^-\}.$$

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Theorem 2.2 (Révész [4]). *With the notation as above,*

$$\alpha = \omega.$$

A short proof. If $\delta\chi_{\{\mathbf{0}\}} - r \in (\mathcal{P} \cap C)^{-}$, then

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})r(\mathbf{n}) \geq \delta \quad \text{for any } f \in C \cap \mathcal{P} \cap \mathcal{H},$$

because in this case

$$0 \geq \sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})(\delta\chi_{\{\mathbf{0}\}}(\mathbf{n}) - r(\mathbf{n})) = \delta f(\mathbf{0}) - \sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})r(\mathbf{n}) = \delta - \sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})r(\mathbf{n}).$$

Therefore, $\omega \leq \alpha$.

On the other hand, if $\beta > \omega$, then $\beta\chi_{\{\mathbf{0}\}} - r \notin (C \cap \mathcal{P})^{-}$, that is, there exists some $f \in C \cap \mathcal{P}$ such that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})(\beta\chi_{\{\mathbf{0}\}}(\mathbf{n}) - r(\mathbf{n})) > 0.$$

This f is necessarily a nonzero element of $\mathcal{P}$ and hence $f(\mathbf{0}) > 0$. Therefore, without loss of generality, we can assume that $f(\mathbf{0}) = 1$. So, there exists $f \in C \cap \mathcal{P} \cap \mathcal{H}$ such that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})(\beta\chi_{\{\mathbf{0}\}}(\mathbf{n}) - r(\mathbf{n})) > 0.$$

That is,

$$\beta > \sum_{\mathbf{n}\in\mathbb{Z}^d} f(\mathbf{n})r(\mathbf{n})$$

for some $f \in C \cap \mathcal{P} \cap \mathcal{H}$, which means that $\beta > \alpha$. We have shown that $\beta > \omega$ implies that $\beta > \alpha$ and, therefore, $\alpha \leq \omega$. The proof is done. □

3. Another application of the intersection formula

The second application concerns integral estimates of nonnegative positive-definite functions. This problem is related to Wiener's problem [8, 10] and to the recent works [1, 9]. The arguments in this section are partially parallel to the arguments presented in the previous section.

Let $L^1_{\mathbb{R},s}(\mathbb{Z})$ denote the real Banach space of all real-valued, symmetric, summable functions on $\mathbb{Z}$ and let us consider its topological dual space $L^\infty_{\mathbb{R},s}(\mathbb{Z})$ endowed with the weak-* topology.

Let us define $C = \{f \in L^1_{\mathbb{R},s}(\mathbb{Z}) \mid f \geq 0\}$ and $\mathcal{P} = \{f \in L^1_{\mathbb{R},s}(\mathbb{Z}) \mid \hat{f} \geq 0\}$. Clearly, C and $\mathcal{P}$ are convex cones in $L^1_{\mathbb{R},s}(\mathbb{Z})$. The closedness of C is obvious; $\mathcal{P}$ is also closed as the Fourier transform is a continuous (moreover, norm-nonincreasing) linear transformation from $L^1_{\mathbb{R},s}(\mathbb{Z})$ into $C_{\mathbb{R},s}(\mathbb{T})$, and the nonnegative functions form a closed set of $C_{\mathbb{R},s}(\mathbb{T})$ with respect to the maximum norm topology. (The symbol $\mathbb{T}$ denotes the additive group of real numbers modulo 2π and $C_{\mathbb{R},s}(\mathbb{T})$ stands for the Banach space of all continuous, symmetric real functions on $\mathbb{T}$.)

Lemma 3.1. *With the notation as above,*

$$C^+ = \{g \in L^\infty_{\mathbb{R},s}(\mathbb{Z}) | \, g \geq 0\}$$

and

$$\mathcal{P}^+ = \{h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}) \mid h \gg 0, \ \textit{that is, } h \textit{ is positive definite}\}.$$

Proof. The first statement of Lemma 3.1 is obvious. The proof of the second statement is very similar to the proof of Lemma 2.1. □

Let L and N be positive integers. Let us define the extremal quantities

$$C(L,N) := \inf\Big\{C \in \mathbb{R} \,\Big|\, \sum_{k=-LN}^{LN} f(k) \leq (C+1) \sum_{k=-N}^{N} f(k) \text{ for any } f \in \mathcal{C} \cap \mathcal{P}\Big\}$$

and

$$K(L,N) := \inf\{h(0) \mid h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}),\ h \gg 0 \text{ and } h(k) \leq -\chi_{\{-LN,\dots,LN\}}(k) \text{ if } |k| > N\}.$$

Let us introduce

$$\mathcal{S}_{L,N} := \{h \in L^\infty_{\mathbb{R},s}(\mathbb{Z}) \mid h(k) \leq -\chi_{\{-LN,\dots,LN\}}(k) \text{ if } |k| > N\}.$$

Observe that $\mathcal{S}_{L,N}$ is closed in the weak-* topology, because it is the intersection of weak-* closed sets. Note that

$$C(L,N) = \inf\{C \in \mathbb{R} \mid (C+1)\chi_{\{-N,\dots,N\}} - \chi_{\{-LN,\dots,LN\}} \in (\mathcal{C} \cap \mathcal{P})^+\}$$

and, by the result of Lemma 3.1,

$$K(L,N) = \inf\{h(0) \mid h \in \mathcal{P}^+ \cap \mathcal{S}_{L,N}\}.$$

Remark 3.2. Let us note that $K(L,N)$ is finite as the set $\mathcal{P}^+ \cap \mathcal{S}_{L,N}$ is not empty. Indeed, one can easily check that the function

$$w_{L,N}(k) := \begin{cases} 2(L-1)N & \text{if } k = 0, \\ -1 & \text{if } N < |k| \leq LN, \\ 0 & \text{otherwise} \end{cases}$$

is positive definite and, therefore, it is an element of $\mathcal{P}^+ \cap \mathcal{S}_{L,N}$.

Theorem 3.3. *With the notation as above,*

$$C(L,N) = K(L,N).$$

Proof. The key idea is the observation that $\mathcal{C}$ and $\mathcal{P}$ are convex sets in $L^1_{\mathbb{R},s}(\mathbb{Z})$ such that $0 \in \mathcal{C} \cap \mathcal{P}$ and $\mathcal{C} \cap \operatorname{int} \mathcal{P} \neq \emptyset$ as $\chi_{\{0\}} \in \mathcal{C} \cap \operatorname{int} \mathcal{P}$. Therefore, the intersection formula

$$(\mathcal{C} \cap \mathcal{P})^+ = \mathcal{C}^+ + \mathcal{P}^+$$

holds.

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On the one hand, if $h \in \mathcal{P}^+ \cap \mathcal{S}_{L,N}$, then

$$h \leq (h(0) + 1)\chi_{\{-N,\dots,N\}} - \chi_{\{-LN,\dots,LN\}}$$

as $h(0) \geq h(n)$ $(n \in \mathbb{Z})$ holds for any positive-definite function $h \in L^\infty_{\mathbb{R},s}(\mathbb{Z})$. Therefore, in this case,

$$(h(0) + 1)\chi_{\{-N,\dots,N\}} - \chi_{\{-LN,\dots,LN\}} \in C^+ + \mathcal{P}^+ = (C \cap \mathcal{P})^+$$

and so $C(L, N) \leq K(L, N)$.

On the other hand, by the intersection formula,

$$C(L, N) = \inf\{C \in \mathbb{R} \mid (C + 1)\chi_{\{-N,\dots,N\}} - \chi_{\{-LN,\dots,LN\}} \in C^+ + \mathcal{P}^+\}$$

and the following argument shows the opposite inequality. If

$$(C + 1)\chi_{\{-N,\dots,N\}} - \chi_{\{-LN,\dots,LN\}} \in C^+ + \mathcal{P}^+,$$

then $(C + 1)\chi_{\{-N,\dots,N\}} - \chi_{\{-LN,\dots,LN\}} = g + h$ for some $g \in C^+$ and $h \in \mathcal{P}^+$. Clearly, this h is an element of $\mathcal{P}^+ \cap \mathcal{S}_{L,N}$ and $h(0) \leq C$; hence, $K(L, N) \leq C(L, N)$. □

Remark 3.4. We have noted (see Remark 3.2) that $K(L, N)$ is finite. Therefore, the result of Theorem 3.3 directly implies the finiteness of $C(L, N)$.

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