Canad. J. Math. Vol. **52** (4), 2000 pp. 673–694

Sums of Two Squares in Short Intervals

Antal Balog and Trevor D. Wooley

Abstract. Let $\mathcal{S}$ denote the set of integers representable as a sum of two squares. Since $\mathcal{S}$ can be described as the unsifted elements of a sieving process of positive dimension, it is to be expected that $\mathcal{S}$ has many properties in common with the set of prime numbers. In this paper we exhibit "unexpected irregularities" in the distribution of sums of two squares in short intervals, a phenomenon analogous to that discovered by Maier, over a decade ago, in the distribution of prime numbers. To be precise, we show that there are infinitely many short intervals containing considerably more elements of $\mathcal{S}$ than expected, and infinitely many intervals containing considerably fewer than expected.

1 Introduction

Until recently it was widely believed that sequences of integers prescribed by reasonable multiplicative constraints should be well-distributed, even amongst intervals short relative to the size of those integers. Thus, for example, it was expected that the number of prime numbers of size about x, lying inside an interval of length y, should be asymptotic to $y/\log x$ provided only that y is at least as large as some fixed power of $\log x$. A little over a decade ago Maier [9] shattered this belief by proving that, for any fixed positive number N, there are arbitrarily large values of x with the property that the interval $[x, x + \log^N x]$ contains a positive proportion more than the expected number, $\log^{N-1} x$, of primes, and similarly such intervals exist containing a positive proportion fewer than the expected number of primes. After a decade of intense effort by a number of authors (see, for example, Granville [4] for a survey of this work), extensive investigations concerning the distribution of primes in short intervals, lying in various sequences, have lifted our platform of knowledge to the position where, with the luxury of hindsight, we may now describe these "unexpected irregularities" in the distribution of prime numbers rather as "expected deviations". In this paper we pursue the philosophy implicit in the latter statement by exhibiting "expected deviations" in the distribution of sums of two squares in short intervals. As far as we are aware, this is the first instance in which such deviations have been investigated for sequences other than those consisting of prime numbers, or special subsequences thereof. We suggest that such deviations are to be expected in the short-scale distribution of integers in any sequence described as the unsifted elements of a sieving process of positive dimension.

Before describing our main theorem it will be useful to discuss some simple properties of the global distribution of sums of two squares, and this will require some notation. Let

Received by the editors July 27, 1999.

The first author's research was supported in part by HNFSR grant T 017433. This was written while the author visited the University of Michigan at Ann Arbor, and enjoyed the benefits of a Fellowship from the David and Lucile Packard foundation. The second author is a Packard Fellow and was supported in part by NSF grant DMS-9622773.

AMS subject classification: 11N36, 11N37, 11N25.

Keywords: sums of two squares, sieves, short intervals, smooth numbers.

$\mathcal{S}$ denote the set of integers representable as a sum of two squares. From time to time it will be convenient to abbreviate the condition that n is representable as a sum of two squares by simply writing $n = \square + \square$. For large x one has the classical asymptotic formula (see, for example, Landau [8, pp. 55–68])

$$\operatorname{card}\{1 \le n \le x : n = \square + \square\} \sim \frac{Bx}{\sqrt{\log x}}, \tag{1.1}$$

where

$$B = \frac{1}{\sqrt{2}} \prod_{p \equiv -1 \pmod 4} (1 - 1/p^2)^{-1/2}. \tag{1.2}$$

When x is large and y is not too small, therefore, the number of elements of $\mathcal{S}$ lying in the interval $(x, x+y]$ is expected to be asymptotic to $By/\sqrt{\log x}$. Recent work of Hooley provides firm evidence that the latter expectation is not too far from the truth. Provided only that $y/\sqrt{\log x} \to \infty$ as $x \to \infty$, Hooley [6] shows that in almost all of the intervals $(x, x+y]$, the number of sums of two squares has precise order $y/\sqrt{\log x}$. Thus there exist positive constants A_1 and A_2 so that, in the sense of natural density, for almost all y one has

$$A_1 y/\sqrt{\log x} \le \operatorname{card}\{x < n \le x+y : n = \square + \square\} \le A_2 y/\sqrt{\log x}. \tag{1.3}$$

For the historical record, it seems appropriate here to remark that the upper bound aspect of this conclusion is contained in earlier work of Friedlander [2], [3]. Moreover, the lower bound aspect of this conclusion was claimed in flawed work of Plaksin [11], which was, however, subsequently claimed to be corrected [12] shortly before the cited work of Hooley [6].

Since our results are intimately related to the theory of the half-dimensional sieve, it is helpful to define some functions associated with the latter sieve in order to announce our conclusions. When s is a positive number, we define the functions $F(s)$ and $f(s)$ to be the unique continuous solutions of the pair of simultaneous differential-difference equations

$$\begin{cases} \big(s^{1/2}F(s)\big)' = \tfrac{1}{2}s^{-1/2}f(s-1), & \text{when } s > 2, \\ \big(s^{1/2}f(s)\big)' = \tfrac{1}{2}s^{-1/2}F(s-1), & \text{when } s > 1, \end{cases} \tag{1.4}$$

subject to the initial conditions

$$\begin{cases} F(s) = 2\sqrt{e^{\gamma}/\pi}\, s^{-1/2}, & \text{when } 0 < s \le 2, \\ f(s) = 0, & \text{when } 0 < s \le 1. \end{cases} \tag{1.5}$$

Here, and in the sequel, $\gamma = 0.577\cdots$ is Euler's constant. It may be verified that $F(s)$ and $f(s)$ are respectively monotone decreasing, and monotone increasing, functions of s, that for positive values of s one has

$$0 \le f(s) < 1 < F(s), \tag{1.6}$$

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

and moreover that when s is large,

$$F(s) = 1 + O(e^{-s}) \quad \text{and} \quad f(s) = 1 + O(e^{-s}) \tag{1.7}$$

(see, for example, Halberstam and Richert [5] or Iwaniec [7]). We remark that one may easily refine (1.7) to show that $F(s)$ and $f(s)$ each tend to 1 with speed $e^{-s\log s}$.

The conclusion of Section 4 of this paper yields the following theorem.

Theorem 1 *Let $N > 0$ be fixed. There is a sequence of real numbers, x^+, tending to infinity, such that with $y = (\log x^+)^N$ one has*

$$\operatorname{card}\{x^+ < n \le x^+ + y : n = \square + \square\} > \frac{By}{\sqrt{\log x^+}}\big(F(N) + o(1)\big).$$

There is also a sequence of real numbers, x^-, tending to infinity, such that with $y = (\log x^-)^N$ one has

$$\operatorname{card}\{x^- < n \le x^- + y : n = \square + \square\} < \frac{By}{\sqrt{\log x^-}}\big(f(N) + o(1)\big).$$

On recalling the inequalities (1.6), the conclusion of Theorem 1 shows that the expectation expressed above, to the effect that

$$\operatorname{card}\{x < n \le x + y : n = \square + \square\} \sim \frac{By}{\sqrt{\log x}},$$

cannot always hold when y is bounded by a fixed power of $\log x$. It follows that Hooley's result (1.3) is best possible, in the sense that one cannot remove the phrase "almost all" from the conclusion, and obtain an asymptotic result at the same time.

Our proof of Theorem 1 avoids explicit application of the half-dimensional sieve in favour of a direct proof which, although a little longer, yields some interesting additional consequences. For example, in the course of our proof we derive an asymptotic formula for the number of y-smooth numbers in $\mathcal{S}$ up to x, which is to say the number of integers up to x whose prime factors are all bounded by y. In order to describe this asymptotic formula, we will require some additional notation. Define $\sigma(s)$ to be the unique continuous solution of the differential-difference equation

$$s\sigma'(s) = -\tfrac{1}{2}\big(\sigma(s) + \sigma(s-1)\big), \quad \text{when } s > 1, \tag{1.8}$$

subject to the initial condition

$$\sigma(s) = s^{-1/2}, \quad \text{when } 0 < s \le 1. \tag{1.9}$$

The integral equation

$$s\sigma(s) = \tfrac{1}{2}\int_{s-1}^{s} \sigma(t)\,dt \quad (s > 1),$$

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

corresponding to (1.8) reveals that the function $\sigma(s)$ is positive, decreasing, and satisfies $\sigma(s) = O(e^{-s\log s})$ for large s. We remark also that a simple calculation shows that when $1 \le s \le 2$, one has

$$\sigma(s) = \frac{1}{\sqrt{s}}\big(1 - \log(\sqrt{s} + \sqrt{s-1})\big) = 1 - \sqrt{s-1} + O(s-1). \tag{1.10}$$

Theorem 2 *Let y and z be real numbers with $y \ge 2$ and $z \ge 2$. Denote by $A(y,z)$ the number of integers $n \in \mathcal{S}$ with $1 \le n \le y$, $n \equiv 1 \pmod 4$, and satisfying the condition that whenever p is prime and $p|n$, then $p \le z$. One has*

$$A(y,z) = \tfrac{1}{2}\sigma\left(\frac{\log y}{\log z}\right)\frac{By}{\sqrt{\log z}} + O\left(\frac{y}{\log z} + \frac{y}{\log^{3/2} y}\right). \tag{1.11}$$

Here the implied constant is absolute.

It is well-known that $n \in \mathcal{S}$ if and only if n can be written in the form $n = md^2$, where m has no prime factor congruent to -1 modulo 4, and d has all of its prime factors in the latter congruence class. It follows easily that the congruence condition $n \equiv 1 \pmod 4$ may be deleted from the definition of $A(y,z)$ in the statement of Theorem 2, so long as the factor $\frac{1}{2}$ is deleted from the conclusion (1.11). A conclusion similar to the latter is contained in work of Moree [10]. We note also that the first error term in (1.11) would appear to be close to best possible, in the sense that a second main term of this size is expected.

In order to relate the sieving function $\sigma(s)$ to the functions $F(s)$ and $f(s)$ defined above, we are forced to derive some integral identities involving Buchstab's function $\omega(s)$, defined for $s \ge 1$ by the differential-difference equation

$$s\omega'(s) = \omega(s-1) - \omega(s), \quad \text{when } s > 2, \tag{1.12}$$

subject to the initial condition

$$\omega(s) = 1/s, \quad \text{when } 1 \le s \le 2. \tag{1.13}$$

It is convenient to define $\omega(s)$ to be zero when $0 < s < 1$. As is well-known (see, for example, Halberstam and Richert [5]), when s is large one has

$$\omega(s) = e^{-\gamma} + O(e^{-s\log s}).$$

In Section 3 we derive the identities contained in the following theorem.

Theorem 3 *When s is a positive real number one has*

$$\sqrt{\pi/e^{\gamma}}F(s) = 2\sigma(s) + \int_0^s \omega(t)\sigma(s-t)\,dt, \tag{1.14}$$

and

$$\sqrt{\pi/e^{\gamma}}f(s) = \int_0^s \omega(t)\sigma(s-t)\,dt. \tag{1.15}$$

In view of the extensive literature concerning the properties of sieve functions associated with differential-difference equations, it would be surprising if the identities (1.14) and (1.15) were genuinely new. However, the authors have been unable to identify any source in the literature which establishes these identities (but see, for example, Wheeler [16] and references cited therein for a discussion of related properties of such functions). It may be interesting to note that on considering the limit as s tends to infinity in (1.15), and making use of the growth properties of $f(s)$, $\sigma(s)$ and $\omega(s)$, one obtains the formula

$$\int_0^\infty \sigma(t)\,dt = \sqrt{\pi e^\gamma}. \tag{1.16}$$

We begin, in Section 2, by establishing Theorem 2 using an identity related to that of Buchstab. Next, in Section 3, we derive the identities described in Theorem 3. Having completed these preliminaries, we are equipped to develop an analogue of Maier's matrix method in Section 4. Unfortunately, since the sieving function $\sigma(s)$ associated with sums of two squares does not have the oscillatory behaviour demonstrated by the function $\omega(s)$ stemming from Maier's treatment of the prime numbers, we are forced to develop separate treatments in order to obtain the positive and negative deviations in the distribution described in Theorem 1. For this purpose we employ an idea of Richards [13], originally used to deduce the existence of unusually large gaps between successive sums of two squares.

Throughout, implicit constants in Vinogradov's notation $\ll$ and $\gg$, and in Landau's notation, will depend at most on the quantities occurring as subscripts to the notation, unless otherwise indicated. We adopt the convention throughout that any variable denoted by the letters p or q is implicitly assumed to be a prime number. Finally, we write $p^r \| n$ to denote that $p^r | n$ but $p^{r+1} \nmid n$.

2 The Distribution of Smooth Sums of Two Squares

Before embarking on the proof of Theorem 2, the main objective of this section, we require an asymptotic formula for the number of sums of two squares in arithmetic progressions. For the latter purpose we appeal to the estimate contained in Lemma 2.1 below, which follows from Iwaniec [7, Corollary 1] and Rieger [14, Satz 1].

Lemma 2.1 *Let k be a positive integer, and let l be an integer satisfying the conditions $(k,l) = 1$ and $l \equiv 1 \pmod{(4,k)}$. Then uniformly in k one has*

$$\sum_{\substack{1\le n\le x\\ n\equiv l \pmod k\\ n=\square+\square}} 1 = \frac{(4,k)}{(2,k)k} \prod_{\substack{p|k\\ p\equiv -1 \pmod 4}} (1+1/p)\frac{Bx}{\sqrt{\log x}}\left(1+O\left(\left(\frac{\log(2k)}{\log x}\right)^{1/5}\right)\right).$$

If k is a fixed integer, moreover, then the exponent $1/5$ arising in the error term may be replaced by the exponent 1.

The Proof of Theorem 2 Let z_0 be a sufficiently large, but fixed, positive number. Then the conclusion of Theorem 2 is trivial for $z \le z_0$ (although, plainly, the implicit constant

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

may depend on our choice of z_0). When $z \geq y/z_0$, on the other hand, Lemma 2.1 implies that

$$A(y,z) = \sum_{\substack{1\leq a\leq y\\ a\equiv 1 \pmod 4\\ a=\square+\square}} 1 + O\Big(\sum_{z<p\leq y} 1\Big)$$
$$= \frac{By}{2\sqrt{\log y}} + O\left(\frac{y}{\log^{3/2} y} + \frac{y}{\log z}\right).$$

On recalling (1.9) and (1.10), we therefore deduce that

$$A(y,z) = \tfrac{1}{2}\sigma(s)\frac{By}{\sqrt{\log z}} + O\left(\frac{y}{\log^{3/2} y} + \frac{y}{\log z}\right),$$

where $s = (\log y)/(\log z)$, and thus the theorem follows also in this case. Henceforth, therefore, we may suppose that $z_0 < z < y/z_0$.

We now develop an iteration procedure based on an identity similar to that of Buchstab [1]. We start by observing that when $p \equiv 1 \pmod 4$, one has that $bp = \square + \square$ if and only if $b = \square + \square$. On the other hand, when $p \equiv -1 \pmod 4$, one has that $bp = \square + \square$ if and only if $p|b$ and $b/p = \square + \square$. Suppose that z_1 is a real number satisfying $z < z_1 \leq y/z_0$. Then by collecting together terms according to their largest prime factor, and making use of the aforementioned criteria, one has

$$A(y,z) = A(y,z_1) - \sum_{z<p\leq z_1} \sum_{\substack{1\leq b\leq y/p\\ q|b\Rightarrow q\leq p\\ bp\equiv 1 \pmod 4\\ bp=\square+\square}} 1$$

(2.1)

$$= A(y,z_1) - \sum_{\substack{z<p\leq z_1\\ p\equiv 1 \pmod 4}} A(y/p,p) - \sum_{\substack{z<p\leq z_1\\ p\equiv -1 \pmod 4}} A(y/p^2,p).$$

But a trivial estimate yields

$$\sum_{\substack{z<p\leq z_1\\ p\equiv -1 \pmod 4}} A(y/p^2,p) \leq \sum_{n>z} y/n^2 \ll y/z,$$

and on substituting the latter bound into (2.1) we obtain

$$A(y,z) = A(y,z_1) - \sum_{\substack{z<p\leq z_1\\ p\equiv 1 \pmod 4}} A(y/p,p) + O(y/z). \tag{2.2}$$

We derive the asymptotic formula (1.11) by iterating (2.2). We begin by considering the situation in which $\sqrt{y} \leq z < y/z_0$, and apply (2.2) with $z_1 = y/z_0$. Write

$$s = \frac{\log y}{\log z} \quad \text{and} \quad s_1 = \frac{\log y}{\log z_1}. \tag{2.3}$$

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

We note first that when $y/p \le p$, one may estimate $A(y/p,p)$ directly by using Lemma 2.1. Thus we deduce that

$$\sum_{\substack{z<p\le z_1\\ p\equiv 1 \pmod 4}} A(y/p,p) = \sum_{\substack{z<p\le z_1\\ p\equiv 1 \pmod 4}} \left(\frac{By}{2p\sqrt{\log(y/p)}} + O\left(\frac{y}{p\log^{3/2}(y/p)}\right)\right). \tag{2.4}$$

By partial summation, moreover, one obtains from the Prime Number Theorem for arithmetic progressions the conclusion that for $r=\frac12$ or $\frac32$,

$$\begin{aligned}\sum_{\substack{z<p\le z_1\\ p\equiv 1 \pmod 4}} \frac{1}{p\log^r(y/p)} &= \tfrac12\int_z^{z_1}\frac{dx}{x\log x\log^r(y/x)} + O\left(\frac{1}{\log z}\right)\\ &= \frac{\mathcal{I}_r(s,s_1)}{2\log^r y} + O\left(\frac{1}{\log z}\right),\end{aligned}\tag{2.5}$$

where here we recall the notation (2.3), and write

$$\mathcal{I}_r(s,t) = \int_t^s \frac{du}{u(1-1/u)^r}. \tag{2.6}$$

Of course, the error term in (2.5) can be substantially improved by using a version of the Prime Number Theorem with a suitably strong error term, but such is unnecessary in the application at hand.

On recalling (2.3), we have that

$$s_1 = 1 + \frac{\log z_0}{\log(y/z_0)},$$

and thus it follows from (2.6) that

$$\begin{aligned}\mathcal{I}_{3/2}(s,s_1) = \int_{s_1}^s \frac{u^{1/2}}{(u-1)^{3/2}}\,du &\le s^{1/2}\int_{s_1}^s (u-1)^{-3/2}\,du\\ &\ll \left(\frac{\log y}{\log z}\right)^{1/2}\left(\frac{\log z_0}{\log(y/z_0)}\right)^{-1/2}.\end{aligned}$$

We therefore deduce from (2.5) that

$$\sum_{\substack{z<p\le z_1\\ p\equiv 1 \pmod 4}} \frac{1}{p\log^{3/2}(y/p)} \ll \frac{1}{\log z}. \tag{2.7}$$

Further, a modest calculation reveals that

$$\mathcal{I}_{1/2}(s,s_1) = \int_{s_1}^s \frac{du}{\sqrt{u(u-1)}} = 2\log\left(\frac{\sqrt{s}+\sqrt{s-1}}{\sqrt{s_1}+\sqrt{s_1-1}}\right),$$

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

so that whenever $1 \leq s_1 < s \leq 2$, one deduces from (1.10) and (2.5) that

$$\sum_{\substack{z<p\leq z_1\\ p\equiv 1 \pmod 4}} \frac{1}{p\log^{1/2}(y/p)} = \frac{\sqrt{s_1}\sigma(s_1)-\sqrt{s}\sigma(s)}{\sqrt{\log y}} + O\left(\frac{1}{\log z}\right). \tag{2.8}$$

On combining (2.4), (2.7) and (2.8), we obtain the estimate

$$\sum_{\substack{z<p\leq z_1\\ p\equiv 1 \pmod 4}} A(y/p,p) = \frac{By}{2\sqrt{\log y}}\big(\sqrt{s_1}\sigma(s_1)-\sqrt{s}\sigma(s)\big) + O\left(\frac{y}{\log z}\right),$$

whence by (2.2), together with the asymptotic formula (1.11) already established when $z = z_1$, we may conclude that whenever $1 < s \leq 2$,

$$\begin{aligned} A(y,z) &= \tfrac{1}{2}\sigma(s_1)\frac{By}{\sqrt{\log z_1}} - \tfrac{1}{2}By\left(\frac{\sigma(s_1)}{\sqrt{\log z_1}} - \frac{\sigma(s)}{\sqrt{\log z}}\right) + O\left(\frac{y}{\log z}\right)\\ &= \tfrac{1}{2}\sigma(s)\frac{By}{\sqrt{\log z}} + O\left(\frac{y}{\log z}\right). \end{aligned}$$

Thus far we have established (1.11) in the range $z \geq \sqrt{y}$. We now establish (1.11) in the range $2 \leq z < \sqrt{y}$ through an inductive argument. When $y \geq 2$ and $z \geq 2$, define $\Delta(y,z)$ by means of the equation

$$A(y,z) = \tfrac{1}{2}\sigma(s)\frac{By}{\sqrt{\log z}} + \frac{y}{\log z}\Delta(y,z), \tag{2.9}$$

where s is defined as in (2.3). When k is a natural number, define

$$\Delta(k) = \max\{1, \sup_{z_0\leq z\leq y\leq z^k} |\Delta(y,z)|\}.$$

We establish by induction that for each natural number k one has

$$\Delta(k) \leq 3\Delta(2). \tag{2.10}$$

That such is the case when $k = 1, 2$ is immediate from our deliberations thus far. Suppose then that k is an integer with $k \geq 2$, and that (2.10) holds. Consider a fixed value of z satisfying the condition

$$z_0^2 \leq z^2 \leq y \leq z^{k+1}.$$

On applying (2.2) with $z_1 = \sqrt{y}$ and recalling the inductive hypothesis, we obtain from (2.9) the equation

$$\begin{aligned} A(y,z) &= \tfrac{1}{2}\sigma(s_1)\frac{By}{\sqrt{\log z_1}} + \frac{y}{\log z_1}\Delta(y,z_1)\\ &\quad - \sum_{\substack{z<p\leq z_1\\ p\equiv 1 \pmod 4}}\left(\tfrac{1}{2}\sigma(s_p-1)\frac{By}{p\sqrt{\log p}} + \frac{y}{p\log p}\Delta(y/p,p)\right) + O(y/z), \end{aligned} \tag{2.11}$$

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

where s_1 is defined as in (2.3), and

$$s_p = \frac{\log y}{\log p} < s \le k + 1.$$

On making the change of variable $y = x^u$ and applying partial summation, it follows from the Prime Number Theorem for arithmetic progressions that

$$\sum_{\substack{z<p\le z_1\\ p\equiv 1 \pmod 4}} \frac{\sigma(s_p-1)}{p\sqrt{\log p}} = \tfrac{1}{2}\int_z^{z_1} \frac{\sigma(s_x-1)}{x\log^{3/2} x}\,dx + O\left(\frac{1}{\log^2 z}\right) = \frac{1}{2\sqrt{\log y}}\int_{s_1}^{s} \frac{\sigma(u-1)}{\sqrt{u}}\,du + O\left(\frac{1}{\log^2 z}\right). \tag{2.12}$$

The equation (1.8), moreover, implies that

$$\sqrt{s_1}\sigma(s_1) - \sqrt{s}\sigma(s) = \tfrac{1}{2}\int_{s_1}^{s} \frac{\sigma(u-1)}{\sqrt{u}}\,du, \tag{2.13}$$

and thus, collecting together (2.9) and (2.11)–(2.13) we arrive at the relation

$$\frac{y}{\log z}\Delta(y,z) = A(y,z) - \tfrac{1}{2}\sigma(s)\frac{By}{\sqrt{\log z}} = \frac{y}{\log z_1}\Delta(y,z_1) - \sum_{\substack{z<p\le z_1\\ p\equiv 1 \pmod 4}} \frac{y}{p\log p}\Delta(y/p,p) + O\left(\frac{y}{\log^2 z}\right). \tag{2.14}$$

But when $z < p \le z_1$ one has

$$\frac{\log(y/p)}{\log p} = s_p - 1 \le k,$$

and thus $|\Delta(y/p,p)| \le \Delta(k)$. Also, by partial summation, it follows from the Prime Number Theorem for arithmetic progressions that

$$\sum_{\substack{z<p\le z_1\\ p\equiv 1 \pmod 4}} \frac{1}{p\log p} = \tfrac{1}{2}\int_z^{z_1} \frac{dx}{x\log^2 x} + O\left(\frac{1}{\log^2 z}\right) = \frac{1}{2\log z} - \frac{1}{2\log z_1} + O\left(\frac{1}{\log^2 z}\right).$$

Thus, by (2.14),

$$|\Delta(y,z)| \le |\Delta(y,z_1)| + \tfrac{1}{2}\Delta(k) + O(1/\log z). \tag{2.15}$$

But z_0 was chosen sufficiently large, and $|\Delta(y,z_1)| \le \Delta(2)$. Then on recalling the inductive hypothesis (2.10), we deduce from (2.15) that

$$|\Delta(y,z)| \le \Delta(2) + \tfrac{3}{2}\Delta(2) + O(1/\log z_0) \le 3\Delta(2),$$

whence

$$\Delta(k+1) \leq \max\{\Delta(2), \sup_{z_0^2 \leq z^2 \leq y \leq z^{k+1}} |\Delta(y,z)|\} \leq 3\Delta(2).$$

Consequently the inductive hypothesis (2.10) holds with $k+1$ in place of k, and the induction is complete. This completes the proof of Theorem 2.

We note that a refinement of the above rather crude approach would permit us to establish asymptotic exponential decay of $\Delta(k)$, thereby extending the range in which Theorem 2 is non-trivial.

3 Some Sieve Function Identities

In order to establish Theorem 3 one has, in principle, merely to check that the sum and difference of the functions on the right hand side of (1.14) and (1.15) satisfy the same differential-difference equations, subject to the same initial conditions, as the functions on the left hand side of the latter equations. However, neither $\omega(t)$ nor $\sigma(t)$ is differentiable at $t = 1$, and thus one is forced to negotiate certain complications in order to execute such a plan. This exercise will be the object of the present section.

We first handle the difference of the functions $F(s)$ and $f(s)$.

Lemma 3.1 *For each $s > 0$ one has*

$$F(s) - f(s) = 2\sqrt{e^\gamma/\pi}\,\sigma(s). \tag{3.1}$$

Proof In view of equations (1.5) and (1.9), the equation (3.1) is satisfied trivially for $0 < s \leq 1$. When $1 < s \leq 2$, moreover, the equations (1.4) yield

$$\big(s^{1/2} f(s)\big)' = \frac{\sqrt{e^\gamma/\pi}}{\sqrt{s(s-1)}},$$

whence by (1.5) and (1.10),

$$\begin{aligned} s^{1/2} f(s) &= \sqrt{e^\gamma/\pi} \int_1^s \frac{dt}{\sqrt{t(t-1)}} = 2\sqrt{e^\gamma/\pi}\log(\sqrt{s} + \sqrt{s-1}) \\ &= 2\sqrt{e^\gamma/\pi}\big(1 - s^{1/2}\sigma(s)\big). \end{aligned}$$

On recalling (1.5), therefore, one finds that equation (3.1) holds also in the range $1 < s \leq 2$. Finally, when $s > 2$ the equations (1.4) yield

$$s\frac{d}{ds}\big((F-f)(s)\big) = -\tfrac{1}{2}\big((F-f)(s) + (F-f)(s-1)\big).$$

Thus $F(s) - f(s)$ is identical with $2\sqrt{e^\gamma/\pi}\,\sigma(s)$ for $0 < s \leq 2$, and, by (1.8), satisfies the same differential-difference equation as does the latter function for $s > 2$. The desired conclusion therefore follows from the continuity of the respective functions.

Next we investigate the sum of the functions $F(s)$ and $f(s)$.

Lemma 3.2 *For each $s > 0$ one has*

$$F(s) + f(s) = 2\sqrt{e^\gamma/\pi}\left(\sigma(s) + \int_0^s \omega(t)\sigma(s-t)\,dt\right).$$

Proof We begin by extending the definition of $\sigma(s)$ provided in the introduction by taking $\sigma(s) = 0$ for $s \leq 0$. It then follows from (1.8) and (1.9) that

$$s\sigma'(s) = -\tfrac{1}{2}\big(\sigma(s) + \sigma(s-1)\big) \tag{3.2}$$

for $s \in \mathbb{R}\setminus\{0,1\}$. Similarly, if we extend the definition of $\omega(s)$ provided in the introduction by taking $\omega(s) = 0$ for $s < 1$, then it follows from (1.12) and (1.13) that

$$s\omega'(s) = \omega(s-1) - \omega(s) \tag{3.3}$$

for $s \in \mathbb{R} \setminus \{1,2\}$. It will be convenient in our argument also to extend the definitions of $\sigma'(s)$ and $\omega'(s)$ so that $\sigma'(1) = -\frac{1}{2}$, $\omega'(1) = -1$, $\omega'(2) = 0$ and $\sigma'(0) = \omega'(0) = 0$. This latter convention results in only cosmetic consequences, the occurrences of these values of the derivatives being smoothed away by integration.

Define the function $G(s)$ for real s by

$$G(s) = \sigma(s) + \int_0^s \omega(t)\sigma(s-t)\,dt. \tag{3.4}$$

Then one has $G(s) = 0$ for $s \leq 0$ and $G(s) = s^{-1/2}$ for $0 \leq s \leq 1$. Moreover when $1 < s \leq 2$, the equations (1.4) and (1.5) together imply that

$$\big(s^{1/2}F(s)\big)' = \tfrac{1}{2}s^{-1/2}f(s-1),$$

whence

$$\big(s^{1/2}(F+f)(s)\big)' = \tfrac{1}{2}s^{-1/2}(F+f)(s-1)$$

for $1 < s \leq 2$. It follows that $F(s) + f(s) = 2\sqrt{e^\gamma/\pi}G(s)$ for $0 < s \leq 1$, and that to complete the proof of the lemma it suffices to show that $G(s)$ satisfies the differential-difference equation satisfied by $F(s) + f(s)$. Thus we aim to show that for $s > 1$,

$$sG'(s) = -\tfrac{1}{2}\big(G(s) - G(s-1)\big). \tag{3.5}$$

Let s be any real number exceeding 1, and consider

$$I(s) = \frac{d}{ds}\int_0^s \omega(t)\sigma(s-t)\,dt. \tag{3.6}$$

One has

$$\begin{aligned} I(s) &= \frac{d}{ds}\left(s\int_0^1 \omega(su)\sigma\big(s(1-u)\big)\,du\right) \\ &= \int_0^1 \omega(su)\sigma\big(s(1-u)\big)\,du + s\lim_{h\to 0}\frac{J(h)}{h}, \end{aligned} \tag{3.7}$$

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

where

$$J(h) = \int_0^1 \omega\big((s+h)u\big)\sigma\big((s+h)(1-u)\big)\,du - \int_0^1 \omega(su)\sigma\big(s(1-u)\big)\,du.$$

Now $\omega(t)$ is zero for $t < 1$, and $\sigma(t)$ is differentiable for $t \neq 0, 1$. Thus, when $h > 0$ one has

$$\begin{aligned} J(h) = \int_{1/s}^1 &\omega\big((s+h)u\big)\sigma\big((s+h)(1-u)\big) - \omega(su)\sigma\big(s(1-u)\big)\,du \\ &+ \int_{1/(s+h)}^{1/s} \omega\big((s+h)u\big)\sigma\big((s+h)(1-u)\big)\,du, \end{aligned}$$

whence

$$\begin{aligned} \lim_{h\to 0+} \frac{J(h)}{h} &= \int_{1/s}^1 u\omega'(su)\sigma\big(s(1-u)\big) + (1-u)\omega(su)\sigma'\big(s(1-u)\big)\,du + \frac{\sigma(s-1)}{s^2} \\ &= \frac{1}{s^2}\int_1^s t\omega'(t)\sigma(s-t) + (s-t)\omega(t)\sigma'(s-t)\,dt + \frac{\sigma(s-1)}{s^2}. \end{aligned} \tag{3.8}$$

For positive values of t, define

$$\tilde{\omega}(t) = \begin{cases} 1/t, & \text{when } 0 < t < 1, \\ \omega(t), & \text{when } t \geq 1. \end{cases}$$

When h is a small number with $h < 0$, one has

$$\begin{aligned} J(h) = \int_{1/s}^1 &\tilde{\omega}\big((s+h)u\big)\sigma\big((s+h)(1-u)\big) - \omega(su)\sigma\big(s(1-u)\big)\,du \\ &- \int_{1/s}^{1/(s+h)} \frac{1}{(s+h)u}\sigma\big((s+h)(1-u)\big)\,du, \end{aligned}$$

and thus one similarly obtains

$$\lim_{h\to 0-} \frac{J(h)}{h} = \frac{1}{s^2}\int_1^s t\omega'(t)\sigma(s-t) + (s-t)\omega(t)\sigma'(s-t)\,dt + \frac{\sigma(s-1)}{s^2}. \tag{3.9}$$

On combining (3.8) and (3.9), and invoking (3.2) and (3.3), we deduce that

$$\begin{aligned} s^2 \lim_{h\to 0} \frac{J(h)}{h} &= \sigma(s-1) + \int_1^s \big(\omega(t-1) - \omega(t)\big)\sigma(s-t)\,dt \\ &\quad - \tfrac{1}{2}\int_1^s \omega(t)\big(\sigma(s-t) + \sigma(s-t-1)\big)\,dt \\ &= \sigma(s-1) - \tfrac{3}{2}\int_1^s \omega(t)\sigma(s-t)\,dt + \tfrac{1}{2}\int_0^{s-1} \omega(u)\sigma(s-1-u)\,du. \end{aligned}$$

We may therefore conclude from (3.7) that

$$sI(s) = \sigma(s-1) - \tfrac{1}{2}\int_0^s \omega(t)\sigma(s-t)\,dt + \tfrac{1}{2}\int_0^{s-1} \omega(u)\sigma(s-1-u)\,du.$$

On recalling (3.2), (3.4) and (3.6), therefore, we deduce that when $s > 1$ one has

$$\begin{aligned} sG'(s) + \tfrac{1}{2}G(s) &= s\sigma'(s) + \tfrac{1}{2}\sigma(s) + sI(s) + \tfrac{1}{2}\int_0^s \omega(t)\sigma(s-t)\,dt \\ &= \tfrac{1}{2}\sigma(s-1) + \tfrac{1}{2}\int_0^{s-1} \omega(t)\sigma(s-1-t)\,dt = \tfrac{1}{2}G(s-1). \end{aligned}$$

The equation (3.5) is thus satisfied for $s > 1$, and the lemma follows.

Theorem 3 is immediate from the conclusions of Lemmata 3.1 and 3.2.

4 Dense and Sparse Rectangles

Our basic strategy now is to construct a rectangle, which is to say a set of disjoint intervals of length $\log^N x$ equally spaced apart in $(1, x]$, which contains $F(N)$ times the expected number of sums of two squares. It follows by a box principle that at least one of the aforementioned intervals contains $F(N)$ times the expected number of sums of two squares, whence the first conclusion of Theorem 1 follows. The second conclusion of Theorem 1 follows via a conjugate argument.

Before describing the key machinery of our argument we require some technical estimates. When y and z are positive real numbers, define

$$B^{\pm}(y,z) = \sum_{\substack{1<b\leq y \\ b\equiv\pm 1 \pmod 4 \\ p|b \Rightarrow p>z}} 1.$$

Lemma 4.1 *Uniformly for $y \geq z \geq 2$, one has*

$$B^{\pm}(y,z) = \frac{\omega(u)y - z}{2\log z} + O\left(\frac{y}{\log^2 z}\right),$$

where $u = \log y/\log z$.

Proof Work of Buchstab [1] would suffice to provide the asymptotic formula stated in the lemma, though with a weaker error term. In order to obtain the stated conclusion, we note on the one hand that by a classical estimate (see, for example, Tenenbaum [15, Theorem 3 of Chapter III.6]),

$$B^{+}(y,z) + B^{-}(y,z) = \sum_{\substack{1<b\leq y \\ p|b \Rightarrow p>z}} 1 = \frac{\omega(s)y - z}{\log z} + O\left(\frac{y}{\log^2 z}\right), \tag{4.1}$$

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

where $s = \log y/\log z$, and the implicit constant is absolute. On the other hand, by using a Buchstab iteration one has

$$\begin{aligned} B^+(y,z) - B^-(y,z) = \sum_{z<p\leq y} \sum_{\substack{1\leq\nu\leq\frac{\log y}{\log p} \\ p^\nu\equiv 1 \pmod 4}} \big(B^+(y/p^\nu,p) + 1 - B^-(y/p^\nu,p)\big) \\ + \sum_{z<p\leq y} \sum_{\substack{1\leq\nu\leq\frac{\log y}{\log p} \\ p^\nu\equiv -1 \pmod 4}} \big(B^-(y/p^\nu,p) - B^+(y/p^\nu,p) - 1\big). \end{aligned} \tag{4.2}$$

When $z \geq y$ one has $B^{\pm}(y,z) = 0$, and when $z < y \leq z^2$ it follows from the Prime Number Theorem for arithmetic progressions that

$$B^+(y,z) - B^-(y,z) \ll y/\log^2 z.$$

Suppose that for some absolute constant C, whenever $z^n \geq y$ one has

$$B^+(y,z) - B^-(y,z) \leq Cy/\log^2 z. \tag{4.3}$$

Then the identity (4.2), combined with standard prime number estimates, shows that whenever $z^{n+1} \geq y$ one has

$$B^+(y,z) - B^-(y,z) \leq Cy \sum_{p>z} \frac{1}{p\log^2 p} + O\left(\frac{y}{z}\right) + o\left(\frac{y}{\log^2 y}\right).$$

By making our initial choice for C sufficiently large, we may plainly suppose without loss of generality that z is sufficiently large. In the latter circumstance, moreover, one may combine a standard partial summation argument together with a version of the Prime Number Theorem with error term to obtain

$$\sum_{p>z} \frac{1}{p\log^2 p} = \frac{\frac{1}{2} + O(1/\log z)}{\log^2 z} < \frac{3}{4\log^2 z}.$$

Consequently (4.3) holds in the wider range $z^{n+1} \geq y$. Thus, by induction, the inequality (4.3) holds uniformly in y and z. The lemma follows on combining (4.1) and (4.3).

We next provide an estimate for weighted sums over the sets $B^{\pm}(y,z)$.

Lemma 4.2 *Let y, z and u be real numbers with $2 \leq z \leq u \leq y$, and let $g(t)$ be a non-negative continuously differentiable function of a real variable t, monotonic on (u,y). Then*

$$\sum_{\substack{u<b\leq y \\ b\equiv\pm 1 \pmod 4 \\ p|b \Rightarrow p>z}} \frac{g(b)}{b} - \frac{1}{2\log z}\int_u^y \frac{g(t)}{t}\omega\left(\frac{\log t}{\log z}\right) dt \ll \frac{g(y)+g(u)}{\log^2 z} + \int_u^y \frac{g(t)}{t\log^2 z}\, dt.$$

Here the implicit constant is absolute.

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

Proof When t is a real number with $u \le t \le y$, define

$$s_t = \frac{\log t}{\log z}.$$

By a standard partial summation argument one has

$$\sum_{\substack{u<b\le y\\ b\equiv\pm1 \pmod 4\\ p|b\Rightarrow p>z}} \frac{g(b)}{b} = \frac{g(y)}{y}B^{\pm}(y,z) - \frac{g(u)}{u}B^{\pm}(u,z) - \int_u^y B^{\pm}(t,z)\left(\frac{g(t)}{t}\right)' dt. \tag{4.4}$$

Moreover, from Lemma 4.1,

(4.5)
$$\int_u^y B^{\pm}(t,z)\left(\frac{g(t)}{t}\right)' dt - \int_u^y \frac{\omega(s_t)t - z}{2\log z}\left(\frac{g(t)}{t}\right)' dt \ll \int_u^y \frac{t}{\log^2 z}\left|\left(\frac{g(t)}{t}\right)'\right| dt.$$

Observe that $\big(g(t)/t\big)' = g'(t)/t - g(t)/t^2$. Furthermore, $\omega(s)$ is piecewise continuously differentiable for $s \in (1,\infty)$, with $|\omega'(s)| \le 1$, and

$$\big(\omega(s_t)t\big)' = \omega'(s_t)/\log z + \omega(s_t).$$

Then applying integration by parts in (4.5), one obtains from (4.4) and Lemma 4.1 that

$$\sum_{\substack{u<b\le y\\ b\equiv\pm1 \pmod 4\\ p|b\Rightarrow p>z}} \frac{g(b)}{b} - \int_u^y \frac{g(t)}{2t\log z}\big(\omega(s_t)t\big)' dt \ll \frac{g(y)+g(u)}{\log^2 z} + \int_u^y \frac{t}{\log^2 z}\left|\left(\frac{g(t)}{t}\right)'\right| dt,$$

whence

$$\sum_{\substack{u<b\le y\\ b\equiv\pm1 \pmod 4\\ p|b\Rightarrow p>z}} \frac{g(b)}{b} - \int_u^y \frac{g(t)}{2t\log z}\omega(s_t)\, dt \ll \frac{g(y)+g(u)}{\log^2 z} + \int_u^y \frac{g(t)}{t\log^2 z}\, dt.$$

This completes the proof of the lemma.

In preparation for the main thrust of our argument, we pause to fix some notation. We take x to be a large real number, and fix real numbers y and z with $2 \le y < x^{1/4}$ and $2 \le z < x^{1/4}$. We write

$$s = \log y/\log z. \tag{4.6}$$

When p is a prime number, we define α_p to be the unique odd integer with the property that

$$p^{\alpha_p} > 4y + 1 \ge p^{\alpha_p - 2},$$

and then define

$$P = \prod_{\substack{p \le z \\ p \equiv -1 \pmod 4}} p^{\alpha_p}. \tag{4.7}$$

Note that by using the Prime Number Theorem for arithmetic progressions, one has

$$sz \ll \log P \ll (s+1)z. \tag{4.8}$$

We define the integers $P_\pm$ by

$$P_\pm = \begin{cases} \frac{1}{4}(P \pm 1), & \text{when } P \equiv \mp 1 \pmod 4, \\ \frac{1}{4}(3P \pm 1), & \text{when } P \equiv \pm 1 \pmod 4. \end{cases} \tag{4.9}$$

Thus $P_\pm$ satisfies the congruence $4P_\pm \equiv \pm 1 \pmod P$. Finally we define our rectangle, $\mathcal{M}_\pm = \mathcal{M}_\pm(x, y, z)$, by taking

$$\mathcal{M}_\pm = \{1 \le n \le x : n \equiv P_\pm + r \pmod P, 1 \le r \le y\}. \tag{4.10}$$

We remark that P_+ has been chosen in such a way that the chance that a residue class $P_+ + r$ contains elements of $\mathcal{S}$ is increased. Thus $\mathcal{M}_+$ should contain more than the "expected" number of elements of $\mathcal{S}$. We undertake a similar construction, using an analogous rectangle based on the number P_-, in order to reduce the chance that a residue class $P_- + r$ contains elements of $\mathcal{S}$. Such an idea has been exploited by Richards [13] to show that there are unusually large gaps, with length the square of the average gap, between successive sums of two squares.

Our aim is to evaluate the number of those integers in $\mathcal{M}_\pm$ representable as a sum of two squares. Thus we evaluate the sum

$$S^\pm(x, y, z) = \operatorname{card}\{n \in \mathcal{M}_\pm(x, y, z) : n = \square + \square\}. \tag{4.11}$$

Lemma 4.3 *With the above hypotheses and notation, one has*

$$S^+(x, y, z) = \frac{Bxy}{P\sqrt{\log x}} \left(F(s) + O\left(\frac{s+1}{\sqrt{\log z}} \right) \right) \left(1 + O\left(\left(\frac{(s+1)z}{\log x} \right)^{1/5} \right) \right)$$

and

$$S^-(x, y, z) = \frac{Bxy}{P\sqrt{\log x}} \left(f(s) + O\left(\frac{s+1}{\sqrt{\log z}} \right) \right) \left(1 + O\left(\left(\frac{(s+1)z}{\log x} \right)^{1/5} \right) \right).$$

Here, the implicit constants are absolute.

Proof Write $S^{\pm}$ for $S^{\pm}(x,y,z)$. Then by (4.10) and (4.11) one has

$$S^{\pm} = \sum_{1\leq r\leq y} \sum_{\substack{1\leq n\leq x\\ n\equiv P_{\pm}+r \ (\mathrm{mod}\, P)\\ n=\Box+\Box}} 1. \tag{4.12}$$

Since P is odd, for each value of r counted in the first summation of (4.12) one has

$$(P_{\pm}+r,P) = (4P_{\pm}+4r,P) = (4r\pm 1,P).$$

But if $p^{\beta}\|(4r\pm 1,P)$ and $\beta>0$, then $p^{\beta}\leq 4y\pm 1$ and $p|P$, whence by (4.7) one has $p^{\beta+1}|P$. Consequently, for each integer n with $n\equiv P_{\pm}+r \pmod P$, whenever $p^{\beta}\|n$ one has either $2|\beta$ or $n\neq\Box+\Box$. Thus, there is either some divisor d of P with $d^2|P$ and $(P_{\pm}+r,P)=d^2$, or else the inner sum of (4.12) is empty. Moreover, since the α_p are odd, when $d^2|P$ the integer P/d^2 has precisely the same prime factors as does P. Furthermore, for values of r making a non-trivial contribution to the inner sum of (4.12), one has $(P_{\pm}+r,P)=d^2$ whenever $d^2|P$ and there is an integer u with $4r\pm 1=d^2u$ and $(u,P)=1$. In this situation the conditions

$$n\equiv P_{\pm}+r \ (\mathrm{mod}\, P) \quad \text{and} \quad n=\Box+\Box,$$

are equivalent to the condition that there is an integer m with

$$n=d^2m,\quad m\equiv (P_{\pm}+r)/d^2 \ (\mathrm{mod}\, P/d^2) \quad \text{and} \quad m=\Box+\Box.$$

Making use of the deliberations of the previous paragraph, we can rewrite (4.12) in the shape

$$S^{\pm} = \sum_{d^2|P} \sum_{\substack{1\leq u\leq (4y\pm 1)/d^2\\ u\equiv \pm 1 \ (\mathrm{mod}\, 4)\\ (u,P)=1}} \sum_{\substack{1\leq m\leq x/d^2\\ 4m\equiv u \ (\mathrm{mod}\, P/d^2)\\ m=\Box+\Box}} 1, \tag{4.13}$$

whence by Lemma 2.1 we deduce that

(4.14)
$$\begin{aligned} S^{\pm} &= \sum_{d^2|P} \sum_{\substack{1\leq u\leq (4y\pm 1)/d^2\\ u\equiv \pm 1 \ (\mathrm{mod}\, 4)\\ (u,P)=1}} \prod_{p|P}(1+1/p)\frac{Bx}{P\sqrt{\log(x/d^2)}}\left(1+O\left(\left(\frac{\log(2P/d^2)}{\log(x/d^2)}\right)^{1/5}\right)\right)\\ &= R^{\pm}\prod_{p|P}(1+1/p)\frac{Bx}{P\sqrt{\log x}}\left(1+O\left(\left(\frac{(s+1)z}{\log x}\right)^{1/5}+\frac{\log y}{\log x}\right)\right), \end{aligned}$$

where we write

$$R^{\pm} = \sum_{d^2|P} \sum_{\substack{1\leq r\leq (4y\pm 1)/d^2\\ r\equiv \pm 1 \ (\mathrm{mod}\, 4)\\ (r,P)=1}} 1. \tag{4.15}$$

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

We remark for future reference that the second error term in (4.14) is majorized by the first one.

We next estimate $R^{\pm}$. Observe first that each integer r counted in the second summation of (4.15) may be written uniquely in the form $r = a'b$, where

$$p|a' \Rightarrow p \le z \quad \text{and} \quad p|b \Rightarrow p > z.$$

Further, the summation condition $(r, P) = 1$ implies that $(a', P) = 1$, whence a' is a product of primes in the congruence class 1 modulo 4. In particular, $a' \equiv 1 \pmod 4$, whence $b \equiv \pm 1 \pmod 4$. Finally, on writing $a = a'd^2$, we may rewrite the expressions rd^2 occurring implicitly in the summation conditions of (4.15) in the shape $rd^2 = ab$, where $a \equiv 1 \pmod 4$, $p|a \Rightarrow p \le z$, and $rd^2 = ab$ is counted in the sum if and only if $a = \square + \square$. Thus we arrive at the conclusion

$$R^{\pm} = \sum_{\substack{1 \le b \le 4y \pm 1 \\ b \equiv \pm 1 \pmod 4 \\ p|b \Rightarrow p > z}} \sum_{\substack{1 \le a \le (4y\pm 1)/b \\ a \equiv 1 \pmod 4 \\ p|a \Rightarrow p \le z \\ a = \square + \square}} 1 = \sum_{\substack{1 \le b \le 4y \pm 1 \\ b \equiv \pm 1 \pmod 4 \\ p|b \Rightarrow p > z}} A\big((4y \pm 1)/b, z\big). \tag{4.16}$$

But when $b > y$ one plainly has that $A\big((4y \pm 1)/b, z\big)$ is either 0 or 1, and thus by Theorem 2 and Lemma 4.1 we have

$$R^{\pm} = \sum_{\substack{1 \le b \le y \\ b \equiv \pm 1 \pmod 4 \\ p|b \Rightarrow p > z}} A\big((4y \pm 1)/b, z\big) + O\big(B^{\pm}(4y \pm 1, z)\big) = T_1^{\pm} + O(T_2^{\pm}), \tag{4.17}$$

where

$$T_1^{\pm} = \frac{B(4y \pm 1)}{2\sqrt{\log z}} \sum_{\substack{1 \le b \le y \\ b \equiv \pm 1 \pmod 4 \\ p|b \Rightarrow p > z}} \frac{1}{b} \sigma\left(\frac{\log\big((4y \pm 1)/b\big)}{\log z}\right) \tag{4.18}$$

and

$$T_2^{\pm} = \frac{y}{\log z} + \sum_{\substack{1 \le b \le y \\ b \equiv \pm 1 \pmod 4 \\ p|b \Rightarrow p > z}} \left(\frac{y}{b \log z} + \frac{y}{b \log^{3/2}\big((4y \pm 1)/b\big)}\right). \tag{4.19}$$

We first estimate the contribution to $R^{\pm}$ arising from $T_2^{\pm}$. Note first that in the summation occurring in (4.19), one has either $b = 1$ or else $b > z$, and indeed the former case does not arise when evaluating T_1^{-} or T_2^{-}. Moreover in the latter case the summation is empty unless $z < y$. Thus two applications of Lemma 4.2, with $g(t) = 1$ and

$g(t) = 1/\log^{3/2}\big((4y \pm 1)/t\big)$, yield the conclusion

$$(4.20)\qquad \begin{aligned} T_2^{\pm} &\ll \frac{y}{\log z} + \sum_{\substack{z<b\le y \\ b\equiv \pm 1 \pmod 4 \\ p\mid b \Rightarrow p>z}} \left(\frac{y}{b\log z} + \frac{y}{b\log^{3/2}\big((4y\pm 1)/b\big)} \right) \\ &\ll \frac{y}{\log z} + \frac{y\log y}{\log^2 z} \ll \frac{(s+1)y}{\log z}. \end{aligned}$$

Next we estimate the main term $T_1^{\pm}$. Observe that $\sigma(s)$ is positive, monotone decreasing, differentiable and has derivative continuous everywhere in $(0,\infty)$ except at $s = 1$. Thus we may divide up the range of summation in (4.18) into at most finitely many intervals, in each of which

$$g^{\pm}(t) = \sigma\left(\frac{\log\big((4y\pm 1)/t\big)}{\log z} \right)$$

satisfies the hypotheses necessary to apply Lemma 4.2. We therefore apply Lemma 4.2 and sum the contributions from each of the latter intervals. Observe that the above choice of $g^{\pm}(t)$ is itself monotonic on (z, y), and hence the error terms arising from our applications of Lemma 4.2 will be majorized by the error terms arising from the whole of the interval (z, y). We may therefore conclude that

$$(4.21)\qquad \sum_{\substack{z<b\le y \\ b\equiv \pm 1 \pmod 4 \\ p\mid b \Rightarrow p>z}} \frac{1}{b}\sigma\left(\frac{\log\big((4y\pm 1)/b\big)}{\log z} \right) = T_3^{\pm} + O(T_4^{\pm}),$$

where $T_3^{\pm}$ is zero unless $y > z$, in which case

$$(4.22)\qquad T_3^{\pm} = \frac{1}{2\log z}\int_z^y \frac{1}{t}\sigma\left(\frac{\log\big((4y\pm 1)/t\big)}{\log z} \right)\omega\left(\frac{\log t}{\log z}\right)\,dt$$

and

$$(4.23)\qquad T_4^{\pm} = \frac{1}{\log^2 z}\left(\sigma\left(\frac{\log\big((4y\pm 1)/y\big)}{\log z} \right) + \int_z^y \sigma\left(\frac{\log\big((4y\pm 1)/t\big)}{\log z} \right)\frac{dt}{t} \right).$$

Make the change of variables $t = z^u$ in (4.22), and write

$$\varepsilon^{\pm} = \frac{\log(4 \pm 1/y)}{\log z}.$$

One obtains

$$T_3^{\pm} = \tfrac{1}{2}\int_1^s \sigma(s + \varepsilon^{\pm} - u)\omega(u)\,du,$$

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press

and thus the estimate $\varepsilon^{\pm} = O(1/\log z)$, together with the properties of $\sigma(t)$ previously discussed, lead to the asymptotic formula

$$T_3^{\pm} = \tfrac{1}{2}\int_1^s \sigma(s-u)\omega(u)\,du + O(1/\log z). \tag{4.24}$$

Similarly, one obtains from (4.23) together with (1.9) and (1.16) the estimate

$$T_4^{\pm} \ll \frac{1}{\log^2 z}\left(\sqrt{\log z} + \log z \int_1^s \sigma(s+\varepsilon^{\pm}-u)\,du\right) \ll \frac{1}{\log z}. \tag{4.25}$$

Finally, on combining (4.18), (4.21), (4.24) and (4.25), we arrive at the conclusions

$$T_1^{+} = \frac{2By}{\sqrt{\log z}}\left(\sigma\left(\frac{\log(4y+1)}{\log z}\right) + \tfrac{1}{2}\int_1^s \omega(u)\sigma(s-u)\,du + O\left(\frac{1}{\sqrt{\log z}}\right)\right)$$

and

$$T_1^{-} = \frac{By}{\sqrt{\log z}}\left(\int_1^s \omega(u)\sigma(s-u)\,du + O\left(\frac{1}{\sqrt{\log z}}\right)\right).$$

On recalling (4.17) and (4.20), we therefore obtain

$$R^{+} = \frac{By}{\sqrt{\log z}}\left(2\sigma(s) + \int_1^s \omega(u)\sigma(s-u)\,du + O\left(\frac{s+1}{\sqrt{\log z}}\right)\right) \tag{4.26}$$

and

$$R^{-} = \frac{By}{\sqrt{\log z}}\left(\int_1^s \omega(u)\sigma(s-u)\,du + O\left(\frac{s+1}{\sqrt{\log z}}\right)\right). \tag{4.27}$$

We may now insert the estimates for $R^{\pm}$ contained in (4.26) and (4.27) into (4.14), and hence, on recalling Theorem 3, obtain

$$S^{\pm} = \mathcal{C}\mathcal{D}\left(\sqrt{\frac{\pi}{e^{\gamma}}}\mathcal{F}^{\pm}(s) + O\left(\frac{s+1}{\sqrt{\log z}}\right)\right)\left(1 + O\left(\left(\frac{(s+1)z}{\log x}\right)^{1/5}\right)\right), \tag{4.28}$$

where $\mathcal{F}^{+}(s)$ denotes $F(s)$ and $\mathcal{F}^{-}(s)$ denotes $f(s)$, and where we write

$$\mathcal{C} = \prod_{p\mid P}(1+1/p) \quad\text{and}\quad \mathcal{D} = \frac{Bx}{P\sqrt{\log x}}\frac{By}{\sqrt{\log z}}.$$

However, by combining Mertens' formula with the formula of Leibniz for $L(1,\chi)$, where χ is the non-principal character modulo 4, on recalling (1.2) we have

$$\begin{aligned}\mathcal{C} &= \prod_{p\mid P}(1-1/p^2)^{1/2}\cdot 2^{-1/2}\prod_{p\le z}(1-1/p)^{-1/2}\prod_{p\le z}\bigl(1-\chi(p)/p\bigr)^{1/2}\\ &= (2B)^{-1}(e^{\gamma}\log z)^{1/2}(\pi/4)^{-1/2}\bigl(1+O(1/\log z)\bigr).\end{aligned}$$

On substituting the latter estimate into (4.28), we finally reach the conclusion

$$S^{\pm} = \frac{Bxy}{P\sqrt{\log x}} \left(\mathcal{F}^{\pm}(s) + O\left(\frac{s+1}{\sqrt{\log z}} \right) \right) \left(1 + O\left(\left(\frac{(s+1)z}{\log x} \right)^{1/5} \right) \right),$$

and the lemma follows immediately.

We are now prepared, at last, to prove Theorem 1. We take N to be a fixed positive number, and take

$$y = \log^N x \quad \text{and} \quad z = \frac{\log x}{\log\log x}, \tag{4.29}$$

so that

$$s = \frac{\log y}{\log z} = N + O\left(\frac{\log\log\log x}{\log\log x} \right),$$

$$F(s) = F(N) + O\left(\frac{\log\log\log x}{\log\log x} \right) \quad \text{and} \quad f(s) = f(N) + O\left(\frac{\log\log\log x}{\log\log x} \right).$$

Further,

$$\frac{N \log x}{\log\log x} \ll \log P \ll \frac{(N+1)\log x}{\log\log x}.$$

By Lemma 4.3, therefore,

$$S^{+}(x,y,z) - S^{+}(\tfrac{1}{2}x,y,z) = \frac{Bxy}{2P\sqrt{\log x}} \Big(F(N) + O_N\big((\log\log x)^{-1/5} \big) \Big) \tag{4.30}$$

and

$$S^{-}(x,y,z) - S^{-}(\tfrac{1}{2}x,y,z) = \frac{Bxy}{2P\sqrt{\log x}} \Big(f(N) + O_N\big((\log\log x)^{-1/5} \big) \Big). \tag{4.31}$$

Moreover $\mathcal{M}_{\pm}(x,y,z)$ consists of $x/P + O(1)$ disjoint intervals of length y. We may therefore conclude from (4.29)–(4.31) that at least one of the intervals of length y included in $\mathcal{M}_{+}(x,y,z)$, but not included in $\mathcal{M}_{+}(\frac{1}{2}x,y,z)$, contains at least

$$\frac{By}{\sqrt{\log x}} \big(F(N) + o_N(1) \big)$$

sums of two squares, and similarly at least one of the intervals of length y included in $\mathcal{M}_{-}(x,y,z)$, but not included in $\mathcal{M}_{-}(\frac{1}{2}x,y,z)$, contains at most

$$\frac{By}{\sqrt{\log x}} \big(f(N) + o_N(1) \big)$$

sums of two squares. Theorem 1 follows immediately.

References

[1] A. A. Buchstab, *An asymptotic estimate of a number theoretic function.* Mat. Sbornik N. S. **44**(1937), 1239–1246.
[2] J. B. Friedlander, *Sifting short intervals.* Math. Proc. Cambridge Philos. Soc. **91**(1982), 9–15.
[3] ———, *Sifting short intervals. II.* Math. Proc. Cambridge Philos. Soc. **92**(1982), 381–384.
[4] A. Granville, *Unexpected irregularities in the distribution of prime numbers.* In: Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, 388–399.
[5] H. Halberstam and H.-E. Richert, *Sieve methods.* London Mathematical Soc. Monographs **4**, Academic Press, London-New York, 1974.
[6] C. Hooley, *On the intervals between numbers that are sums of two squares, IV.* J. Reine Angew. Math. **452**(1994), 79–109.
[7] H. Iwaniec, *The half dimensional sieve.* Acta Arith. **29**(1976), 69–95.
[8] E. Landau, *Handbuch der Lehre der Verteilung der Primzahlen, Bd. 2.* Teubner, Leipzig-Berlin, 1909.
[9] H. Maier, *Primes in short intervals.* Michigan Math. J. **32**(1985), 221–225.
[10] P. Moree, *On the number of y-smooth natural numbers $\leq x$ representable as a sum of two integer squares.* Manuscripta Math. **80**(1993), 199–211.
[11] V. A. Plaksin, *The distribution of numbers that can be represented as the sum of two squares.* Izv. Akad. Nauk SSSR Ser. Mat. **51**(1987), 860–877.
[12] ———, *Letter to the editors: "The distribution of numbers that can be represented as the sum of two squares"* [*Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987), 860–877, 911*]. Izv. Akad. Nauk SSSR Ser. Mat. **56**(1992), 908–909.
[13] I. Richards, *On the gaps between numbers which are sums of two squares.* Adv. in Math. **46**(1982), 1–2.
[14] G. J. Rieger, *Über die Anzahl der als Summe von zwei Quadraten darstellbaren und in einer primen Restklasse gelegenen Zahlen unterhalb einer positiven Schranke II.* J. Reine Angew. Math. **217**(1965), 200–216.
[15] G. Tenenbaum, *Introduction to analytic and probabilistic number theory.* Cambridge Stud. Adv. Math. **46**, Cambridge University Press, Cambridge, 1995.
[16] F. S. Wheeler, *Two differential-difference equations arising in number theory.* Trans. Amer. Math. Soc. **318**(1990), 491–523.

Mathematical Institute
Budapest 1364
Hungary
email: balog@math-inst.hu

Department of Mathematics
University of Michigan
East Hall
525 East University Avenue
Ann Arbor, MI 48109-1109
USA
email: wooley@math.lsa.umich.edu

https://doi.org/10.4153/CJM-2000-029-6 Published online by Cambridge University Press