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DUALITY BETWEEN DISTANT POINT AND MEDIAN OF A TREE NETWORK SPACE

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Abstract

A point on a tree network space is said to be a distant point if it maximises its minimum weighted distance from any of its vertices. The median minimises the sum of its weighted distances from the vertices. In this paper two constrained problems are discussed. The first problem is to maximise the minimum of the weighted distances from the vertices subject to an upper bound value of the sum of the weighted distances from the vertices, while the second problem is to minimise the sum of the weighted distances subject to a lower bound value of the minimum weighted distance to any of its vertices. It is shown that these two constrained problems are duals of each other in a well defined sense.

1. Introduction

A graph is defined to be an ordered pair (V, E) where V is a set of vertices and E is a set of (binary relations defined on the elements of V) edges. A tree is a connected graph that does not contain any loop, and therefore any two vertices of a connected tree are connected by the unique path. The concept of a tree (and graph also) is essentially discrete in nature. In this paper we consider a metric space T defined on an undirected connected tree (V, E) which is essentially continuous in nature. A positive weight $w(v)$ is associated to each vertex $v \in V$. A point $x \in T$ may be a vertex or a point on an edge. For any two points $x, y \in T$, $\phi(x, y)$ denotes the length (distance or metric) of the unique path connecting these two points (as discussed in [6]) in T. Whenever the end vertices u and v of an edge e need to be specifically identified, the edge will be denoted by $e(u, v)$ and its length by $\phi(u, v)$. A number of real life situations suggest that a network space is a more faithful representation of the reality as compared to the

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Euclidean space. For example, in a road network or a communication network or a pipeline system, the travel occurs along the underlying edges rather than along the Euclidean distances (straight lines) between two point locations.

A system of roads, gas pipelines or transportations link between specified points of interest is often modelled as a network space for purposes of analysis, planning and evaluation. A network space is considered here, where the nodes correspond to locations of existing facilities and the edges represent the transportation links between the existing facilities. Each edge of the network space has a known positive length that corresponds to the travel distance or cost between the two nodes. Location problems on a network space involve determining the locations of the new facilities with respect to the existing facilities. The new facilities may be located at some point along the existing edges (links) of the transportation system or may coincide with the existing facilities and therefore the network space is necessarily continuous. The metric space $T = (V, E)$ is known as tree network space and for the sake of brevity we call this simply a "tree" for further discussions.

Three important points on a tree namely, centre, median (see [6]) and obnoxious (see [2]) (denoted by c, m and 0 respectively) are well known. The points $c, m, 0 \in T$ satisfy the following relations:

$$Z_c(c) = \min_{x \in T}[Z_c(x)], \tag{1}$$

$$Z_m(m) = \min_{x \in T}[Z_m(x)], \tag{2}$$

$$Z_m(0) = \max_{x \in T}[Z_m(x)], \tag{3}$$

where for a point $x \in T$,

$$Z_c(x) = \max_{v \in V}[w(v)\phi(x, v)],$$

$$Z_m(x) = \sum_{v \in V} w(v)\phi(x, v).$$

We introduce a fourth important point on a tree which maximises its minimum weighted distance from any vertex of the tree and call such a point a distant point $d \in T$. Obviously the location of the distant point is important from the point of view of the optimal location of some facilities involving significant degree of pollution (such as airports, chemical plants) or some facilities involving high risk of radiation (such as nuclear reactor, uranium processing plant). A distant point d satisfies the following relations:

$$Z_d(d) = \max_{x \in T}[Z_d(x)], \tag{4}$$

where

$$Z_d(x) = \min_{v \in V}[w(v)\phi(x, v)]$$

is called the distant function, for each point $x, x \in T$.

Some physical situations exist where the median function and the centre function (or the median function and the distant function) are both important at the same time. Usually, however, the two goals are antagonistic and therefore it is necessary to devise an approach to combine these two objective functions. One of these approaches is to optimise one of the two objective functions subject to a lower limit or upper limit imposed on the other objective function as a measure of safeguard and control. In this direction, Halpern [5], has studied the dual relation between a constrained median problem and a constrained centre problem. In this paper we deal with the problems of similar nature with regard to the functions $Z_d(x)$ and $Z_m(x)$. We wish to consider the following problems.

P.1 $\max_{x \in T}[Z_d(x)]$.

P.2 For a given number $\mu \geq Z_m(m)$, the constrained distant point problem $\mathrm{CDP}(\mu)$ is defined as

$$\mathrm{CDP}(\mu)\colon \max_{x \in T}[Z_d(x)\colon Z_m(x) \leq \mu].$$

Let $x_d(\mu) \in T$ be an optimal solution to $\mathrm{CDP}(\mu)$ and $Z_d\{x_d(\mu)\}$ be the maximum value of the objective function $Z_d(x)$. For $\mu = \infty$, the problem becomes unconstrained and $x_d(\infty) = d$, the optimal solution to problem P.1.

P.3 For a given $\delta \leq Z_d(d)$, the constrained median problem $\mathrm{CMP}(\delta)$ is defined as

$$\mathrm{CMP}(\delta)\colon \min_{x \in T}[Z_m(x)\colon Z_d(x) \geq \delta].$$

Let $x_m(\delta)$ be an optimal solution and $Z_m\{x_m(\delta)\}$ be the minimal value of $Z_m(x)$ for $\mathrm{CMP}(\delta)$. When $\delta = 0$ the problem becomes unconstrained and $x_m(0) = m$, the optimal solution to (2).

2. Mathematical formulation

With the view to claiming a certain function (defined on T) to be piecewise linear or concave or quasi-concave it is necessary to define the set of points (contained in T) properly. In this regard, we assume that the set of points contained in T is a metric space with $\phi(x, y)$ as its metric which satisfies all the four standard properties as mentioned in [3] (p. 106). Construction of such a metric space associated with some tree in Euclidean space is described in [1]. A concave function is defined on a convex set and therefore we assume that the set of points in each e is a closed convex set which is a subset of one dimensional Euclidean space R. As discussed in [4] a vertex $m \in T$ which possesses the minimisation property of the median is assumed to be known for further discussions. Let V' be a subset of V such that exactly one edge is incident on each vertex of V'. Suppose V contains n vertices, $v_1, v_2, \ldots, v_n$; then without loss of generality we assume that V' contains the first l, $(l < n)$ vertices, $v_1, v_2, \ldots, v_k, \ldots, v_l$. For every $v_k \in V'$ there exists a unique path P_k (containing a set of points which is a subset of T) between m and v_k, comprising of a set of edges E_k (a subset of E) and a set of vertices V_k (including m). Note that

$$\bigcup_{k=1}^{l} E_k = E, \qquad \bigcup_{k=1}^{l} V_k = V, \qquad \bigcup_{k=1}^{l} P_k = T.$$

For each $v_k \in V'$ we have $m \in V_k$, and therefore for each P_k, m is considered as origin for further discussions.

Let $e(v_{ik}, v_{jk})$ be some edge which belongs to E_k, v_{ik}, $v_{jk} \in V_k$, $(k = 1, \ldots, l)$, and this edge for further treatment will be denoted by e_k. The set of points contained in e_k is a convex set which is a straight line segment between v_{ik} and v_{jk}. Since this convex set is a subset of the one dimensional Euclidean space real line R, it is obvious that P_k can be mapped into a convex set which is a subset of R with m as origin. Clearly such an assumption does not affect the properties of $Z_d(x)$, $Z_m(x)$ and $Z_c(x)$. Henceforth it will be assumed that for a given k $(k = 1, \ldots, l)$ the set of vertices $V_k = \{v_{ik}, i = 1, \ldots, h_k\}$ exist on R with m $(m = v_{1k})$ as origin. In this way the problem of studying the behaviour of $Z_d(x)$, $Z_m(x)$ and $Z_c(x)$ on T is converted into l subproblems where $x \in P_k$ which is a convex set. The metric space T provides a natural setting for formulating the network location problem as a mathematical optimisation problem and defining $Z_m(x)$, $Z_d(x)$ and $Z_c(x)$ on a set of convex set $P_k (k = 1, \ldots, l)$ will help in utilising the properties of linear, concave, convex or quasi-convex functions in studying the behaviour of $Z_m(x)$, $Z_d(x)$ and $Z_c(x)$. From this point it is assumed that $Z_m(x)$, $Z_d(x)$, $Z_c(x)$ are defined for every $x \in P_k$, $(k = 1, \ldots, l)$. It is assumed that for every $v \in V$, $\phi(x, v)$ is a concave or a linear function when x belongs to the edge $e(u, v)$, $u \in V$.

Consider a set of concave functions $f_i(x)$, $(i = 1, \ldots, r)$, and let $f(x) = \sum_{i=1}^{r} w_i f_i(x)$, $w_i \geq 0$ $(i = 1, \ldots, r)$ and $w_i > 0$ for at least one i, then $f(x)$ is said to be a non-negative linear combination of the functions $f_i(x) (i = 1, \ldots, r)$ and is a concave. Therefore, $Z_m(x) = \sum_{v \in V} w(v)\phi(x, v)$ is a concave function when $x \in e_i$. Let $g(x) = \min_{i=1,\ldots,r}[f_i(x)]$ then $g(x)$ is a concave function and therefore $Z_d(x) = \min_{v \in V}[w(v)\phi(x, v)]$ is a concave function of $x \in e_k, e_k \in E_k (k = 1, \ldots, l)$. Note that $\phi(x, v)$, $x \in e_k$ can be a linear function and in those cases $Z_m(x)$ and $Z_d(x)$ will be piecewise linear concave functions for $x \in e_k$.

3. Determination of a distant point

Step 1. For each pair of vertices $p, q \in V$ find all $x_{p,q} \in T$ such that $w(p)\phi(x_{pq}, p) = w(q)\phi(x_{pq}, q)$.

Step 2. Determine a pair of vertices $p^\circ, q^\circ \in V$ and a point $x_{p^\circ q^\circ} \in T$ such that $w(p)\phi(x_{pq}, p) \leq w(p^\circ)\phi(x_{p^\circ q^\circ}, p^\circ)$ for all x_{pq} such that $w(p)\phi(x_{pq}, p) = Z_d(x_{pq})$ then $x_{p^\circ q^\circ} = d$.

4. Approach to determine $x_d(\mu)$ for P2

In CDP(μ), the problem of maximisation with respect to all $x \in T$ is divided into l subproblems by considering that $x \in P_k$, $(k = 1, \ldots, l)$. Now we assume that $x \in P_k$.

Step 1. Obtain $Z_d(x)$ on P_k without considering any restriction on $Z_m(x)$. Note the value of $Z_m(x)$ is a minimum at $x = m$, and it is a concave function over any $e \in E_k$.

Step 2. For a given μ, obtain an optimal point $x_d(\mu, k) \in P_k$ such that

$$Z_d\{x_d(\mu, k)\} = \max_{x \in P_k}[Z_d(x) : Z_m(x) \leq \mu].$$

Step 3. Repeat steps 1 and 2 for $k = 1, \ldots, l$.

Step 4. Let P_k^* be a path such that

$$\max_{k=1,\ldots,l}[Z_d\{x_d(\mu, k)\}] = Z_d\{x_d(\mu, k^*)\};$$

then

$$x_d(\mu, k^*) = x_d(\mu)$$

and

$$Z_d\{x_d(\mu, k^*)\} = Z_d\{x_d(\mu)\}.$$

Note $Z_d\{x_d(\mu)\}$ is a nondecreasing function of μ and will be denoted by $Z_d(\mu)$.

5. Approach for the determination of $x_m(\delta)$ for P3

In CMP(δ), the problem of minimisation with respect to all $x \in T$ is divided into l subproblems by considering $x \in P_k$ $(k = 1, \ldots, l)$.

Step 1. Obtain $Z_m(x)$ on P_k without considering any restriction on $Z_d(x)$.

Step 2. For a given δ, obtain an optimal point $x_m(\delta, k) \in P_k$ such that

$$Z_m\{x_m(\delta, k)\} = \min_{x \in P_k}[Z_m(x) : \delta \leq Z_d(x)].$$

Step 3. Repeat steps 1 and 2 for $k = 1, \ldots, l$.

Step 4. Let P_{k^0} be a path such that

$$\min_{k=1,\ldots,l}[Z_m\{x_m(\delta,k)\} = Z_m[x_m(\delta,k^0)];$$

then

$$x_m(\delta,k^0) = x_m(\delta)$$

and

$$Z_m\{x_d(\delta,k^0)\} = Z_m\{x_m(\delta)\}.$$

From this point on, $x_m(\delta)$ and $Z_m[x_m(\delta)]$ are assumed to be known for a given δ. Note $Z_m\{x_m(\delta)\}$ is a nonincreasing function of δ and will be denoted by $Z_m(\delta)$.

6. Duality of the CDP(μ) and CMP(δ)

Let $Z_d = [0, Z_d(d)]$ and $Z_m = [Z_m(m), Z_m(d)]$ be two sets contained in R. Note that values of $\delta \in Z_d$ and values of $\mu \in Z_m$ are the only relevant values in order to obtain a complete range of solutions for CMP(δ) and CDP(μ) respectively, and $Z_d\{x_d(\mu)\} \in Z_d$ and $Z_m\{x_m(\delta)\} \in Z_m$ for the relevant values of δ and μ. Let the horizontal axis be assigned to values of μ and the vertical axis to values of $Z_d\{x_d(\mu)\}$, and plot its graph for all values of $\mu \in Z_m$. Assume now that the vertical axis is assigned to the values of δ and the horizontal axis for values of $Z_m\{x_m(\delta)\}$ whose graph is plotted for all values of $\delta \in Z_d$. The duality relationship between CDP(μ) and CMP(δ) is demonstrated by the fact that the two above-mentioned graphs coincide. This result is proved in the following theorem.

THEOREM. (a) *For any* $\mu \in Z_m$, *if*

$$Z_d\{x_d(\mu)\} = \delta, \tag{5}$$

then

$$Z_d[x_d\{Z_m(x_m(\delta)\}] = \delta.$$

(b) *For any* $\delta \in Z_d$, *if*

$$Z_m\{x_m(\delta)\} = \mu,$$

then

$$Z_m[x_m\{Z_d(x_d(\mu)\}] = \mu.$$

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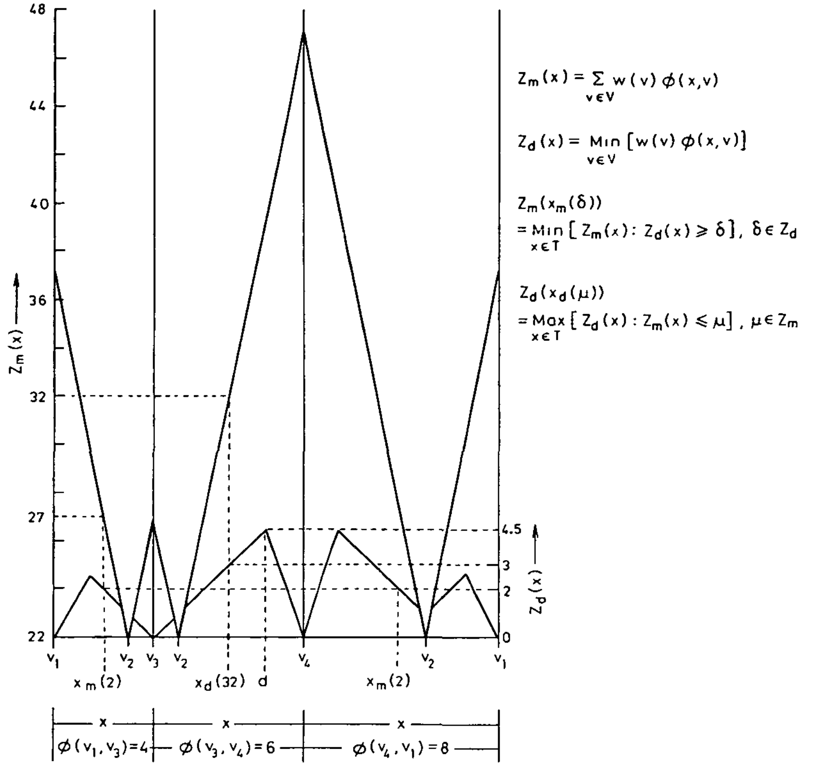

FIGURE 1

PROOF. Due to symmetry it suffices to prove (a).

Let $Z_d(x_d(\mu)) = Z_d(x^*)$, for some $x^* \in T$ for which $Z_m(x^*) \leq \mu$. Hence

$$\begin{aligned} Z_m(x_m(Z_d(x_d(\mu)))) &= \min_{x \in T}[Z_m(x) : Z_d(x) \geq Z_d(x_d(\mu)) = Z_d(x^*)] \\ &\leq Z_m(x^*) \leq \mu. \end{aligned} \tag{6}$$

Similarly let $Z_m(x_m(\delta)) = Z_m(x^0)$, for some $x^0 \in T$ for which $Z_d(x^0) \geq \delta$. Hence

$$\begin{aligned} Z_d(x_d(Z_m(x_m(\delta)))) &= \max_{x \in T}[Z_d(x) : Z_m(x) \leq Z_m(x_m(\delta)) = Z_m(x^0)] \\ &\geq Z_d(x^0) \geq \delta. \end{aligned} \tag{7}$$

From (5) and (6) we obtain

$$Z_m(x_m(\delta)) \leq \mu. \tag{8}$$

Since $Z_d(x_d(\mu))$ is obtained as a result of maximisation, and the feasible region in T increases with μ, this implies that $Z_d(x_d(\mu))$ is a nondecreasing function.

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Hence applying (8) and (5) we get

$$Z_d(x_d(Z_m(x_m(\delta)))) \le Z_d(x_d(\mu)) = \delta. \tag{9}$$

Obviously (7) and (9) imply

$$Z_d(x_d(Z_m(x_m(\delta)))) = \delta.$$

7. Illustration

Let $V = (v_1, v_2, v_3, v_4)$, $E = \{e(v_1, v_2), e(v_2, v_3), e(v_2, v_4)\}$, $\phi(v_1, v_2) = 3$, $\phi(v_2, v_3) = 1$, $\phi(v_2, v_4) = 5$, $w(v_1) = 2$, $w(v_2) = 5$, $w(v_3) = 1$, $w(v_4) = 3$.

As reported earlier, $Z_m(x)$ assumes its minimum at some vertex. In this case,

$$Z_m(v_1) = 43, \quad Z_m(v_2) = 22, \quad Z_m(v_3) = 31, \quad Z_m(v_4) = 47.$$

Hence $m = v_2$ (median).

In Figure 1, $Z_m(x)$ and $Z_d(x)$ are shown for $x \in P(v_1, v_3)$, $x \in P(v_3, v_4)$ and $x \in P(v_4, v_1)$. Clearly $Z_m(x)$ is a minimum at the point $x = v_2$, $Z_m(v_2) = 22$, and $Z_d(x)$ is a maximum when $x = d$, $Z_d(d) = 4.5$, where $\phi(v_2, d) = 3.5$ and $\phi(v_4, d) = 1.5$. Hence $Z_m(d) = 39.5$, $Z_d = [0, 4.5]$, $Z_m = [22, 39.5]$. For $\mu = 32$, (note $\mu \in Z_m$), we have $Z_d(x_d(32)) = 3$ and $x_d(32)$ is given by $\phi(v_2, x_d(32)) = 2$ and $\phi(v_4, x_d(32)) = 3$. Also for $\delta = 2$, (note $\delta \in Z_d$) we have $Z_m(x_m(2)) = 27$ and $x_m(2)$ (not unique)is given by $\phi(v_2, x_m(2)) = 1$ and $\phi(v_4, x_m(2)) = 4$ and also by $\phi(v_2, x_m(2)) = 1$, and $\phi(v_1, x_m(2)) = 2$. In this case, the two graphs of $Z_d(x_d(\mu))$, $\mu \in Z_m$ and $Z_m(x_m(\delta))$, $\delta \in Z_d$ are shown to be identical (see Figure 2) and this is the result which is proved in the theorem for the general case. In this particular case one can easily show that

$$Z_d(x_d(\mu)) = \begin{cases} \mu - 22, & 22 \le \mu \le 23.25, \\ (\mu - 17)/5, & 23.25 \le \mu \le 39.5 \end{cases}$$

and

$$Z_m(x_m(\delta)) = \begin{cases} \delta + 22, & 0 \le \delta \le 5/4, \\ 5\delta + 17, & 5/4 \le \delta \le 4.5. \end{cases}$$

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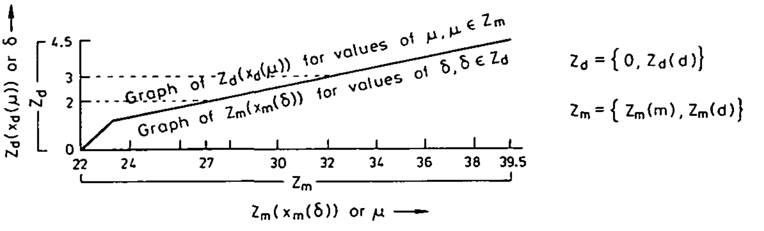

FIGURE 2

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