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ONTARIO MATHEMATICAL MEETINGS

The fourth Ontario Mathematical Meeting was held Saturday, March 25, 1967 at Sidney Smith Building, University of Toronto. Research papers were presented in the morning, followed by lunch at Sir Daniel Wilson Residence, followed by a one-hour address by Professor Irving Kaplansky (University of Chicago) entitled: Quadratic Forms. Abstracts of research papers which were presented are as follows:

67.7   J. Brenner and L. Beasley (University of British Columbia)
Bounds for Certain Permanents and Determinants.

If a matrix has dominant main diagonal, a bound can be given for the permanent of the matrix in terms of the absolute values of its elements. The result is similar to that of Price (Proc. Amer. Math. Soc. 2 (1951), 497 - 502), but the proof is different; relations among the permanents of the various minors of the matrix must be used (see Duke J., 1959). For a matrix block diagonally dominant, (definition given in Proc. Nat. Acad. Sci. 40 (1954), 452-454), bounds for the determinant and in some cases also for the permanent can be given.

67.8   D. Klarner (McMaster University)
The Number of Non-Isomorphic K-Coloured Graphs.

A k-coloured graph of type $(n_1, \ldots, n_k)$ is a set of coloured nodes $N_{n_1}^1 \cup \ldots \cup N_{n_k}^k$, where $N_r^s = \{ (s,j) : j=1, \ldots, r \}$, and an edge set E contained in the complete edge set $\{\{(c,j),(c',j')\} : (c,j) \varepsilon N_{n_{c'}}^{c'}, (c',j') \varepsilon N_{n_{c'}}^{c'}, 1 \leq c < c' \leq k \}$; the complete k-coloured graph $G(n_1, \ldots, n_k)$ of type $(n_1, \ldots, n_k)$ has the complete edge set as its edge set.

Let S_n denote the symmetric group on $N_n = \{1, \ldots, n\}$. Two k-coloured graphs X and Y of type $(n_1, \ldots, n_k)$ are <u>isomorphic</u> if there is a $\pi \, \varepsilon \, S_k$ such that $n_i = n_{\pi i}$, $i = 1, \ldots, k$, and if there are $\varphi_1 \, \varepsilon \, S_{n_1}, \ldots, \varphi_k \, \varepsilon \, S_{n_k}$ such that

(1) $\quad (\pi : \varphi_1, \ldots, \varphi_k)^* : \{(c, j), (c', j')\} \to \{(\pi c, \varphi_c j), (\pi c', \varphi_{c'} j')\}$

transforms the edge set of X into that of Y .

The cycle index of a permutation θ of degree n written as the product of disjoint cycles is $Z(\theta : \bar{x}) = x_1^{i_1} x_2^{i_2}, \ldots,$ where θ has exactly i_t cycles of length t . The cycle index of a permutation group G of degree n is
$$|G|^{-1} Z(G : \bar{x}) = |G|^{-1} \Sigma \, Z(\theta : \bar{x}) \, , \quad \theta \, \varepsilon \, G \, .$$

Let $\Gamma^* G(n_1, \ldots, n_k)$ denote the permutation group of edge automorphisms (as defined as in (1)) of the complete k-coloured graph of type $(n_1, \ldots, n_k)$. According to Pólya's theorem, the number of non-isomorphic k-coloured graphs of type $(n_1, \ldots, n_k)$ with exactly e edges is the coefficient of x^e in the polynomial

$$|\Gamma^* G(n_1, \ldots, n_k)|^{-1} \, Z(\Gamma^* G(n_1, \ldots, n_k) : x+1, x^2+1, \ldots) \, .$$

Our result is an algorithm for computing the cycle index of $\Gamma^* G(n_1, \ldots, n_k)$ in terms of the cycle indices of certain symmetric groups. We have calculated these cycle indices for all tricoloured graphs of type (n_1, n_2, n_3) , where (n_1, n_2, n_3) is a partition of n , $3 \leq n \leq 9$.

(Harary, On the Number of Bi-Coloured Graphs, Pacific J. of Math. 8 (1958), 743-755, described an algorithm for computing the relevant cycle indices for two coloured graphs).

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67.9   S.M. Fakhruddin (Queen's University)
        <u>Generalized length of a module</u>.

Let R be a commutative ring with unity, $\mathcal{C}(R)$ the
category of all left R-modules. Let Γ be a positively
fully ordered semigroup with zero which is sup-complete.
Definition: - A generalised length function on R is a
"mapping" $L: \mathcal{C}(R) \rightarrow \Gamma \cup \{\infty\}$ s.t

(1) $L(\bar{0}) = 0$ $\bar{0} \in \mathcal{C}(R)$

(2) $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ exact $\Longrightarrow L(E) = L(E') + L(E'')$

(3) $L(E) = \sup \{ L(X) | X \subseteq E$, X   a finitely generated
      module$\}$.

This generalises the concept introduced by D.G. Northcott
and M. Reufel (A generalisation of the concept of length,
Quart. Jour. of Math. Oxford, Vol.16 (1965), 297-321).
We determine all the generalised length functions on a
valuation ring R with valuation v , and extend some of
their main theorems.

<u>Sample theorems</u>.

THEOREM 1.   Let R be an integral domain, L a length
function on R to Γ with $0 < L(R) < \infty$. Then for every
$E \in \mathcal{C}(R)$, $L(E) = \text{Rank } E \cdot L(R)$.

THEOREM 2.   Let (R, v, Γ) be a valuation ring,   L a
length function on R to $\bar{\Gamma}$ such that
$\{\alpha | \alpha \in R , L(\frac{R}{\alpha R}) = \infty\} = (0)$ and $\{\alpha | \alpha \in R , L(\frac{R}{\alpha R}) > 0\}$
is the maximal ideal of R . Then there exists a map
$\varphi: \Gamma^+ \rightarrow \bar{\Gamma}$ s.t $(\varphi \circ v)(\alpha) = L(\frac{R}{\alpha R})$ and $\varphi \circ v$ is an equivalent
valuation to v . Further, for every ideal A in R ,
$L(\frac{R}{A}) = \inf_{\alpha \in A} L(\frac{R}{\alpha R})$.

THEOREM 3.   Let (R, v, Γ) be a valuation ring where Γ
is a totally ordered group. Let $(\Gamma^+)^* = \bar{\Gamma}$ be the com-
pletion of the positive part of Γ . Then there exists a
length function on R to $\bar{\Gamma}$ such that for every R-ideal
A , $L(\frac{R}{A}) = \inf_{\alpha \in A} v(\alpha)$.

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67.10 R.A. Smith (University of Toronto)
The Circle Problem in an Arithmetic Progression.

We prove that if k is <u>odd and</u> $0(X^{2/3})$, <u>then</u>

$$\sum_{\substack{1 < n < X \\ n \equiv b \,(\text{mod } k)}} r(n) = \frac{\pi}{4} H_k(B) \frac{X}{k} + 0\,(X^{2/3} k^{-1/2} d_3(k))$$

$$+ 0(X^{(2/3)+\beta_k-(1/2)(1+3\beta)}(b,k)^{1/2} d_3(k)) ,$$

for $0 < \beta < 1/2$, where

$$H_k(b) = \sum_{q \mid k} \frac{X(q)}{q} \; C_q(b) ,$$

$X(q)$ is the non-principal character modulo 4 , $C_q(b)$ is
the Ramanujan sum and $d_3(k)$ is the number of represen-
tations of k as a product of three factors. A similar
result holds for k even. A direct consequence of this is
that for (b, k) = 1 , there exists a solution to

$$x^2 + y^2 \equiv b(\text{mod } k)$$

such that $x, y = 0(k^{3/4})$, provided k has a bounded
number of factors.

67.11 Rudolf Wille (McMaster University)
Affine Coordinatization of Abstract Geometries.

An <u>abstract geometry</u> is an order pair (S, C) consisting
of a set S and an algebraic closure operator C on the
power set of S under which the empty set and singletons
are closed (B. Jónsson, <u>Lattice-theoretic approach to
projective and affine geometry</u>, Symposium on the Axiom-
atic Method (1959), 188-203). (S, C) is said to be <u>affine
coordinatizable</u> if and only if the lattice of all subspaces
(i.e. all $X \subseteq S$ with C(X) = X) is isomorphic to the
lattice of all congruence classes of an universal algebra,
which is therefore called an <u>affine coordinatization</u> algebra.

THEOREM 1.   <u>Every universal algebra is an affine
coordinatization algebra.</u>

For characterization of the affine coordinatizable geo-

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metries we generalize the concepts which are used to coordinatize affine geometries (E. Artin, Geometric algebra, 1957). A weak parallelism on an abstract geometry (S, C) is a reflexive binary relation on the set of all subspaces, the domain of which is the set of all lines (i.e. all $C(p, q)$ with $p \neq q$ in S), furthermore:

(i) If $L \Pi P$, $P \supseteq M$ and $M \Pi Q$, then there is an $R \supseteq Q$ with $L \Pi R$.

(ii) If $L \Pi P$ and $p \epsilon P$, then $P \subseteq C(p, L)$.

(iii) For L and p there is at most one P with $L \Pi P$ and $p \epsilon P$.

A Π-dilatation is a mapping δ from S into itself such that $p\delta \neq q\delta$ implies there is a P with $C(p, q) \Pi P$ and $p\delta$, $q\delta \epsilon P$.

THEOREM 2. <u>An abstract geometry</u> (S, C) <u>is affine coordinatizable if and only if</u> (S, C) <u>satisfies the following two conditions:</u>

(i) $C(p, q, r) \subseteq X$ <u>whenever</u> $p, q, r \epsilon X$ implies $C(X) = X$.

(ii) There exists a weak parallelism Π <u>such that</u> $p \epsilon C(q, r, s)$ <u>implies there are</u> π-<u>dilatations</u> $\delta_1, \dots, \delta_n$

<u>with</u> $q \epsilon \{q, r, s\}\delta_1$, $\{q, r, s\}\delta_i \bigcap \{q, r, s\}\delta_{i+1} \neq \phi$ <u>for</u>

$i = 1, \dots, n-1$ <u>and</u> $p \epsilon \{q, r, s\} \delta_n$.

THEOREM 3. <u>An affine geometry</u> (S, C) <u>is affine co-ordinatizable if and only if the dilatation group generated by all dilatations</u> δ <u>with</u> $p\delta = q$ <u>is transitive on</u> $C(p, q)$.

THEOREM 4. <u>A projective geometry</u> (S, C) <u>is affine coordinatizable if and only if</u> S <u>is a line or every line consists of exactly two points.</u>

Theorem 4 states that the lattice of all congruence classes of a non-simple universal algebra is modular if and only if it is distributive.

67.12 S. Golab (University of Waterloo)
<u>On Inner Product of Two Vectors</u>.

Suppose that in a vector space the norm $\Omega(x)$ of the vector x fulfills only the axioms:

(1) $\Omega(x) \geq 0$,

(2) $\Omega(x) = 0 \iff x = \theta$ (θ denotes the zero vector)

(3) $\Omega(\lambda\ x) = \lambda\ \Omega(x)$ for $\lambda \geq 0$.

We set

$$\varphi(x, y) = \frac{1}{2}\{\Omega^2(x) + \Omega^2(y) - \Omega^2(y-x)\} , \quad \phi(x) = \Omega^2(x) .$$

The question: 'when does $\varphi(x+y, z) = \varphi(x, z) + \varphi(y, z)$?' leads to the functional equation: $\phi(x+y) + \phi(z) - \phi(z-x-y) = \phi(x) + \phi(z) - \phi(z-x) + \phi(y) + \phi(z) - \phi(z-y)$. We show that because of the non-negativity of ϕ , all solutions of the above equation determine the euclidean metric in our vector space (we use a result of J. Aczél; the theorem of Jordan and von Neumann cannot be applied here).

67.13 F.V. Atkinson (University of Toronto)
<u>Sums of powers of complex numbers</u>.

The problem of determining

$$M_n = \min \{ \max_{1 \leq k \leq n} |\sum_{r=1}^{n} z_r^k | \} \text{ for all } z_1, \ldots, z_n$$

with $\max |z_r| = 1$,

was raised by P. Turán who conjectured that $M_n \to 1$ as $n \to \infty$.

I showed (On sums of powers of complex numbers, Acta. Math. Acad. Sci. Hung. 12 (1961), 185-188) that $M_n > 1/6$, and subsequently (On sums of powers of complex numbers; an improved estimate, Math. Research Center, Technical Summary Report #428, Madison, December 1963) that $M_n > 1/3$. In the present investigation the method of these papers is pushed to its natural limit, to give the Theorem. For large n , M_n exceeds the root, lying between $1/3$ and $\pi/8$, of the equation

$$\frac{s^2}{\pi} \int_0^\infty | \exp 2s \int_0^{\varphi/2} | \frac{\sin x}{x} | dx - 1 |^2 | \int_\varphi^\infty \frac{e^{-iu}}{u} du |^2 d\varphi = 1 .$$

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67.14   K. Murasugi (University of Toronto)
      <u>Interchangeability of Links.</u>

A link ℓ consisting of two knots K and L in 3-sphere S^3 is said to be <u>interchangeable</u> if there exists an auto-homeomorphism of S^3 that maps K onto L and L onto K. If ℓ is interchangeable then K and L must be of the same type. This is obviously a necessary but insufficient condition.

THEOREM.   Suppose that a link $\ell = K \cup L$ is interchangeable. Assume that K is a trivial knot and that the linking number of K and L is not zero. Then necessarily the Alexander polynomial of ℓ is symmetric, i.e. $\Delta(x, y) = \Delta(y, x)$.

67.15   T.M. Viswanathan (Queen's University)
      <u>o-Tensor Products.</u>

Let A be an ordered ring and M_A, $_A N$ ordered modules over the ordered ring A . We construct an ordered abelian group $M_A \overset{o}{\underset{A}{\otimes}} {}_A N$, called the o-tensor product of the ordered modules M and N , as the solution of a universal problem. An example is given to show that $M \overset{o}{\underset{A}{\otimes}} N$ may not coincide (as an abelian group) with $M \underset{A}{\otimes} N$. We determine a set of sufficient conditions for $M \overset{o}{\underset{A}{\otimes}} N$ and $M \underset{A}{\otimes} N$ to coincide and deduce thereby that the o-tensor product of two semi-closed abelian groups coincides with the usual tensor product. We prove that $M \overset{o}{\underset{A}{\otimes}} N$ satisfies properties analogous to those of $M \underset{A}{\otimes} N$ - for example $B^M {}_A \overset{o}{\underset{A}{\otimes}} {}_A{}^N$ is an ordered B-module in a natural way, when B is a directed ring.

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