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Ordering the Representations of S_n Using the Interchange Process

Gil Alon and Gady Kozma

Abstract. Inspired by Aldous' conjecture for the spectral gap of the interchange process and its recent resolution by Caputo, Liggett, and Richthammer, we define an associated order $\prec$ on the irreducible representations of S_n. Aldous' conjecture is equivalent to certain representations being comparable in this order, and hence determining the "Aldous order" completely is a generalized question. We show a few additional entries for this order.

1 Aldous' Order

Let G be a finite graph with vertex set $\{1, \dots, n\}$, and equip each edge $\{i, j\}$ with an alarm clock that rings with exponential rate $a_{i,j}$. Put a marble in every vertex of G, all different, and whenever the clock of $\{i, j\}$ rings, exchange the two marbles. Each marble therefore does a standard continuous-time random walk on the graph but the different walks are dependent. This process is called *the interchange process* and is one of the standard examples of an interacting particle system, related to exclusion processes (where the marbles have only a few possible colors), but typically more complicated. Furthermore, when one considers the evolution of the permutation taking the initial positions of the marbles to their positions at time t, one gets a continuous-time random walk on a (weighted) *Cayley graph* of the group of permutations S_n.

The first landmark in the understanding of this process was the work of Diaconis and Shahshahani [5]. For the case of G being the complete graph they diagonalized the relevant $n! \times n!$ matrix completely using representation theory and achieved very fine results on the mixing properties.

If one cannot get the whole spectrum, the second eigenvalue (the so-called spectral gap) allows one to get significant partial information on the process. In 1992 Aldous made the bold conjecture that the spectral gap of the interchange process is in fact equal to the spectral gap of the simple random walk on G, for every G. This was the focus of much research [2, 4, 6, 9, 12, 15] and was finally resolved by Caputo, Liggett, and Richthammer [3]. However, our focus in this paper is the spectrum as a whole, and for this we need to discuss the problem from a representation theoretical point of view. More details on representation theory will be given below in §2; for now we continue assuming that the reader has basic familiarity with the subject.

Let $n \in \mathbb{N}$ and let $A = \{a_{i,j}\}_{1 \le i < j \le n}$ with all $a_{i,j}$ non-negative. Examine the

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following *formal sum of permutations with real coefficients*[1]

$$\Delta_A = \sum_{i<j} a_{i,j}(\mathrm{id} - (i\,j)),$$

where id stands for the identity permutation. Let ρ be any representation of S_n. Then

$$\rho(\Delta_A) = \sum_{i<j} a_{i,j}\big(\rho(\mathrm{id}) - \rho((i\,j))\big) \tag{1.1}$$

is some $\dim\rho \times \dim\rho$ matrix. It is well known that $\rho(\Delta_A)$ is a positive-semidefinite matrix, indeed $(i\,j)$ is an involution so all the eigenvalues of $\rho((i\,j))$ are ± 1 and the eigenvalues of $\rho(\mathrm{id} - (i\,j))$ are in $\{0, 2\}$, so each term in (1.1) is positive and hence so is their sum. We shall denote the eigenvalues of $\rho(\Delta_A)$ by

$$\lambda_1(A;\rho) \le \cdots \le \lambda_{\dim(\rho)}(A;\rho).$$

We will occasionally drop the A from the notation.

Now the irreducible representations of S_n are indexed by partitions of n. Namely, for each integer sequence $r_1 \ge r_2 \ge \cdots \ge r_k > 0$ with $\sum_{i=1}^k r_i = n$ there exists a unique irreducible representation, which we shall denote by $[r_1, \dots, r_k]$. Such a partition has a nice graphical representation given by the associated *Young diagram*, obtained by drawing each r_i as a line of boxes from top to bottom,

$$[5,1] = \ydiagram{5,1} \qquad [3,2,1] = \ydiagram{3,2,1} \qquad [2,1^3] = \ydiagram{2,1,1,1}$$

and we will occasionally use it. We may now state Aldous' conjecture.

Theorem 1.1 (Caputo, Liggett, and Richthammer [3]) *For any A and any irreducible ρ different from the trivial representation $[n]$,*

$$\lambda_1(A;[n-1,1]) \le \lambda_1(A;\rho). \tag{1.2}$$

The formulation of the result in [3] does not use representation theory. As discussed above, they showed that the interchange process has the same spectral gap as the simple random walk. See [4] for how to get from one formulation to the other.

Faced with (1.2), one is tempted to generalize the question. Define the *Aldous order* on irreducible representations by $\rho \preceq \sigma \iff \forall A\ \lambda_1(A;\rho) \ge \lambda_1(A;\sigma)$, where again by $\forall A$ we mean for all $n \times n$ matrices with non-negative coefficients. It is rather unfortunate that the *largest* representation in the $\prec$ order has the *smallest* λ_1, but we wish to make $\prec$ consistent with the *domination order*, for which there is already an established direction. We say that $\rho \prec \sigma$ if $\rho \preceq \sigma$ and they are different, and remark that $\rho \preceq \sigma$ and $\rho \succeq \sigma$ imply that $\rho = \sigma$; see Remark 3.3

[1] Alternatively, element of the group ring $\mathbb{R}[S_n]$.

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As can be seen in Figure 1, $\prec$ is an interesting object, and it seems to be correlated with the domination order $\lhd$. We say that $\sigma \lhd \rho$ if σ can be obtained from ρ by a sequence of steps such that in each step one box is dropped to the row below it in a way that leaves a Young diagram. Cesi [4] remarked that it would be nice if Aldous' order were identical to the domination order, but also noted a counterexample in $n = 4$: one has $\ydiagram{2,2} \rhd \ydiagram{2,1,1}$ but $\ydiagram{2,2} \not\succ \ydiagram{2,1,1}$. See [4, Counterexample 8.2]. Such a counterexample exists for every $n \geq 4$: we will see in Corollary 3.2 that $[2, 2, 1^{n-4}] \not\succ [2, 1^{n-2}]$, although clearly $[2, 2, 1^{n-4}] \rhd [2, 1^{n-2}]$. The graph demonstrating this is a star. Shannon Starr discovered numerically another asymptotic family of counterexamples for even sizes, $[n + 1, n - 1] \not\succ [n, n]$ (private communication, checked numerically up to size 14, $n = 7$). In this case the graph demonstrating it is the cycle. We remark also that the results of Diaconis and Shahshahani [5] imply that if $\sigma \lhd \tau$, then $\sigma \not\succ \tau$. We do not know if it is generally true that if two representations are $\lhd$-incomparable, then they are also $\prec$-incomparable, so we cannot quite state that $\prec$ is a sub-order of $\lhd$, though it is a very natural conjecture.

Despite the similarity with the completely explicit $\lhd$, it is not easy to prove any entry in the Aldous order. In fact, the only entries in the Aldous order known previous to this paper are $[n] \succeq \rho \succeq [1, \dots, 1]$ for all ρ (this is easy once one identifies these representations; see Corollary 4.2), a result of Bacher [2] that the hook-shaped diagrams are ordered among themselves

$$[n] \succ [n - 1, 1] \succ [n - 2, 1^2] \succ \cdots \succ [2, 1^{n-1}] \succ [1^n],$$

(we provide some details about this in the Appendix), and of course the Caputo et al. result, $[n - 1, 1] \succ \rho$ for all $\rho \neq [n], [n - 1, 1]$.

We may now state our result.

Theorem 1.2 *Let $n \geq 4k^2 + 4k$. Let τ be an irreducible representation whose Young diagram has $\geq n - k$ boxes in the first row; and let σ be an irreducible representation whose Young diagram has $\geq n - k$ boxes at the the leftmost column. Then $\tau \succ \sigma$.*

Again we see the relationship with $\rhd$. What we show is that if "$\tau \ggcurly \sigma$" *i.e.,* if τ is much larger in the domination order than σ, then $\tau \succ \sigma$.

Let us end this introduction by returning to the work of Diaconis and Shahshahani and to the mixing time of Markov chains. Better understanding of Aldous' order will allow extending their results to other graphs. Of particular interest are the hook-shaped representations because for them all eigenvalues are explicitly known [2]. Determining which representations are $\prec$ a given hook-shaped representation will allow quickly estimating their contribution to the mixing of the process. Let us formulate some modest questions.

Question *Describe the representations $\sigma \prec [n - 2, 1^2]$. The natural generalization of Aldous conjecture is that $\sigma \lhd [n - 2, 1^2]$ implies $\sigma \prec [n - 2, 1^2]$. Is this true? If not, maybe there exists some absolute constant K such that any $\sigma \lhd [n - K, K - 1, 1]$ satisfies $\sigma \prec [n - 2, 1^2]$?*

There are, of course, many other natural questions about this order. How many entries does it have? What is the longest chain? Is it really a subset of the domination

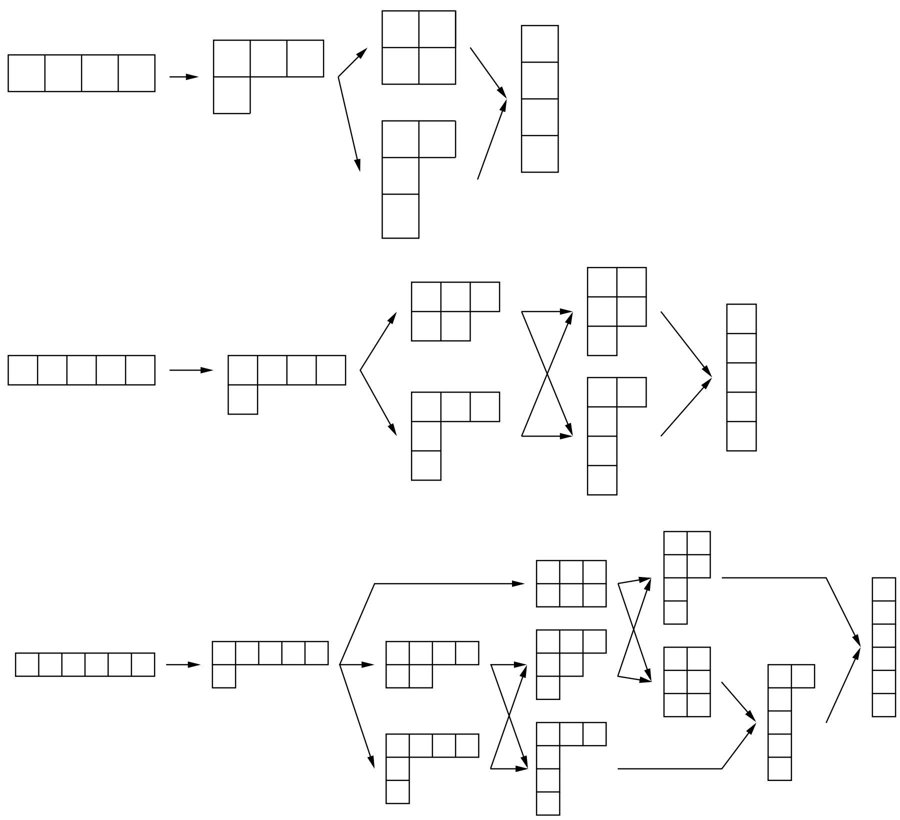

Figure 1: The Aldous order for $n = 4, 5, 6$. Arrows are drawn from the larger representations to the smaller ones. For $n = 4$ everything is proved, but the others are results of computer simulations. What can be trusted in the diagrams are the *non-arrows* — if the arrows do not imply a relationship between two diagrams that means a computer search found two examples proving no relationship may exist.

order? What is the longest chain in the domination order which is completely incomparable in Aldous order? The simulation results seem to indicate that $\sigma \succ \rho \triangleright \tau$ implies $\sigma \succ \tau$. We see no particular reason for this to be true, but if it is, it would be interesting. We chose to highlight the question above because we believe it has relevance to questions which do not need representation theory to state, *e.g.*, mixing time and the quantum Heisenberg ferromagnet (see [1] for the latter).

2 Some Representation Theory

This section will contain only the minimal set of facts needed for the paper. For a thorough introduction to the topic see one of the books [7, 8, 10, 14] and the influential paper [13]. A representation of S_n is a group homomorphism $\rho\colon S_n \to \mathrm{GL}(V)$, where V is some linear space over $\mathbb{C}$ and $\mathrm{GL}(V)$ is the space of all linear transformations of V. We will assume throughout that V is finite-dimensional. It can be assumed [14, Theorem 1.5.3] that $\rho(g)$ is a unitary matrix, and so we will assume this *a priori* for all our representations. We denote $\dim \rho = \dim V$.

Given two representations $\rho_i\colon S_n \to \mathrm{GL}(V_i)$, $i = 1, 2$ one may construct their direct sum $\rho_1 \oplus \rho_2\colon S_n \to \mathrm{GL}(V_1 \oplus V_2)$ by

$$(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}.$$

On the other hand, if there is a decomposition $V = V_1 \oplus V_2$ such that for any $g \in S_n$, $\rho(V_i) \subset V_i$ for $i = 1, 2$ then one may construct $\rho_i(g) = \rho(g)|_{V_i}$ and get $\rho \cong \rho_1 \oplus \rho_2$. If no such decomposition exists, we say that ρ is irreducible. Every representation can be written as a direct sum of irreducible representations, and the isomorphism classes of the factors are unique up to order [14, Proposition 1.7.10]. Recall also Schur's Lemma, which states that a linear map from an irreducible V to V which commutes with the action of every $g \in S_n$ is a constant multiple of the identity [14, Corollary 1.6.8].

2.1 Young Diagrams

We will use one specific method that constructs all the irreducible representations as explicit subspaces of the group ring R. The construction is somewhat abstract, but we will only need a few properties which will be easy to deduce. We will do so in Lemma 2.1 and (2.2) below and forget about the actual definition of the representations.

Recall that the group ring $\mathbb{R}[S_n]$ is simply the collection of all formal sums $\sum_{g \in S_n} a_g g$ with coefficients $a_g \in \mathbb{R}$. We will denote $R = \mathbb{R}[S_n]$. Then S_n acts on R by $h(\sum a_g g) = \sum a_g hg$ which makes R into a (left) representation known as the *regular representation*. It is true generally for any finite group that any irreducible representation can be embedded into the regular representation [14, Proposition 1.10.1]. For a general finite group this requires working over $\mathbb{C}$, but as we will see shortly, the specific structure of the representation of S_n allows working over $\mathbb{R}$, which is more natural in our setting.

Now let $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_m$ with $\sum \tau_i = n$. We define H to be the group of permutations h that preserve the rows of the diagram $[\tau_1 \ldots, \tau_m]$ in the sense that

$$i \in [1, \tau_1] \Rightarrow h(i) \in [1, \tau_1], \quad i \in [\tau_1 + 1, \tau_1 + \tau_2] \Rightarrow h(i) \in [\tau_1 + 1, \tau_1 + \tau_2], \tag{2.1}$$

etc. Let V be the group of permutations that preserve the columns of the diagram $[\tau_1, \ldots, \tau_m]$, *e.g.*, any $v \in V$ must preserve the set $\{1, \tau_1+1, \tau_1+\tau_2+1, \ldots, n-\tau_m+1\}$. We define the following elements of the group ring R,

$$a_\tau = \sum_{h \in H} h \quad b_\tau = \sum_{v \in V} \operatorname{sign}(v)v \quad c_\tau = a_\tau b_\tau.$$

Then the representation $[\tau_1, \ldots, \tau_m]$ is defined to be $Rc_\tau = \{rc_\tau : r \in R\}$, with the group acting by multiplication from the left. This is a subspace of R which is easily seen to be closed under the action of S_n. By [7, Theorem 4.3] these representations are irreducible and exhaust all the irreducible representations of S_n.

To shed a little light on the definition, let us take two examples. The first is $[n]$. In this case $H = S_n$ and $V = \{\text{id}\}$. Hence $c_\tau = \sum_{g \in S_n} g$, so $hc_\tau = c_\tau$ for any $h \in S_n$. This means that $[n]$ is one-dimensional with a trivial action of S_n. This representation is also known as the *trivial representation*. A second example is $[1^n]$. In this case $H = \{\text{id}\}$ and $V = S_n$, so this time $c_\tau = \sum_{g \in S_n} \operatorname{sign}(g)g$. We get that $hc_\tau = \operatorname{sign}(h)c_\tau$, so this representation is also one-dimensional, but this time the action of S_n is by multiplication with the sign of the permutation, the so-called *sign representation*, which we will denote by **sgn**. In general, if τ is any Young diagram and if τ' is the diagram one gets by reflecting τ along the main diagonal (so that the lengths of the rows of τ become the lengths of the columns of τ'), then

$$[\tau'] = [\tau] \otimes \mathbf{sgn}. \tag{2.2}$$

See [10, 2.1.8].

Now Ra_τ is also a representation. Generally it is reducible, but it is much more convenient to work with. Indeed, ga_τ is simply $\sum_{h \in gH} h$, so Ra_τ is isomorphic to the natural action of S_n on the set of cosets $\{gH : g \in S_n/H\}$. Furthermore, each coset can be thought of as a coloring of n by m colors with exactly τ_1 numbers colored in the first color, exactly τ_2 numbers colored in the second color, etc. Formally we define

$$Q = Q(\tau) = \{q \colon \{1, \ldots, n\} \to \{1, \ldots, m\} : \#q^{-1}(i) = \tau_i\}$$

and let $L^2(Q)$ be a representation of S_n with the natural action $(gq)(i) = q(g^{-1}i)$. We will mainly work with these representations, and we relate them to the irreducible ones by the following lemma.

Lemma 2.1 *Let $\sigma_1 \geq \ldots \geq \sigma_m$ with $\sum \sigma_i = n$. Then*

(i) *$[\sigma_1, \ldots, \sigma_m]$ can be embedded in $L^2(Q(\sigma))$.*

(ii) *For any $q \in Q$ there is a non-zero element of this embedding which is invariant under any permutation ϕ that preserves the coloring q,* i.e., *to any ϕ for which $q(\phi(i)) = q(i)$ for all i.*

For example, for $[n-1,1]$ we have that $m=2$ and an element $q \in Q$ is uniquely identified by $q^{-1}(2)$, which is an element of $\{1,\dots,n\}$. Hence $|Q| = n$ and $L^2(Q)$ can be thought of as $\mathbb{R}^n$ with S_n acting by permutation matrices (this representation is known as the *standard representation* of S_n). Clearly the constant vectors form a one-dimensional invariant subspace, and so is their orthogonal complement, the vectors whose entries sum to 0. It is not difficult to see (directly from the definition) that both are irreducible representations. The first is the trivial one, hence the second is $[n-1,1]$.

Now the second clause of the lemma in this example is as follows. Take some $q \in Q$, *i.e.*, $q(i) = 1$ for all $i \in \{1,\dots,n\}$ except one k for which $q(k) = 2$. A permutation ϕ preserves q if and only if $\phi(k) = k$. An element of $L^2(Q)$ invariant under any such ϕ must be constant on $\{1,\dots,n\}\setminus\{k\}$, and the only such element (up to multiplication by constants) in the subspace isomorphic to $[n-1,1]$ is $(1,\dots,1,1-n,1,\dots,1)$, where the position of the negative entry is k (we are not interested in the uniqueness, only in the existence).

We will prove Lemma 2.1 immediately after this simple claim.

Lemma 2.2 *Let $\rho\colon S_n \to \mathrm{GL}(V)$ be a representation and let $V_1,\dots,V_m$ be subspaces of V invariant under the action of S_n. Let W be an irreducible component of $\sum V_i$. Then W is isomorphic to a component of one of the V_i.*

Proof Clearly, it is enough to prove this for just two subspaces V_1 and V_2. Denote $U = V_1 \cap V_2$. Then U is invariant under the action of S_n. Since every invariant subspace is complemented [14, Proposition 1.5.2], we can write $V_i = U \oplus U_i$ and $V_1 + V_2 = U \oplus U_1 \oplus U_2$. The lemma now follows by the uniqueness of decomposition into irreducible representations. ∎

Proof of Lemma 2.1 Examine $Ra_\sigma R = \sum_{g\in S_n} Ra_\sigma g$. It is easy to check that each $Ra_\sigma g$ is a representation which is isomorphic to Ra_σ (g is an invertible element of the group ring R). Since $[\sigma] \cong Rc_\sigma \subset Ra_\sigma R$, we get by Lemma 2.2 that $[\sigma]$ can be embedded into Ra_σ. By the discussion before the statement of the lemma, $Ra_\sigma \cong L^2(Q)$, so the first claim of Lemma 2.1 is proved.

For the second claim, note that the permutations ϕ as above form a group, which we will denote by H_q. Now it is not important which q one takes, since if v is invariant under the action of H_q, then gv is invariant under the action of gH_qg^{-1}; and $gH_qg^{-1} = H_{gq}$ which can give any $q' \in Q$. So we will verify the claim for the H defined in (2.1). But in this case it is clear that c_σ itself is invariant under the action of H. Since the property of existence of a vector invariant under H is an abstract property of a representation, then it is not important that the vector c_σ is not necessarily in Ra_σ but just in some isomorphic copy. ∎

It is interesting to note that the irreducible components of $L^2(Q)$ are known and are all $[\tau]$ with $\tau \trianglerighteq \sigma$ [10, Lemma 2.1.10]. But we will not use this fact.

3 Star Graphs and the Gelfand–Tsetlin Basis

For $n \ge k \ge 1$, let $K_{n,k}$ be the graph with vertices $\{1,\dots,n\}$ where all the vertices $\{1,\dots,k\}$ are connected with each other and the remaining $n-k$ vertices are isolated.

By abuse of notation, we will also denote by $K_{n,k}$ the adjacency matrix of this graph. For any matrix A we denote by $\mathrm{Wt}(A) = \sum_{i<j} a_{i,j}$. We put $\mathrm{Star}_{n,k} = K_{n,k} - K_{n,k-1}$, a star graph having the vertex k connected with each of $1, \dots, k-1$. We remark that as elements of the group ring, $\mathrm{Star}_{n,k}$ are known as the Jucys–Murphy elements.

In this section, we shall find all the eigenvalues of $\sigma(\Delta_{\mathrm{Star}_{n,k}}) = \sum \sigma(\mathrm{id}) - \sigma((i\,j))$ for any irreducible representation σ of S_n. These will serve as useful examples, but more importantly will be used in the proof of the main theorem in Section 4. We will use the Gelfand–Tsetlin basis, an idea also used in [6].

Now the theory of Gelfand–Tsetlin pairs and bases is very deep with many analogs in different categories (see [11] for a survey) but we will not need any of it here. For our purposes it is enough to define the basis inductively as follows: if $n = 1$, then the representation must be one-dimensional, and we take a nonzero vector as the basis vector. For $n > 1$, we consider the natural embedding $S_{n-1} \hookrightarrow S_n$ whose image is $\{\pi \in S_n : \pi(n) = n\}$. We decompose the restriction $\sigma|_{S_{n-1}}$ into irreducible representations of S_{n-1} as $\bigoplus_i V_i$, take the Gelfand–Tsetlin basis of each V_i and define our basis as the union of the resulting bases.

If $\sigma = [\alpha]$, where α is a Young diagram of size n, then $\sigma|_{S_{n-1}} = \bigoplus_\beta [\beta]$, where β goes over all the Young diagrams of size $n-1$, obtained by removing, in all possible ways, one box from α. See [14, §2.8]. Hence, the elements of the Gelfand–Tsetlin basis of $[\alpha]$ are in one-to-one correspondence with the sequences

$$\alpha = \alpha_n, \alpha_{n-1}, \dots, \alpha_1 = \square$$

of Young diagrams, in which each α_i for $i < n$ is obtained from α_{i+1} by removing one box.

Lemma 3.1 *Let $G = \mathrm{Star}_{n,k}$ for some $2 \le k \le n$, and let σ be an irreducible representation of S_n.*

- (i) *The Gelfand–Tsetlin basis of σ is a basis of eigenvectors for e_G acting on σ.*
- (ii) *Let $\alpha = \alpha_n, \alpha_{n-1}, \dots, \alpha_1 = \square$ be a sequence of Young diagrams, where α_{i-1} is obtained from α_i by removing the x_i-th box at the y_i-th row, and $\sigma = [\alpha_n]$. Let v be the Gelfand–Tsetlin basis element corresponding to the above sequence. Then the eigenvalue of Δ_G with respect to v is $(k-1) + y_k - x_k$.*

Proof (i) Let us first prove that each vector in the Gelfand–Tsetlin basis of σ is an eigenvector for each of $K_{n,1}, K_{n,2}, \dots, K_{n,n}$. We will do this by induction on n. For $n = 1$, the claim is vacuous. For $n > 1$, we decompose $\sigma|_{S_{n-1}}$ into irreducible representations and, using the induction hypothesis, conclude that the Gelfand–Tsetlin basis vectors are eigenvectors of $K_{n,1}, \dots, K_{n,n-1}$. As for $K_{n,n}$, the element $\Delta_{K_{n,n}}$ lies in the center of $\mathbb{R}[S_n]$ (being a linear combination of the sum of all transpositions and the identity). Hence, by Schur's Lemma it acts as a scalar on each irreducible representation of S_n. This finishes the inductive step. Hence, each vector of the Gelfand–Tsetlin basis is an eigenvector of $G = K_{n,k} - K_{n,k-1}$.

(ii) Let us now find the scalar by which $K_{n,i}$ acts on the irreducible S_i-representation $[\alpha_i]$, assuming that α_i has row lengths $l_1, \dots, l_m$. For that matter, we invoke the trace formula from [5, Lemma 7], by which, the trace of a transposition acting on

$[\alpha_i]$ is

$$\frac{\dim[\alpha_i]}{\binom{i}{2}}\sum_{j=1}^{m}\left(\binom{l_j-j+1}{2}-\binom{j}{2}\right),$$

where we define $\binom{x}{2}=\frac{1}{2}x(x-1)$ also for $x<2$. Hence, $\Delta_{K_{n,i}}=\sum_{1\le j<k\le i}(\mathrm{id}-(jk))$ acts on $[\alpha_i]$ via the scalar

$$c_i := \mathrm{Wt}(K_{n,i})-\sum_{j=1}^{m}\left(\binom{l_j-j+1}{2}-\binom{j}{2}\right).$$

Let us find the eigenvalue of $\Delta_G=\Delta_{K_{n,k}}-\Delta_{K_{n,k-1}}$ with respect to our basis vector v. This eigenvalue is equal to c_k-c_{k-1}, and we will now simplify it. Recall that passing from α_k to α_{k-1} is done by removing the rightmost box from the y_k-th row. We distinguish two cases.

Case 1: α_k and α_{k-1} have the same number of rows. In that case, all the summands except the y_k-th one cancel out, and we are left with

$$\begin{aligned}\mathrm{Wt}(\mathrm{Star}_{n,k})-\left(\binom{l_{y_k}-y_k+1}{2}-\binom{l_{y_k}-y_k}{2}\right)&=\mathrm{Wt}(\mathrm{Star}_{n,k})-(l_{y_k}-y_k)\\&=k-1-(x_k-y_k).\end{aligned}$$

Case 2: α_k is obtained from α_{k-1} by removing the unique box in the y_k-th row. In that case, we have $l_{y_k}=1$, and the scalar is

$$\begin{aligned}&\mathrm{Wt}(\mathrm{Star}_{n,k})-\left(\binom{l_{y_k}-y_k+1}{2}-\binom{y_k}{2}\right)\\&\qquad=\mathrm{Wt}(\mathrm{Star}_{n,k})-\left(\binom{2-y_k}{2}-\binom{y_k}{2}\right)\\&\qquad=\mathrm{Wt}(\mathrm{Star}_{n,k})-(1-y_k)=k-1-(x_k-y_k).\end{aligned}$$

Hence, $\Delta_{\mathrm{Star}_{n,k}}v=(k-1-(x_k-y_k))v$, and the result follows. ∎

Corollary 3.2 *For any n, $[2,2,1^{n-4}]\not\succ[2,1^{n-2}]$.*

Proof Examine the star graph $G=\mathrm{Star}_{n,n}$. We get that $\lambda_1(G;\rho)=\min\{(n-1)+y-x\}$, where the minimum is taken over all boxes (x,y) which can be removed to get a legal Young diagram. For $[2,2,1^{n-4}]$ these boxes are $(2,2)$ and (if $n>4$) $(1,n-2)$. The minimum is achieved at $(2,2)$ so $\lambda_1(G;[2,2,1^{n-4}])=n-1$. For $[2,1^{n-2}]$ the minimum is achieved at $(2,1)$ giving that $\lambda_1(G;[2,1^{n-2}])=n-2$. ∎

Since $[2,2,1^{n-4}]\rhd[2,1^{n-2}]$, this shows that $\succ$ and $\rhd$ differ for every n. Similarly one can show that $[2^i,1^{n-2i}]\not\succ[2^j,1^{n-2j}]$ for any $j<i\le n/2$ giving a chain of length $\lfloor n/2\rfloor$ in the domination order, no two elements of which are comparable in the Aldous order.

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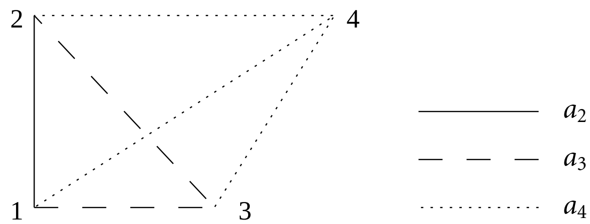

Figure 2: A quasi-complete graph.

3.1 Quasi-Complete Graphs

Let $a_2, a_3, \dots, a_n$ be non-negative numbers and examine $\sum_{k=2}^{n} a_k \operatorname{Star}_{n,k}$. This is not just any combination of stars, but stars formed in a special order, each added vertex connected to all existing ones. We call such graphs *quasi-complete graphs*. See Figure 2. Since the Gelfand–Tsetlin basis of $[\alpha]$ is a basis for of eigenvalues for $\operatorname{Star}_{n,k}$ for all k, it is also a basis of eigenvalues for their linear combination. Further, the basis element corresponding to a sequence $\alpha_n, \dots, \alpha_1$ as in Lemma 3.1 gives the eigenvalue

$$\operatorname{Wt}(G) - \sum_{k=2}^{n} a_k(x_k - y_k). \tag{3.1}$$

Let us make three remarks about quasi-complete graphs.

Remark 3.3 (1) Taking a_k to be very fast decreasing (taking $a_k = n^{-2k}$ is good enough), it is easy to see that the minimal eigenvalue is achieved when the boxes are removed as follows: first remove the lowest row completely, then the second lowest row completely, etc. This shows that if $\alpha < \beta$ in the lexicographical order, then $\alpha \not\succeq \beta$. As a corollary we get that $\alpha \preceq \beta$ and $\beta \preceq \alpha$ imply $\alpha = \beta$.

(2) This family of graphs is not rich enough to determine the Aldous order. For example, take the fact that $[n+1, n-1] \not\succeq [n, n]$, which can be verified for small $n \geq 3$ by direct calculation for the circle graph (we have no proof that it holds for all n, but this is not relevant at this point). Quasi-complete graphs cannot demonstrate this fact, because any sequence of Young diagrams as in Lemma 3.1 for $[n, n]$ must start by removing the box at $(n, 2)$, while a sequence for $[n+1, n-1]$ *may* start by removing the box at $(n+1, 1)$, and continue by mimicking the first sequence, since in both cases $\alpha_{2n-1} = [n, n-1]$. So we see that $\lambda_1(G; [n+1, n-1]) \leq \lambda_1(G; [n, n])$ for any quasi-complete graph G.

(3) As an approximation to the Aldous order, one may ask whether two Young diagrams σ, τ of size n satisfy $\lambda_1(G; \sigma) \leq \lambda_1(G; \tau)$ for all quasi-complete graphs G. Using formula (3.1), the answer can be put in terms of the following combinatorial game: let players A and B get the Young diagrams σ and τ, respectively. Both players fill out their diagrams as follows: on each square at position (i, j) they write the number $j - i$. The game has n steps. At each step of the game, player B breaks off a square from his diagram in a way that leaves a legal Young diagram, and announces the number on that square. Then player A does the same. We say that player A wins

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the game if at each step, her number is no less than player B's number. It is not hard to see that player A has a winning strategy if and only if $\lambda_1(G;\sigma) \le \lambda_1(G;\tau)$ for all quasi-complete graphs G.

4 Proof of the Main Theorem

The proof requires that we examine closely the *maximal* eigenvalue of $\sigma(\Delta_A)$ (which is of course also its norm as an L^2 operator, as this is a positive matrix). We denote it by $\lambda_{\max}(A;\sigma)$. From the previous section we keep the notation $\mathrm{Wt}(A) = \sum_{i<j} a_{i,j}$. For $k < n$, we denote by $\mathcal{S}_k$ the set of representations corresponding to Young diagrams of size n such that the first row has $\ge n-k$ boxes (so all the other rows combined have $\le k$ boxes). Denote $\mathcal{S}_k \otimes \mathbf{sgn} = \{\sigma \otimes \mathbf{sgn} : \sigma \in \mathcal{S}_k\}$, which by (2.2) is also the set of representations corresponding to Young diagrams of size n such that the leftmost column has $\ge n-k$ boxes.

As in the previous section, we will use "matrix" and "weighted graph" interchangeably, understanding that all our matrices have non-negative entries, and that the corresponding graph has an edge at any (i, j) for which $a_{i,j} \ne 0$. In particular, when we subtract graphs and weighted graphs, we are in fact subtracting the corresponding matrices. Let us start with two standard facts which we prove for the convenience of the reader.

Lemma 4.1 *For any A with entries a_{ij} and any representation σ*

$$\lambda_{\max}(A;\sigma) = 2\,\mathrm{Wt}(A) - \lambda_1(A;\sigma\otimes\mathbf{sgn}), \tag{4.1}$$

$$\lambda_{\max}(A;\sigma) \le 2\,\mathrm{Wt}(A). \tag{4.2}$$

Proof The action of $\Delta_A = \sum_{i<j} a_{ij}(\mathrm{id} - (ij))$ on σ is linearly isomorphic to the action of $\sum_{i<j} a_{ij}(\mathrm{id} + (ij)) = 2\,\mathrm{Wt}(A) - \Delta_A$ on $\sigma\otimes\mathbf{sgn}$. This gives (4.1). The second part follows immediately since $\lambda_1(A;\sigma\otimes\mathbf{sgn}) \ge 0$. ∎

Corollary 4.2 *For any σ, $[n] \succeq \sigma \succeq [1^n]$.*

Proof Recall that $[n]$ is the trivial representation, so $\lambda_1(A;[n]) = 0$ which shows $[n] \succeq \sigma$. In the other direction, $[1^n] = \mathbf{sgn}$, so

$$\lambda_1(A;[1^n]) = \lambda_{\max}(A;[1^n]) = 2\,\mathrm{Wt}(A) \ge \lambda_{\max}(A;\sigma) \ge \lambda_1(A;\sigma). \qquad ∎$$

Lemma 4.3 *Assume $n \ge 4k$ and let $\sigma \in \mathcal{S}_k$. Let G be a graph with $2k$ disjoint edges,* i.e., *$4k$ vertices have degree 1, and the remaining $n-4k$ vertices are isolated. Then $\lambda_{\max}(G;\sigma) \le 2k$.*

Proof Recall the representation $L^2(Q)$ of Lemma 2.1, namely, if $\sigma_1 \ge \cdots \ge \sigma_m > 0$ are the lengths of the rows of (the Young diagram corresponding to) σ, then

$$Q = Q(\sigma) = \{q : \{1,\dots,n\} \to \{1,\dots,m\} : \#q^{-1}(i) = \sigma_i\}.$$

By Lemma 2.1 we know that the representation σ can be embedded in $L^2(Q)$. Hence it is enough to show that $\lambda_{\max}(G;L^2(Q)) \le 2k$. Let $f \in L^2(Q)$ be an eigenvector for

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$\lambda_{\max}$, and let $q \in Q$ be the point where the maximum of $|f|$ is attained. Fix one edge $(i, j) \in G$ and examine

$$((\mathrm{id} - (i\,j))f)(q) = f(q) - f((i\,j)q). \tag{4.3}$$

If $q(i) = q(j) = 1$ (recall that the elements q of Q are themselves functions), then $(i\,j)q = q$ and (4.3) is zero. However, by definition of Q the number of i such that $q(i) \neq 1$ is $\leq k$. Because the degree of G is ≤ 1, we get that for any i there can be at most one j such that $(i, j) \in G$; hence there are a totality of no more than k edges $(i, j) \in G$ for which (4.3) is non-zero. Hence we get

$$\sum_{(i,j)\in G} ((\mathrm{id} - (i\,j))f)(q) \leq 2k|f(q)|.$$

Since f is an eigenfunction of $\lambda_{\max}$, we also have

$$\sum_{(i,j)\in G} ((\mathrm{id} - (i\,j))f)(q) = \lambda_{\max} f(q). \qquad \blacksquare$$

Lemma 4.4 *Let $\sigma \in \mathcal{S}_k$ and let $G = \mathrm{Star}_{n,l+1}$,* i.e., *a star graph with l edges and the rest of the vertices isolated. Then $\lambda_{\max}(G; \sigma) \leq l + k$.*

Proof This is a corollary of Lemma 3.1. The eigenvector for the maximal eigenvalue corresponds to an element of the Gelfand–Tsetlin basis, which corresponds to a sequence of boxes (x_i, y_i) of σ as in Lemma 3.1. Since σ has no more than $k + 1$ rows, we have $y_i \leq k + 1$ for all i, and we always have $x_i \geq 1$. According to Lemma 3.1,

$$\lambda_{\max}(G; \sigma) = ((l + 1) - 1) + (y_{l+1} - x_{l+1}) \leq l + k + 1 - 1 = l + k. \qquad \blacksquare$$

Lemma 4.5 *Let A be a weighted star,* i.e., *assume there are some $a_2 \geq \cdots \geq a_n \geq 0$ such that $(1, i)$ has weight a_i but (i, j) has weight 0 when both $i > 1$ and $j > 1$. Let $\sigma \in \mathcal{S}_k$. Then $\lambda_{\max}(A; \sigma) \leq 2a_2 + \cdots + 2a_{k+1} + a_{k+2} + \cdots + a_n$.*

Proof Write $A = A_2 + \cdots + A_n$, where A_i is a weighted star with weights

$$\begin{array}{ll} \overbrace{a_i - a_{i+1}, \ldots, a_i - a_{i+1}}^{i-1 \text{ times}}, \overbrace{0, \ldots, 0}^{n-i \text{ times}} & \qquad i = 2, \ldots, n-1 \\ \overbrace{a_n, \ldots, a_n}^{n-1 \text{ times}} & \qquad i = n. \end{array}$$

Since $\lambda_{\max}$ is a norm (recall that $\sigma(\Delta_A)$ is a positive matrix, so $\lambda_{\max}$ is its norm as an L^2 operator), we have that

$$\lambda_{\max}(A; \sigma) \leq \sum_{i=2}^{n} \lambda_{\max}(A_i; \sigma).$$

Each summand may be estimated by Lemma 4.4 and we get

$$\lambda_{\max}(A_i;\sigma) \le (i-1+k)(a_i - a_{i+1}). \quad (\text{define } a_{n+1} := 0).$$

However, for $i \le k$ we actually get a better estimate from the trivial bound (4.2),

$$\lambda_{\max}(A_i;\sigma) \le 2(i-1)(a_i - a_{i+1}).$$

Summing we get

$$\begin{aligned}\lambda_{\max}(A;\sigma) &\le \sum_{i=2}^{k} 2(i-1)(a_i - a_{i+1}) + \sum_{i=k+1}^{n} (i-1+k)(a_i - a_{i+1}) \\ &= \sum_{i=2}^{k+1} 2a_i + \sum_{i=k+2}^{n} a_i\end{aligned}$$

as was to be proved. ■

Lemma 4.6 *Let A, H be two weighted graphs with n vertices, and let σ, τ be two irreducible representations of S_n. If*

$$\lambda_1(H;\tau) \ge \lambda_{\max}(H;\sigma) \tag{4.4}$$

and $\lambda_1(A;\tau) \ge \lambda_1(A;\sigma)$, then $\lambda_1(A+H;\tau) \ge \lambda_1(A+H;\sigma)$.

Proof Recall the variational characterization of λ_1 which states that for any positive matrix M, its lowest eigenvalue is the minimum over all vectors v of $\langle Mv, v\rangle$. Hence we may bound $\lambda_1(A+H;\sigma)$ above with any v, and we choose v to be a unit eigenvector corresponding to $\lambda_1(A;\sigma)$. Then

$$\begin{aligned}\lambda_1(A+H;\sigma) &\le \langle(\Delta_A + \Delta_H)v, v\rangle \\ &= \lambda_1(A;\sigma) + \langle \Delta_H v, v\rangle \le \lambda_1(A;\sigma) + \lambda_{\max}(H;\sigma) \\ &\le \lambda_1(A;\tau) + \lambda_1(H;\tau) \le \lambda_1(A+H;\tau).\end{aligned}$$ ■

We remark that even though Lemma 4.6 works for any matrix H, we will apply it only to matrices whose entries take two values (one of which is 0), *i.e.,* to graphs whose weights are all the same.

Definition 4.7 Let G, H be two weighted graphs, and σ, τ representations of S_n.

(i) We call H a *reducing graph* for σ, τ if H satisfies (4.4).
(ii) We say that G is H*-irreducible* if there does not exist a graph H' isomorphic to H and a number $\epsilon > 0$ such that $\epsilon H' \le G$.

The identities (2.2) and (4.1) show that equation (4.4) is equivalent to

$$\lambda_{\max}(H;\sigma) + \lambda_{\max}(H;\tau \otimes \mathbf{sgn}) \le 2\,\mathrm{Wt}(H).$$

We will use this reformulation to show that a given graph H is reducing.

Lemma 4.6 is the basis to our strategy of reduction: in proving that $\sigma \succ \tau$ if H is a reducing graph, it is enough to prove that $\lambda_1(A;\sigma) \le \lambda_1(A;\tau)$ only for H-irreducible matrices A. Indeed, if A is not H-irreducible, then we can find $H' \cong H$ and $\epsilon > 0$ such that $A - \epsilon H'$ has nonnegative weights on the edges, and fewer nonzero weights than A. According to Lemma 4.6, it is enough to prove the inequality for $A - \epsilon H'$. Repeating this procedure, we reduce the problem to H-irreducible graphs.

Proof of the Theorem Let $n \ge 4k^2 + 4k$ and let $\sigma \in \mathcal{S}_k$ and $\tau \in \mathcal{S}_k \otimes \mathbf{sgn}$. The claim of the theorem is that under these conditions $\sigma \succ \tau$. Let H be the graph with $2k$ disjoint edges, *i.e.,* as a matrix its coefficients h_{ij} are given by

$$h_{ij} = \begin{cases} 1 & i = 2k \text{ and } j = 2k+1 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 4.3, $\lambda_{\max}(H; \tau \otimes \mathbf{sgn}) + \lambda_{\max}(H;\sigma) \le 4k = 2\,\mathrm{Wt}(H)$. Hence, H is a reducing graph for σ and τ. It is hence enough to prove that $\lambda_1(A,\sigma) \le \lambda_1(A,\tau)$ for any H-irreducible matrix A.

It is well known that an H-irreducible graph can be written as a union of $4k-2$ weighted stars. Indeed, choose an edge e of A arbitrarily and remove from A the two stars centered at the two vertices of e. If the resulting graph is non-empty, choose again some edge arbitrarily and remove two stars. This process must stop after $2k-1$ steps, since otherwise we would have found $2k$ disjoint edges in A. Hence we wrote A as a union of $4k-2$ stars. Denote

$$A = \sum_{i=1}^{4k-2} S_i.$$

We now use Lemma 4.5 for each of the S_i and sum over i. Recall that the k edges with the largest weights of a weighted star play a special role in Lemma 4.5 — their weights were multiplied by 2 rather than by 1. Collecting these special edges for the $4k-2$ stars gives a total of $k(4k-2)$ special edges. Denote them by e_i. Thus the conclusion of Lemma 4.5 is

$$\lambda_{\max}(A, \tau \otimes \mathbf{sgn}) \le \sum_{i<j} a_{ij} + \sum_{i=1}^{k(4k-2)} a_{e_i},$$

(where if $e = (i,j)$, then we denote $a_e = a_{ij}$). The edges e_i combined have no more than $(k+1)\cdot(4k-2) = 4k^2 + 2k - 2$ vertices.

Let us now move to the estimate of $\lambda_1(A;\sigma)$. For that matter, pick $2k$ vertices which do not belong to any of the e_i's (here we use the condition $n \ge 4k^2 + 4k$). Denote these vertices by $v_1, \dots, v_{2k}$. For every i, let $W(i)$ be the weight of v_i, *i.e.,*

$$W(i) = \sum_{j=1}^{n} a_{v_i j}.$$

Assume without loss of geneality that the v_i are arranged so that $W(i)$ are increasing. Examine again the representation $L^2(Q)$ from Lemma 2.1. By clause (ii) of that lemma, since $\sigma \in \mathcal{S}_k$, for any set of $n-k$ vertices there exists a nonzero element $f \in V \subset L^2(Q)$, where V is an invariant subspace of $L^2(Q)$ isomorphic to σ such that f is invariant under permutations of elements from this set. Normalize f to have $\|f\| = 1$. We choose the set to be all vertices except $v_1, \dots, v_k$. Examine now $\sum_{i<j} a_{ij}(\mathrm{id} - (ij))f$. If both i and j are different from $v_1, \dots, v_k$, then $(ij)f = f$ and the contribution to the sum is 0. Otherwise, we simply estimate $\|a_{ij}(\mathrm{id} - (ij))f\| \le 2a_{ij}$ (here $\|\cdot\|$ is the norm in $L^2(Q)$) and we get

$$\Big\| \sum_{i<j} a_{ij}(\mathrm{id} - (ij))f \Big\| \le 2\sum_{i=1}^{k} W(i) \le \sum_{i=1}^{2k} W(i),$$

where the second inequality comes from the fact that we chose the $W(i)$ increasing. This bounds $\lambda_1(A;\sigma)$ and we get

$$\lambda_1(A;\sigma) + \lambda_{\max}(A;\tau \otimes \mathbf{sgn}) \le \sum_{i=1}^{2k} W(i) + \sum_{i<j} a_{ij} + \sum_{i=1}^{k(4k-2)} a_{e_i}.$$

Since the vertices $v_1, \dots, v_{2k}$ are different from the vertices on the edges e_i, the above sum contains each edge no more than twice. Hence,

$$\lambda_1(A;\sigma) \le 2\,\mathrm{Wt}(A) - \lambda_{\max}(A;\tau \otimes \mathbf{sgn}) = \lambda_1(A;\tau). \qquad \blacksquare$$

A The Hook-Shaped Diagrams

In this appendix we prove the claim appearing in the introduction that

$$[n] \succ [n-1,1] \succ \cdots \succ [1^n]. \tag{A.1}$$

This is basically a result of Bacher [2], who showed that the eigenvalues

$$\lambda_i(A;[n-k,1^k])$$

are simply all the sums of all k-tuples of the eigenvalues $\lambda_i(A;[n-1,1])$. This immediately implies (A.1) (recall that the eigenvalues are all non-negative). However, he used a different description of these representations, as wedge products of $[n-1,1]$. We will now prove that the two descriptions coincide. This was known before (for example, it is mentioned without proof in [3] in the penultimate paragraph of the introduction), but we found no proof in the literature.

Lemma A.1 $\wedge^k[n-1,1] \cong [n-k,1^k]$.

The proof will use the Murnaghan–Nakayama formula for the characters of the irreducible representations of S_n. See [14, Theorem 4.10.2]. We will not give a full description of this formula here.

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Proof Recall that for a representation ρ the character is a function $\chi\colon G \to \mathbb{C}$ defined by $\chi(g) = \operatorname{tr}\rho(g)$ and that two representations are isomorphic if and only if their characters coincide [14, Corollary 1.9.4(5)]. Thus it is enough to prove that the two representations have the same character. Let us denote the character of the representation on the left-hand side by $\chi_k^{\wedge}$ and the right-hand side by χ_k^{Γ} (the letter Γ reminds us of a hook). We will prove that $\chi_k^{\wedge} + \chi_{k-1}^{\wedge} = \chi_k^{\Gamma} + \chi_{k-1}^{\Gamma}$. Since the lemma is true for $k = 1$, that will be enough.

Let V be the n-dimensional Euclidean space, with the standard basis $e_1, \dots, e_n$, viewed as the standard representation of S_n. Since $V \cong [n-1, 1] \oplus [n]$ and $[n]$ is one-dimensional, it can easily be seen that

$$\wedge^k V \cong \wedge^{k-1}[n-1,1] \oplus \wedge^k[n-1,1].$$

Therefore $\chi_{\wedge^k V} = \chi_k^{\wedge} + \chi_{k-1}^{\wedge}$. Recall now the standard basis for $\wedge^k V$: for any subset $K = \{i_1 < i_2 < \cdots < i_k\}$ of $\{1, 2, \dots, n\}$ let $e_K = e_{i_1} \wedge \cdots \wedge e_{i_k}$. We will calculate the trace of a permutation g (acting on $\wedge^k V$) using this basis. Let $g \in S_n$ have cycles of lengths $c_1 \geq c_2 \geq \cdots \geq c_r$. We have $ge_K = \pm e_{g(K)}$. The only contribution to the trace comes from e_K's for which $g(K) = K$, and in this case the $\pm$ above is simply $\operatorname{sgn}(g|_K)$. Hence we get

$$\chi_{\wedge^k V}(g) = \sum_{\substack{\epsilon_1,\dots,\epsilon_r \in \{0,1\} \\ \sum \epsilon_i c_i = k}} (-1)^{\sum \epsilon_i (c_i - 1)}.$$

Let us now calculate $\chi_k^{\Gamma} + \chi_{k-1}^{\Gamma}$. We use the Murnaghan–Nakayama rule, which takes a nice form for hook-shaped diagrams. For a hook-shaped diagram γ, let $\mathbf{S}(\gamma)$ be the set of all sequences of Young diagrams $\gamma = \gamma_1, \dots, \gamma_{r+1} = \varnothing$ such that for $1 \leq i \leq r$, γ_{i+1} is obtained from γ_i by removing c_i consecutive boxes. Note that, except for the last stage, all the removed parts do not contain the corner $(1,1)$ and hence are either horizontal or vertical bars. We will call the removed parts of the $r-1$ first stages the *bars of the sequence*. For any set of consecutive boxes we define its *height* to be the number of rows it occupies. And for an element $(\gamma_1, \dots, \gamma_{r+1})$ of $\mathbf{S}(\gamma)$, we define its height to be $\sum_{i=1}^{r}(1 + \operatorname{height}(\gamma_i \setminus \gamma_{i+1}))$. Then, according to the Murnaghan–Nakayama rule,

$$\chi_k^{\Gamma}(g) = \sum_{s \in \mathbf{S}([n-k,1^k])} (-1)^{\operatorname{height}(s)}.$$

Let f be the bijective function from the Young diagram $[n-k, 1^k]$ to the Young diagram $[n-k+1, 1^{k-1}]$ taking the box at position $(1,i)$ to the box $(1, i-1)$ for all $i > 1$, and taking the box at $(i,1)$ to $(i+1,1)$ for all i. See Figure 3. Let $\mathbf{A}$ be the subset of $\mathbf{S}([n-k,1^k])$ of all sequences for which no bar has $(1,2)$ as an endpoint, and let $\mathbf{B}$ be the subset of $\mathbf{S}([n-k+1,1^{k-1}])$ of all sequences for which no bar has $(2,1)$ as an endpoint. Let $F\colon \mathbf{A} \to \mathbf{B}$ be the function defined by applying f to each stage of the sequence. Then F is well defined because the condition that $(1,2)$ is not an end point ensures that the image is indeed a set of legal Young diagrams, and F is a bijection since an inverse function can be defined using f^{-1}. Moreover, the corresponding

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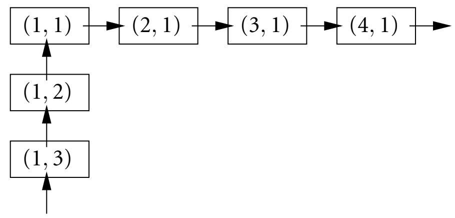

Figure 3: The mapping $f\colon [n-k, 1^k] \to [n-k+1, 1^{k-1}]$.

terms for s and $F(s)$ cancel out in the sum for $\chi_k^\Gamma + \chi_{k-1}^\Gamma$. Hence,

$$\chi_k^\Gamma + \chi_{k-1}^\Gamma = \sum_{s\in \mathbf{S}([n-k,1^k])\setminus \mathbf{A}} (-1)^{\mathrm{height}(s)} + \sum_{s\in \mathbf{S}([n-k+1,1^{k-1}])\setminus \mathbf{B}} (-1)^{\mathrm{height}(s)}.$$

But this is equal to $\chi_V(g)$; a sequence $\epsilon_1, \dots, \epsilon_r \in \{0, 1\}$ such that $\sum \epsilon_i c_i = k$ determines a unique term in one of the above two summands in the following way. At each stage, we remove a vertical bar of size c_i if $\epsilon_i = 1$ and a horizontal one if $\epsilon_i = 0$. We get a sequence of Young diagrams in the first summand when $\epsilon_r = 0$, and in the second summand when $\epsilon_r = 1$. Furthermore, it is easy to check that all terms in both summands are obtained in this way, and that the corresponding term is $(-1)^{\sum \epsilon_i(c_i - 1)}$. ■

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Division of Mathematics, The Open University of Israel, Raanana 43107, Israel
e-mail: gilal@openu.ac.il

Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel
e-mail: gadyk@weizmann.ac.il