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SUM-FREE SETS, COLOURED GRAPHS AND DESIGNS

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Abstract

Sum-free sets may be used to colour the edges of a complete graph in such a way as to avoid monochromatic triangles. We discuss the automorphism groups of such graphs. Embedding of colourings is considered. Finally we illustrate a way of constructing colourings using block designs.

1. Introduction

Suppose G is a finite graph. A *proper r-colouring* of G is an assignment of r colours to the edges of G in such a way that the resulting coloured graph contains no monochromatic triangle. A proper r-colouring is equivalent to a decomposition of G into r edge-disjoint subgraphs $G_1, G_2, \cdots, G_r$ — the monochromatic subgraphs, G_i consisting of all the edges with colour i — none of which contains a triangle.

We write K_n for the complete graph on n vertices. It is clear from Ramsey's Theorem that, given r, there exists an integer $R_3(3,2)$, or simply R_r, such that K_n has a proper r-colouring if and only if $n < R_r$. (See Wallis, Street and Wallis (1972).) The numbers R_r are not easily calculated. It is well known that $R_2 = 6$ and it has been proved (Greenwood and Gleason (1955)) that $R_3 = 17$. We know of R_4 only that $51 \leqq R_4 \leqq 65$; see Chung (1973, 1974), Folkman (1967), Whitehead (1973).

We can construct proper colourings of graphs using sum-free sets. If H is any group, a subset S of H is called *sum-free* if and only if it does not contain elements x, y, z which satisfy $xy = z$. A *sum-free r-partition* of H means a partition of the set H^* of non-identity elements of H into r sets, each of which is sum-free.

Suppose that $H = \{x_1, x_2, \cdots, x_n\}$ is a group of order n and that a sum-free

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r-partition of H into the sets $H_1, H_2, \cdots, H_r$ is known. With each element of H_i, we associate the ith colour. A proper r-colouring of K_n may be constructed as follows. First, the elements of H are ordered in some way; for example,

$$x_1 < x_2 < \cdots < x_n.$$

Then the vertices of K_n are labelled $x_1, x_2, \cdots, x_n$ in some fashion. Finally, if $x_i < x_j$ in the given ordering, then edge (x_i, x_j) is coloured with the colour associated with $x_i x_j^{-1}$; in the terminology of the decomposition into subgraphs, the graph G_k consists of all edges (x_i, x_j) such that $x_i < x_j$ and $x_i x_j^{-1}$ belongs to the set H_k. If the induced colouring were to contain a monochromatic triangle with vertices x_i, x_j, x_k, where $x_i < x_j < x_k$, then we would have $x_i x_j^{-1}$, $x_i x_k^{-1}$ and $x_j x_k^{-1}$ all belonging to the same set H_l of the partition, which is impossible because

$$(x_i x_j^{-1})(x_j x_k^{-1}) = x_i x_k^{-1},$$

but H_l is a sum-free set. So the colouring is proper.

This use of sum-free partitions to construct proper colourings has been widely studied in the case of abelian groups (whence the word "sum-free" rather than "product-free") whose orders are prime to 3. A sum-free partition of H is called *symmetric* if and only if x and x^{-1} always belong to the same set H. If a sum-free partition is symmetric, then $x_i x_j^{-1}$ and $x_j x_i^{-1}$ always belong to the same set, and the induced colouring is independent of the ordering imposed on the group. However, if 3 divides n, symmetric sum-free partitions cannot exist, because H must contain at least one element, say y, of order 3, and $yy = y^{-1}$, so a set containing both y and y^{-1} cannot be sum-free. Therefore it is interesting to discuss non-symmetric cases also.

It should be observed that not every proper r-colouring of K_n comes from a sum-free r-partition of a group of order n. The first example of this phenomenon occurs among the proper 2-colourings of K_4. There are two such colourings possible, as shown in Figure 1. There are two groups of order 4, namely $Z_4 = \langle x \mid x^4 = 1\rangle$ and $Z_2 \times Z_2 = \langle a, b \mid a^2 = b^2 = [a, b] = 1\rangle$.

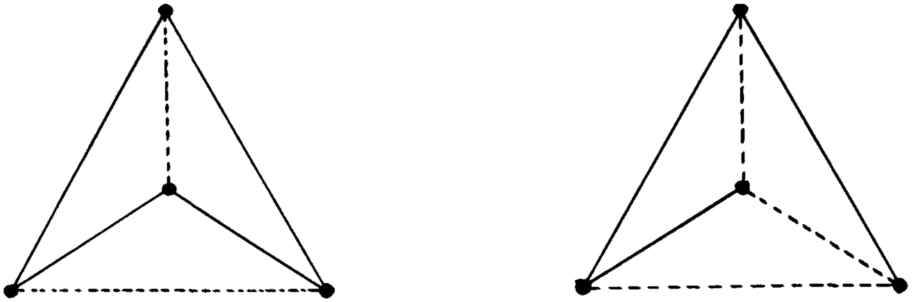

Figure 1

There is only one sum-free 2-partition of Z_4, namely

$$\{x, x^3\},\ \{x^2\},$$

and there are three such partitions of $Z_2 \times Z_2$, all of which are isomorphic to

$$\{a, b\},\ \{ab\}.$$

In both cases the partitions are symmetric and give rise to the colouring of Figure 1(a).

Thus the colouring of Figure 1(b) does not arise from a partition of a group of order 4; however, it does arise in the following way. Consider

$$Z_5 = \langle x \mid x^5 = 1\rangle,$$

which has only one sum-free 2-partition, namely

$$\{x, x^4\},\ \{x^2, x^3\}.$$

This partition is also symmetric and leads to the colouring of Figure 2(a).

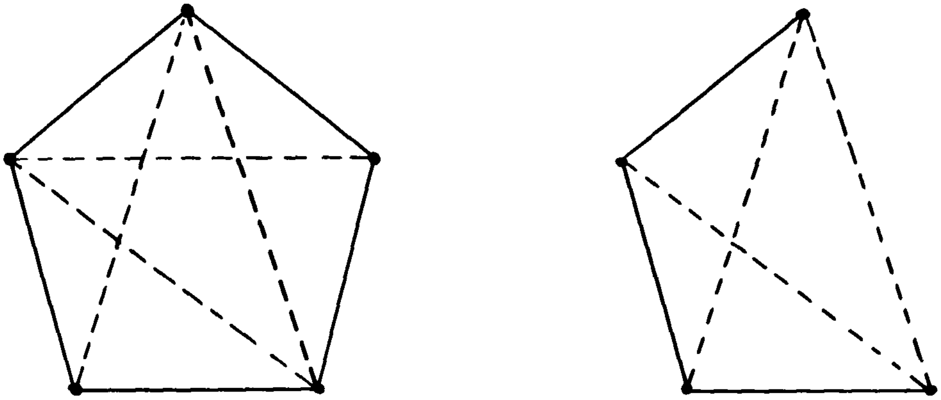

Figure 2

Deletion of any vertex and the four edges incident with it gives the colouring of K_4 shown in Figure 2(b), which is isomorphic with that in Figure 1(b). This raises the following interesting question.

Suppose we have a proper r-colouring of K_n, not induced by a sum-free r-partition of a group of order n. Is it always possible to embed this in a proper s-colouring of K_m, for some $s \geqq r$ and for some $m > n$, which is induced by a sum-free s-partition of some group of order m? Or (less hopefully) under what circumstances is such an embedding possible and what can we say about s and m as functions of r and n?

In Section 2, we discuss the proper 3-colourings of K_{16}, the largest complete graph for which three colours are sufficient. In a subsequent paper, we shall discuss the proper 3-colourings of K_6, the smallest complete graph for which three colours are necessary; at present we merely note that there are 332 non-isomorphic 3-colourings, 75 of which are induced by sum-free 3-partitions of the groups of order 6.

In Section 3, we consider the automorphism group of coloured graphs with colourings induced by sum-free partitions; in Section 4, we look briefly at the possibility of embedding a proper r-colouring of K_n in a proper r-colouring of K_{n+1} and finally in Section 5, we discuss some relationships between block designs and proper colourings.

2. The proper 3-colourings of K_{16}

Kalbfleisch and Stanton (1968) have shown that there exist precisely two non-isomorphic proper 3-colourings of K_{16}; their edge-colourings are listed in Table's 1 and 5. We refer to them as X and Y respectively.

(a) The colouring X.

1																
2	*G*															
3	*B*	*G*														
4	*B*	*B*	*G*													
5	*G*	*B*	*B*	*G*												
6	*B*	*G*	*R*	*R*	*G*											
7	*G*	*B*	*G*	*R*	*R*	*B*										
8	*R*	*G*	*B*	*G*	*R*	*R*	*B*									
9	*R*	*R*	*G*	*B*	*G*	*R*	*R*	*B*								
10	*G*	*R*	*R*	*G*	*B*	*B*	*R*	*R*	*B*							
11	*G*	*R*	*B*	*B*	*R*	*R*	*B*	*G*	*G*	*B*						
12	*R*	*G*	*R*	*B*	*B*	*B*	*R*	*B*	*G*	*G*	*R*					
13	*B*	*R*	*G*	*R*	*B*	*G*	*B*	*R*	*B*	*G*	*G*	*R*				
14	*B*	*B*	*R*	*G*	*R*	*G*	*G*	*B*	*R*	*B*	*G*	*G*	*R*			
15	*R*	*B*	*B*	*R*	*G*	*B*	*G*	*G*	*B*	*R*	*R*	*G*	*G*	*R*		
16	*R*	*R*	*R*	*R*	*R*	*G*	*G*	*G*	*G*	*G*	*B*	*B*	*B*	*B*	*B*	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 1

The colouring X was originally studied by Greenwood and Gleason (1955), who showed that it could be induced by a sum-free 3-partition of $(Z_2)^4$, the additive group of $GF[2^4]$. If we consider $GF[2^4]$ as the set of polynomials in x over $GF[2]$, modulo $x^4 = x + 1$, then the correspondence between the labelled vertices in K_{16} and the field elements is given in Table 2 and the sets of the sum-free partition are the cyclotomic classes with respect to the cubic residues, so that

$$S_R = C_0 = \{1, x^3, x^3 + x, x^3 + x^2, x^3 + x^2 + x + 1\},$$

$$xS_R = S_G = C_1 = \{x, x + 1, x^3 + x + 1, x^2 + x + 1, x^3 + x^2 + 1\},$$

$$x^2S_R = S_B = C_2 = \{x^2, x^2 + x, x^2 + 1, x^3 + x^2 + x, x^3 + 1\}.$$

1	x^3+x^2+x+1	5	x^3	9	$x+1$	13	x^2+x
2	x^3+x^2	6	x^3+x+1	10	x^3+x^2+1	14	x^3+1
3	1	7	x	11	x^2	15	x^2+1
4	x^3+x	8	x^2+x+1	12	x^3+x^2+x	16	0

Table 2

More recently, Whitehead (1975) has shown that the group

$$(4,4 \mid 2,2) = \langle r, s \mid r^4 = s^4 = (rs)^2 = (r^{-1}s)^2 = 1\rangle$$

has a symmetric sum-free 3-partition, with

$$S_R = \{r, r^3, s, s^3, r^2s^2\},$$

$$S_G = \{r^2, rs^3, r^3s, r^2s, r^2s^3\},$$

and $S_B = \{s^2, rs, r^3s^3, rs^2, r^3s^2\}$.

This partition also induces the colouring X of K_{16}, where the correspondence of labelled vertices to group elements is given in Table 3.

1	r^2s^2	5	s^3	9	r^2s^3	13	rs
2	s	6	r^2	10	r^3s	14	r^3s^3
3	r	7	rs^3	11	s^2	15	rs^2
4	r^3	8	r^2s	12	r^3s^2	16	1

Table 3

We calculate Aut(X), the automorphism group of X. Kalbfleisch and Stanton (1968) have already observed that the following maps belong to Aut(X):

(i) $\psi_\alpha : i \mapsto i + \alpha$ for each $\alpha \in GF[2^4]$, which is transitive and preserves colours;

(ii) $\chi: i \mapsto xi$ where x generates $GF[2^4]$ which, considered as a permutation of vertices, becomes

$$\chi = (1,10,14,3,7,11,5,9,13,2,6,15,4,8,12)$$

and permutes the colours in the fashion (RGB);

(iii) $\phi = (1,2,5,4)\ (6,12,10,14)\ (7,15,9,11)\ (8,13)$ which also stabilises 16 and permutes the colours (BG).

Since $\operatorname{Aut}(X)$ is transitive, we concentrate on the stabiliser of 16. By (ii) and (iii), it is transitive on colours and can move the classes $R = \{1,2,3,4,5\}$, $G = \{6,7,8,9,10\}$, $B = \{11,12,13,14,15\}$ in any fashion.

Now we consider the stabiliser of $\{R, G, B, 16\}$. Since $\chi^3 = (1,3,5,2,4)$ $(10,7,9,6,8)$ $(14,11,13,15,12)$, this stabiliser is transitive on B. Hence it is sufficient to look at the stabiliser of $\{R, B, G, 11, 16\}$.

Consider the adjacencies to 16 and 11 in X. These are described by Table 4, where, for example, the entry $\{3,4\}$ in position $(11B, 16R)$ means that both 3 and 4 are joined to 11 by a blue edge and to 16 by a red edge.

	$16R$	$16G$	$16B$
$11R$	$\{2,5\}$	$\{6\}$	$\{12,15\}$
$11G$	$\{1\}$	$\{8,9\}$	$\{13,14\}$
$11B$	$\{3,4\}$	$\{7,10\}$	

Table 4

If ρ is a vertex permutation which stabilises $\{R, G, B, 11, 16\}$, then ρ must stabilise every entry of this array. Hence ρ is a product of some of the transpositions (2,5), (3,4), (8,9), (7,10), (12,15), (13,14).

Suppose we apply (2,5) but not (3,4) on the edge-colouring array of X in Table 1. The top left block changes from

$$\begin{array}{c} -GBBG \\ G-GBB \\ BG-GB \\ BBG-G \\ GBBG- \end{array} \qquad \text{to} \qquad \begin{array}{c} -GBBG \\ G-BGB \\ BB-GG \\ BGG-B \\ GBGB- \end{array}$$

and similarly, applying (3,4) but not (2,5) causes a change. So the contribution from these transpositions to any automorphism must be either (2,5) (3,4) or (1). Similarly, we must have (7,10) (8,9) together and (12,15) (13,14). Checking the remaining possibilities shows that the only non-trivial permutation to stabilise $\{R, B, G, 11, 16\}$ is

$$\zeta = (2,5)(3,4)(7,10)(8,9)(12,15)(13,14).$$

Then $\mathrm{Aut}(X)$ has order $16 \cdot 6 \cdot 5 \cdot 2 = 960$ and is generated by ψ_α, χ, ϕ and ζ. It is doubly-transitive in its natural permutation representation on the vertices. Since X contains 80 *trichromatic and 480 bichromatic triangles,* $\mathit{Aut}(X)$ *cannot* be triply transitive (for this would force all triangles in X to be chromatically identical). So the group is precisely doubly-transitive.

The Sylow 2-subgroups of $\mathrm{Aut}(X)$ are of order 64. We consider one such subgroup, H, where $H = \langle \psi_1, \psi_x, \psi_{x^2}, \psi_{x^3}, \phi \rangle$. H has three subgroups of order 32, namely

$$L = \langle \psi_1, \psi_x, \psi_{x^2}, \psi_{x^3}, \phi^2 \rangle,$$
$$M = \langle \psi_1, \psi_x, \psi_{x^2}, \phi \rangle$$
and
$$N = \langle \psi_1, \psi_x, \psi_{x^2}, \phi\psi_{x^3} \rangle.$$

H contains 16 elements of order 8 (all of which belong to N and none of which belongs to L or M), 28 elements of order 4 and 19 of order 2. The six subgroups of H of order 16 are as follows:

$$\Phi(H) = \langle \psi_1, \psi_x, \psi_{x^2}, \phi^2 \rangle = L \cap M \cap N \cong Z_2 \times D_4;$$
$$A = \langle \psi_1, \psi_{x^2+x}, \psi_{x^3}, \phi^2 \rangle \cong Z_2 \times D_4 \cong B = \langle \psi_1, \psi_{x^2+x}, \psi_{x^3+x}, \phi^2 \rangle;$$
$$C = \langle \psi_1, \psi_x, \psi_{x^2}, \phi^2\psi_{x^3} \rangle \cong (4,4 \mid 2,2) \cong$$
$$D = \langle \psi_1, \psi_{x^2+x}, \psi_{x^3}, \phi^2\psi_x \rangle;$$
$$E = \langle \psi_1, \psi_x, \psi_{x^2}, \psi_{x^3} \rangle \cong (Z_2)^4.$$

(Here D_4 is the dihedral group of order 8 and $Z_2 \times D_4$ denotes its direct product with Z_2.) Each subgroup of order 16 is contained in L; the only subgroup of order 16 contained in M or N is the Frattini subgroup $\Phi(H)$.

(b) The colouring Y.

1																
2	*G*															
3	*B*	*B*														
4	*B*	*G*	*G*													
5	*G*	*B*	*G*	*B*												
6	*B*	*G*	*R*	*R*	*G*											
7	*G*	*B*	*G*	*R*	*R*	*B*										
8	*R*	*G*	*G*	*B*	*R*	*R*	*B*									
9	*R*	*R*	*B*	*G*	*G*	*R*	*R*	*B*								
10	*G*	*R*	*R*	*G*	*B*	*B*	*R*	*R*	*B*							
11	*G*	*R*	*B*	*B*	*R*	*R*	*B*	*G*	*G*	*B*						
12	*R*	*B*	*R*	*B*	*G*	*B*	*R*	*G*	*B*	*G*	*R*					
13	*B*	*R*	*G*	*R*	*B*	*G*	*B*	*R*	*B*	*G*	*G*	*R*				
14	*B*	*B*	*R*	*G*	*R*	*G*	*G*	*B*	*R*	*B*	*G*	*G*	*R*			
15	*R*	*G*	*B*	*R*	*B*	*B*	*G*	*B*	*G*	*R*	*R*	*G*	*G*	*R*		
16	*R*	*R*	*R*	*R*	*R*	*G*	*G*	*G*	*G*	*G*	*B*	*B*	*B*	*B*	*B*	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Table 5

The colouring Y shown in Table 5 was initially found by Kalbfleisch and Stanton (1968) who worked directly with the graph of K_{16}. It was subsequently pointed out by Whitehead (1971) that it could be induced by the sum-free 3-partition of $Z_4 \times Z_4$, where

$$S_R = \{22,32,23,21,12\},\ S_G = \{02,33,30,10,11\} \text{ and } S_B = \{20,01,13,31,03\},$$

with the correspondence of labelled vertices to group elements given in Table 6.

1	22	5	12	9	10	13	13
2	32	6	02	10	11	14	31
3	23	7	33	11	20	15	03
4	21	8	30	12	01	16	00

Table 6

More recently, Whitehead (1975) has shown that the same colouring is induced by the sum-free 3-partition of $Z_2 \times D_4$, where

$$Z_2 \times D_4 = \langle a, b, c \mid a^4 = b^2 = c^2 = [a, c] = [b, c] = 1, ba^3 = ab \rangle,$$

and $$S_R = \{a^2c, ac, a^2bc, a^2b, a^3c\},$$

$$S_G = \{c, a^3bc, a, a^3, ab\}$$

and $$S_B = \{a^2, b, abc, a^3bc, bc\}.$$

Table 7 gives the correspondence of vertices to elements.

1	a^2c	5	a^3c	9	a^3	13	abc
2	ac	6	c	10	ab	14	a^3b
3	a^2bc	7	a^3bc	11	a^2	15	bc
4	a^2b	8	a	12	b	16	1

Table 7

The following maps belong to Aut(Y):

(i) $\psi_\alpha : i \mapsto i + \alpha$ for each $\alpha \in Z_4 \times Z_4$, which is transitive;

(ii) $\beta : xy \mapsto (-y)x$, where the vertex is labelled with the group element $xy \in Z_4 \times Z_4$, which may be written in terms of permutation of vertices as (2,3,5,4) (6,11) (7,13,10,14) (8,15,9,12) and acts on the colours (GB);

(iii) $\gamma = (1,11)\,(2,13,5,14)\,(3,12,4,15)\,(7,8,10,9)$ which permutes the colours (RB).

Since Aut(Y) is transitive, we concentrate on the stabiliser of 16. This contains both β and γ and is therefore transitive on the colours. It can have at most two orbits on vertices, namely $\{1,6,11\}$ and $\{2,3,4,5,7,8,9,10,12,13,14,15\}$; since a search shows that no element of Aut(Y) stabilises 16 and maps $1 \mapsto 2$, these two orbits do in fact exist, so Aut(Y) has rank 3 and is not doubly-transitive.

If we now consider the stabiliser of $\{1,6,11,16\}$, we find (as in the discussion of Aut(X)) that the only non-trivial permutation in this stabiliser is the map ζ defined previously. Hence Aut(Y) has order $16 \cdot 6 \cdot 2 = 192$ and is generated by ψ_α, β and γ. (Since $\zeta = \beta^2 = \gamma^2$, we may omit it from the set of generators.)

Again the Sylow 2-subgroups of Aut(Y) have order 64. We consider one such subgroup $P = \langle \psi_{10}, \psi_{01}, \beta \rangle$, which has 44 elements of order 4 and 19 elements of order 2. P has two subgroups of order 32, namely

$$Q = \langle \psi_{10}, \psi_{01}, \zeta \rangle \text{ and } R = \langle \psi_{11}, \psi_{02}, \beta \rangle.$$

The six subgroups of P of order 16 are as follows:

$$\Phi(P) = \langle \psi_{11}, \psi_{02}, \zeta \rangle = Q \cap R \cong Z_2 \times D_4;$$

$$S_1 = \langle \psi_{10}, \psi_{02}, \zeta \rangle \cong Z_2 \times D_4 \cong S_2 = \langle \psi_{01}, \psi_{20}, \zeta \rangle;$$

$$T_1 = \langle \psi_{20}, \psi_{02}, \beta \rangle \cong (4,4 \mid 2,2) \cong T_2 = \langle \psi_{20}, \psi_{02}, \beta\psi_{11} \rangle;$$

$$U = \langle \psi_{10}, \psi_{01} \rangle \cong Z_4 \times Z_4.$$

The subgroups $\Phi(P)$, S_1, S_2 and U are contained in Q; $\Phi(P)$, T_1 and T_2 are contained in R. S_1 is conjugate to S_2 in P and T_1 to T_2.

A computer check run by Whitehead (1975) showed that of the groups of order 16, only $(Z_2)^4$, $Z_4 \times Z_4$, $Z_2 \times D_4$ and $(4,4 \mid 2,2)$ have sum-free 3-partitions. All the partitions are symmetric.

3. Concerning automorphism groups

Suppose G is a group of order n, with elements $x_1, x_2, \cdots, x_n$ which has a sum-free r-partition

$$G^* = G_1 \cup G_2 \cup \cdots \cup G_r.$$

Let K denote a copy of K_n in which vertices are labelled with the elements of G. Consider the proper colouring of K induced by the given partition and the ordering

$$x_1 < x_2 < \cdots < x_n.$$

The effect of the vertex map $\phi_g: x_i \mapsto x_i g$, where g is any element of G, is to transform the proper colouring into the one induced by the ordering

$$x_1 g < x_2 g < \cdots < x_n g.$$

But these two colourings are the same since

$$x_i g (x_j g)^{-1} = x_i x_j^{-1}.$$

This means that ϕ_g is an automorphism of the colouring. As $\{\phi_g \mid g \in G\} \cong G$, it follows that the automorphism group of the colouring must contain a subgroup isomorphic to G.

In the last section, we computed the subgroups of order 16 in Aut(X) and Aut(Y), where X and Y are the proper 3-colourings of K_{16}. We found only four groups, namely $(Z_2)^4$, $Z_4 \times Z_4$, $Z_2 \times D_4$ and $(4,4 \mid 2,2)$, confirming the result of Whitehead (1975) that no other group of order 16 has a sum-free 3-partition.

It should be noted that our observations do not bar the possibility that a partition of $Z_2 \times D_4$ could give rise to the colouring X, or a partition of $(4,4 \mid 2,2)$ to the colouring Y. However, neither of these in fact occurs.

4. Embedding

As was pointed out in the Introduction, we would like to know when it is possible to embed a proper colouring of a given graph in a proper colouring of a larger graph. This seems to be a difficult question: even in a case as small as K_4, one of the two proper colourings can be embedded in a proper colouring of K_5, while the other cannot.

Most of the results so far obtained have hinged on elementary analysis of the monochromatic subgraphs, as in the following proof.

THEOREM. *Suppose that a proper r-colouring of* K_n *is induced by the symmetric sum-free r-partition*

$$G^* = S_1 \cup S_2 \cup \cdots \cup S_r, \tag{1}$$

where G is a group of order n and G^* *the set of its non-identity elements. Then this colouring may be embedded in a proper r-colouring of* K_{n+1} *if and only if there exists an associated r-partition*

$$G = T_1 \cup \cdots \cup T_r, \tag{2}$$

such that $(T_i - T_i) \cap S_i = \phi$, $i = 1, 2, \cdots, r$.

PROOF. (a) Suppose that the associated partition (2) exists. Label as ∞ the $(n+1)$st vertex in K_{n+1} and complete the colouring of K_{n+1} by assigning the colour i to the edge (∞, x) if $x \in T_i$.

If the monochromatic graph in colour i contains a triangle, it must be of the form $\{\infty, x, y\}$, for the colouring of K_n is proper. Since the edges (∞, x) and (∞, y) are coloured i, we have $x, y \in T_i$. Hence $x - y \in T_i - T_i$, so $x - y \notin S_i$. But this means that (x, y) is not i-coloured, so we have a contradiction.

(b) Suppose the embedding exists and that the $(n+1)$st vertex of K_{n+1} is labelled ∞. Define the set $T_i \subseteq G$ by

$$T_i = \{x \mid x \in G, (\infty, x) \text{ is coloured with colour } i\}.$$

Since no triangle is monochromatic, we see that if (∞, x) and (∞, y) are coloured i, then (x, y) is coloured in some other colour, or in other words, if $x, y \in T_i$ then $x - y \notin S_i$. This completes the proof.

As an application of the Theorem, we consider two non-isomorphic symmetric sum-free 3-partitions of Z_{13} and the non-isomorphic proper 3-colourings of K_{13} which they induce, neither of which can be embedded in a proper 3-colouring of K_{14}.

Our first partition P consists of the sets

$$S_R = \{1,5,8,12\},$$
$$S_B = \{2,3,10,11\},$$
$$S_G = \{4,6,7,9\}.$$

These sets are isomorphic to each other (since $2S_G = S_R$ and $2S_R = S_B$), not in arithmetic progression and in each case

$$S_i + S_i = \overline{S_i},$$

so that $|S_i| = 4$, $|S_i + S_i| = 9$. Hence if the associated 3-partition exists, we have

$$|T_R| + |T_B| + |T_G| = 13$$

and $$|T_i - T_i| \leqq 9,\ i = R, B, G.$$

By the Cauchy-Davenport theorem [Wallis, Street and Wallis (1972), p. 187, Theorem 6.4] we know that

$$2|T_i| - 1 \leqq |T_i - T_i| \leqq 9,$$

so that $|T_i| \leqq 5$, and by Vosper's theorem [Wallis, Street and Wallis (1972), p. 188, Theorem 6.9] this implies that $|T_i| = 5$ if and only if T_i is in arithmetic progression. But T_i in arithmetic progression implies that $T_i - T_i$ is also in arithmetic progression; if $|T_i| = 5$, then $T_i - T_i' = \overline{S_i}$ which forces $\overline{S_i}$ and hence S_i to be in arithmetic progression. But this is false for each of the given sets. Hence $|T_i| \leqq 4$ and $|T_R| + |T_B| + |T_G| \leqq 12$, which is a contradiction.

So we see that P induces a proper 3-colouring of K_{13} which cannot be extended to a proper 3-colouring of K_{14}. In particular, if we consider a monochromatic subgraph of P, say the red graph, then at most four of its vertices can be joined by red edges to the extra vertex, ∞, without forming a monochromatic triangle.

Our second partition, π, consists of the sets

$$\Sigma_R = \{1,4,9,12\},$$
$$\Sigma_B = \{2,3,10,11\},$$
$$\Sigma_G = \{5,6,7,8\}.$$

$\Sigma_B = S_B$ is not in arithmetic progression, but Σ_G and Σ_R are isomorphic to each other and both are in arithmetic progression ($\Sigma_G = 5\Sigma_R$). So π is certainly not isomorphic to P. If we have an associated 3-partition $\tau_R \cup \tau_B \cup \tau_G$, then by the previous argument we have

$$|\tau_B| \leqq 4,\ |\tau_R| \leqq 5,\ |\tau_G| \leqq 5,$$

but since $|\tau_R| + |\tau_B| + |\tau_G| = 13$, we must have either $|\tau_R| = |\tau_G| = 5$, $|\tau_B| = 3$, or $|\tau_R| = |\tau_B| = 4$, $|\tau_G| = 5$ (without loss of generality).

Since $|\tau_G| = 5$, we must have $\tau_G - \tau_G = \overline{\Sigma_G}$, and by Wallis Street and Wallis (1972), p. 191, Lemma 6.11, this implies that

$$\tau_G = \{a, a+1, a+2, a+3, a+4\}$$

for some $a \in Z_{13}$. Hence for $b = a+5$, we have

$$\tau_R \cup \tau_B = \{b, b+1, \cdots, b+7\}.$$

If $|\tau_R| = 5$, then $\tau_R - \tau_R = \overline{\Sigma_R}$, and the same argument shows that

$$\begin{aligned}\tau_R &= \{c, c+5, c+10, c+15, c+20\}\\ &= \{c, c+2, c+5, c+7, c+10\}\end{aligned}$$

for some $c \in Z_{13}$. Since no such set is contained in $\tau_R \cup \tau_B$, we must have $|\tau_R| = |\tau_B| = 4$. Suppose $x \in \tau_R$. Then $x+1, x+4, x+9, x+12 \notin \tau_R$. If $x+1 \notin \tau_B$, then $x = b+7$, so that $b+3, b+6 \in \tau_B$. But $(b+6)-(b+3) = 3 \in \Sigma_B$ which is a contradiction, so $x+1 \in \tau_B$. If also $x+4 \in \tau_B$, then again $3 \in (\tau_B - \tau_B) \cap \Sigma_B$ so $x+4 \notin \tau_R \cup \tau_B$. Hence $x = b+4$ or $b+5$ or $b+6$. But in any of these cases, $x+12 \in \tau_B$ also, so that $(x+1)-(x+12) = 2 \in (\tau_B - \tau_B) \cap \Sigma_B$, again a contradiction. Hence no such partition exists.

Again π induces a proper 3-colouring of K_{13} which cannot be extended to a proper 3-colouring of K_{14}, but the colouring induced by π is not isomorphic to that induced by P. For the red graph, in this second colouring, can have five of its vertices joined by red edges to the extra vertex, ∞, without forming a triangle.

5. Block designs and proper colourings

A (*balanced incomplete*) *block design* with parameters v, b, r, k, λ is a way of selecting b subsets each of size k from a v-set of objects so that every object occurs in r sets and every pair of objects occurs together in λ sets. If we interpret the objects as vertices and each k-set as the complete graph on the k vertices, then the union of all the k-sets is the complete graph on v vertices. Moreover, if $\lambda = 1$, this interpretation of a block design is equivalent to a decomposition of the complete graph K_v into edge-disjoint complete subgraphs K_k.

We consider the following method of constructing proper colourings. First, a block design with $\lambda = 1$ and first parameter v is found. Second, the K_k representing each k-set is properly coloured in r colours. Finally, the union is taken of these b copies of K_k. The result is an r-colouring of K_v. The colouring may not be proper, because of interaction between the different copies of K_k

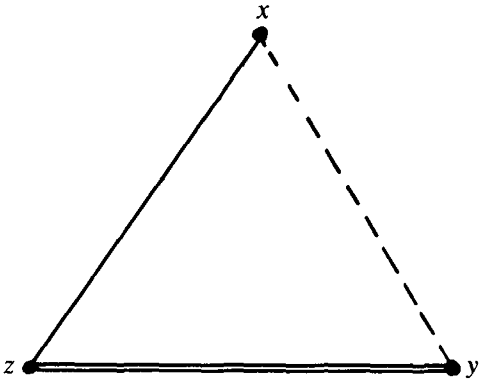

Figure 3

but, if it is proper, then interesting properties may result. In particular, by using the ways in which block designs give rise to other larger block designs, it may be possible to use proper colourings to construct proper colourings of larger graphs.

We have applied this technique to a number of small designs. If we consider the (7,7,3,3,1) design with 3-sets

$$g + \{1,2,4\}, \quad g = 0,1,\cdots,6 \pmod 7)$$

and we write (x, y, z) to mean the 3-coloured K_3 of Figure 3, then the union

$$\bigcup_{g=0}^{6} (1,2,4) + g, \text{ with addition modulo } 7,$$

is the proper 3-colouring shown in Figure 4. This is the same colouring induced

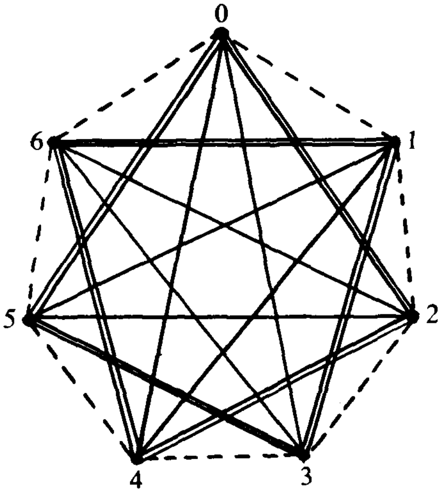

Figure 4

by the symmetric sum-free 3-partition of Z_7, with $S_R = \{1,6\}$, $S_G = \{2,5\}$, $S_B = \{3,4\}$. This colouring can be embedded in a proper 3-colouring of K_8, by applying the Theorem of the previous section with, say, $T_R = \{1,3,5\}$, $T_G = \{0,6\}$, $T_B = \{2,4\}$.

Similarly, consider the (13,13,4,4,1) design with 4 sets

$$g + \{0,1,3,9\}, \quad g = 0,1,\cdots,12 \pmod{13}$$

and let $[x, y, z, t]$ denote the 3-coloured K_4 shown in Figure 5.

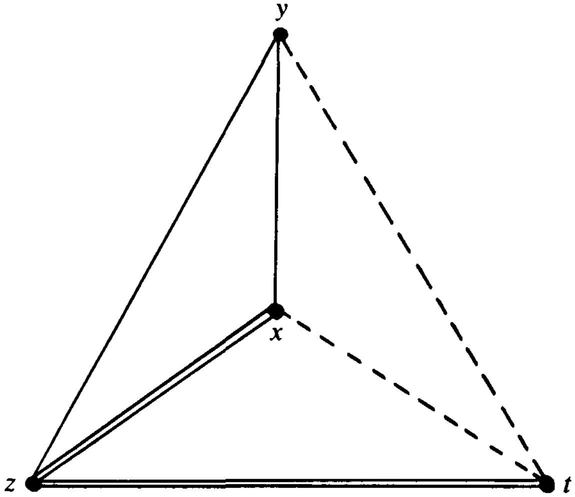

Figure 5

Then the union $\bigcup_{g=0}^{12} [0,1,9,3] + g$, with addition modulo 13, is the colouring of K_{13} induced by the partition P of Section 4, whereas $\bigcup_{g=0}^{12} [2,0,9,3] + g$ is the colouring of K_{13} induced by the partition π. As we have already seen, these colourings are not isomorphic.

Both the proper colourings of K_{16} can be derived from designs. Again let $[x, y, z, t]$ denote the colouring of Figure 5 and let (x, y, z, t) denote the colouring of Figure 6.

Then the following coloured graph is isomorphic to X, derived from a (16,20,5,4,1) block design:

$$(1,15,2,13) \cup (5,7,6,3) \cup (8,10,11,12) \cup (9,14,16,4)$$
$$\cup (1,12,14,5) \cup (2,10,4,6) \cup (13,8,16,3) \cup (15,11,7,9)$$
$$\cup (1,16,11,6) \cup (2,9,8,5) \cup (13,14,10,7) \cup (15,4,12,3)$$
$$\cup (1,8,7,4) \cup (2,11,3,14) \cup (13,12,6,9) \cup (15,10,5,16)$$
$$\cup (1,9,10,3) \cup (2,16,12,7) \cup (13,4,11,5) \cup (15,14,8,6).$$

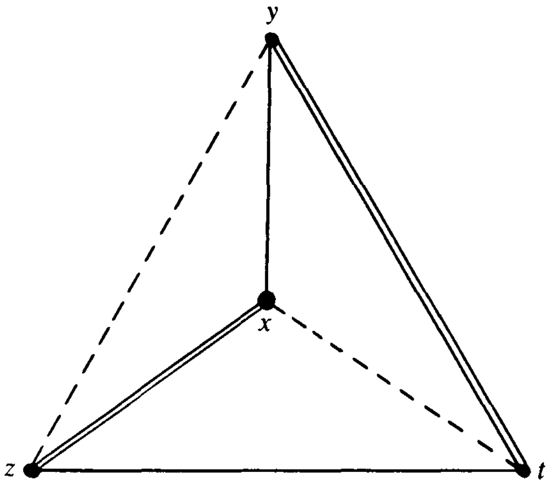

Figure 6

To obtain Y it is necessary to use both the given colourings of K_4, again with the (16,20,5,4,1) design:

$$
\begin{aligned}
&[1,7,3,9] \cup [16,10,12,2] \cup [6,13,15,8] \cup [11,14,4,5]\\
&\cup [10,1,5,15] \cup [7,16,8,4] \cup [14,6,2,3] \cup [13,11,9,12]\\
&\cup [8,12,14,1] \cup [2,15,7,11] \cup [5,3,13,16] \cup [9,4,10,6]\\
&\cup [4,2,1,13] \cup [12,5,6,7] \cup [3,8,11,10] \cup [15,9,16,14]\\
&\cup (1,16,11,6) \cup (7,10,14,13) \cup (3,12,4,15) \cup (9,2,5,8).
\end{aligned}
$$

Y can also be obtained from a (16,16,6,6,2) design. Write $[x, y, z, t, u, v]$ for the coloured graph in Figure 7. Then Y is

$$
\begin{aligned}
&[1,2,8,16,10,14] \cup [1,2,15,9,11,3] \cup [1,5,9,16,7,13] \cup [1,5,12,8,11,4]\\
&\cup [1,7,15,12,10,6] \cup [2,4,9,16,6,12] \cup [2,4,10,13,15,5] \cup [2,6,13,11,8,7]\\
&\cup [3,4,14,6,13,1] \cup [3,5,6,16,8,15] \cup [3,5,12,14,7,2] \cup [3,8,12,10,13,9]\\
&\cup [4,3,7,16,10,11] \cup [4,9,15,7,14,8] \cup [5,6,14,11,9,10] \cup [11,13,15,12,14,16].
\end{aligned}
$$

Every edge is coloured twice under this formula, but the colour is the same in every case.

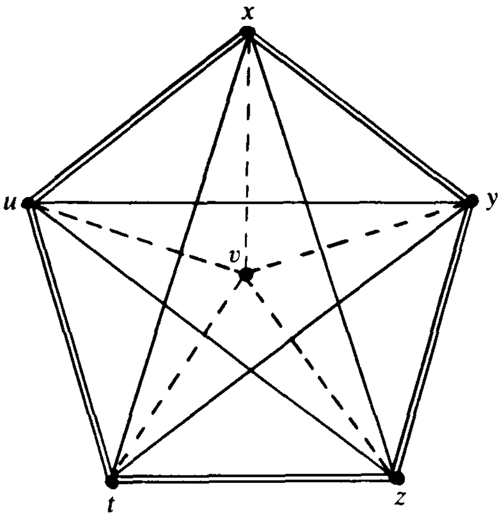

Figure 7

Finally, we have found a (49,56,8,7,1) design which gives rise to a proper 4-colouring of K_{49}. Again we write the elements of $Z_7 \times Z_7$ in the form xy where x and y are integers modulo 7. We let $[a, b, c, d, e, f, g]$ denote the colouring of K_7 shown in Figure 4 (with, say, $a = 0$, $b = 1, \cdots, g = 6$) and $\{a, b, c, d, e, f, g\}$ the colouring shown in Figure 8.

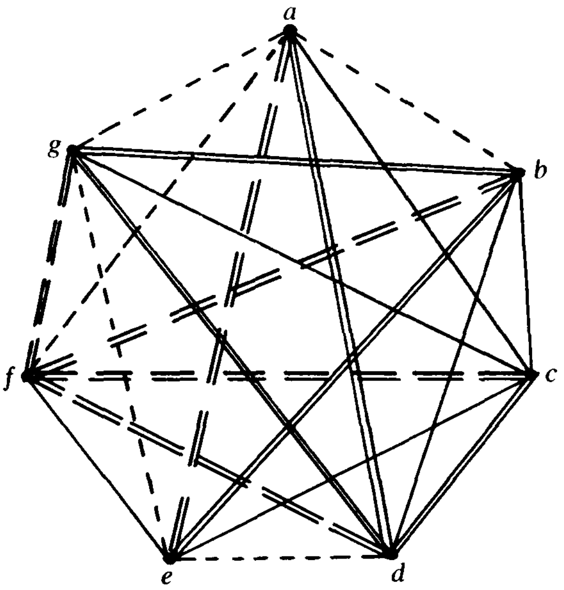

Figure 8

If solid, broken, double-broken and double-solid lines correspond to the colours C_1, C_2, C_3 and C_4 respectively, then the union of the 49 graphs

$$\{00,04,12,14,31,32,61\} + xy, \quad xy \in Z_7 \times Z_7,$$

with the 7 graphs

$$[00,11,22,33,44,55,66] + 0y, \qquad y \in Z_7,$$

is precisely the 4-colouring of K_{49} due to Whitehead (1973) in the form given in Wallis, Street and Wallis (1972), p. 263.

Added 5 *September*, 1975

The questions asked at the end of Section 1 about embedding in larger sum-free colourings have been answered, by Katherine Heinrich and by Anne Penfold Street. Their papers will appear in the Proceedings of the Fourth Australian Conference on Combinatorial Mathematics.

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